UNIVERSITY OF NORTH CAROLINA
Department of Statistics
Chapel Hill, N. C.

NEGATIVE LATIN SQUARE DESIGNS

by

Dale M. Mesner
Purdue University and University of North Carolina

November 1964

This research was supported in part by the National Science Foundation, Grant GP 16-60.

Institute of Statistics
Mimeo Series No. 410

NEGATIVE LATIN SQUARE DESIGNS[1]

by

Dale M. Mesner
Purdue University and University of North Carolina

1. <u>General properties of designs and association schemes</u>. In a balanced or partially balances incomplete block design, a collection of b subsets, called blocks, is chosen from a set of v objects, commonly called varieties or treatments, in such a way that every block contains the same number k of objects, every object occurs in the same number r of blocks, and a further regularity condition holds for the number of occurrences together within blocks of pairs of distinct objects. In a balanced incomplete block (BIB) design this number has the same value λ for all pairs of distinct objects. In an m-class partially balanced incomplete block (PBIB) design [5, 18], any two distinct objects are related as first, second, ..., or m-th associates in accordance with rules to be stated in (1.1), and all pairs of objects which are i-th associates occur together in the same number λ_i of blocks. The arrangement of pairs of distinct objects into associate classes is called an m-class association scheme and involves parameters n_i, p^i_{jk}, i, j, k = 1, 2, ..., m. We denote by P_i the matrix whose element in the j,k position is p^i_{jk}. Association schemes have been found useful in the combinatorial study of PBIB designs, as well as in the analysis of data from experiments in which these designs are applied. To each association scheme there corresponds a family of designs which share

[1] Prepared with the partial support of the National Science Foundation, Grant GP-1660.

this association scheme and have common values of certain parameters, including v, n_i, p^i_{jk}, but which differ in the arrangement of objects into blocks and in values of b, r, k, and λ_i.

An m-class association scheme with v objects is defined by the following conditions [6].

(1.1)
(i) Any two distinct objects are either first, second, ..., or m-th associates.

(ii) Each object has n_i i-th associates, $i = 1, \ldots, m$.

(iii) For any pair of the v objects which are i-th associates, the number p^i_{jk} of objects which are j-th associates of the first and k-th associates of the second is independent of the pair of i-th associates with which we start.

The following are well-known identities which can be derived from this definition.

$$\sum_{i=1}^{m} n_i = v - 1,$$

$$p^i_{jk} = p^i_{kj},$$

$$\sum_{k=1}^{m} p^i_{jk} = n_i,\ j \neq i, \tag{1.2}$$

$$= n_i - 1,\ j = 1,$$

$$n_i p^i_{jk} = n_j p^j_{ik}.$$

These relations among the parameters make it possible to simplify the definition. A two-class association scheme with v objects may be defined by the following conditions [3].

(1.3) (i) Any two objects are either first or second associates.

(ii) Each object has n_1 first associates.

(iii) Given any two objects which are i-th associates, i = 1, 2, there are exactly p_{11}^i other objects which are first associates of both.

Then, defining other parameters by

(1.4)
$$n_1 + n_2 = v - 1,$$
$$p_{12}^1 = p_{21}^1,\ p_{12}^2 = p_{21}^2,$$
$$p_{11}^1 + p_{12}^1 + 1 = p_{11}^2 + p_{12}^2 = n_1,$$
$$p_{12}^1 + p_{22}^1 = p_{12}^2 + p_{22}^2 + 1 = n_2,$$

each object has n_2 second associates and, given any two objects which are i-th associates, there are p_{jk}^i other objects which are j-th associates of the first and k-th associates of the second. Also,

(1.5)
$$n_1 p_{12}^1 = n_2 p_{11}^2,\quad n_1 p_{22}^1 = n_2 p_{12}^2.$$

If N is the v x b incidence matrix of objects and blocks in the design, then the v x v symmetric matrix NN^T has only three distinct characteristic roots θ_0, θ_1, θ_2, with multiplicities α_0, α_1, α_2 respectively, where $\Sigma\, \alpha_i = v$. θ_0 may be expressed

(1.6)
$$\theta_0 = r + n_1\lambda_1 + n_2\lambda_2\ ,$$

and $\alpha_0 = 1$ if NN^T is irreducible (equivalently if the design is connected). Then

(1.7)
$$\alpha_1 + \alpha_2 = v - 1\ .$$

If we define

(1.8) $\gamma = p_{12}^2 - p_{12}^1,$

$\Delta = \gamma^2 + 2p_{12}^1 + 2p_{12}^2 + 1,$

$\sigma = (\Delta^{\frac{1}{2}} - \gamma - 1)/2,$

$\tau = (\Delta^{\frac{1}{2}} + \gamma - 1)/2,$

then it has been shown [13] that

(1.9) $\theta_1 = r + \lambda_1\tau + \lambda_2(-\tau - 1),$

$\theta_2 = r + \lambda_1(-\sigma - 1) + \lambda_2\sigma,$

(1.10) $\alpha_1 = [\sigma n_1 + (\sigma + 1)n_2]/\Delta^{\frac{1}{2}},$

$\alpha_2 = [(\tau + 1)n_1 + \tau n_2]/\Delta^{\frac{1}{2}}.$

The parameters γ, Δ, σ, τ, α_1, α_2 depend only on the association scheme and not on blocks. Other known relations [17] that will be needed later are

(1.11) $vn_1n_2 = \Delta\,\alpha_1\,\alpha_2,$

(1.12) $p_{12}^1 = \sigma(\tau + 1),$

$p_{12}^2 = \tau(\sigma + 1).$

If in a two-class association scheme we interchange the designation of first and second associates we obtain another association relation which satisfies (1.1). Two association schemes related in this way will be/complements of each other. If a scheme has parameters

said to be

$$v, \quad \begin{matrix} n_1 = \ell, \\ n_2 = m, \end{matrix} \quad P_1 = \begin{bmatrix} c & d \\ d & e \end{bmatrix}, \quad P_2 = \begin{bmatrix} f & g \\ g & h \end{bmatrix},$$

its complement will have parameters

$$v, \quad \begin{matrix} n_1 = m, \\ n_2 = \ell, \end{matrix} \quad P_1 = \begin{bmatrix} h & g \\ g & f \end{bmatrix}, \quad P_2 = \begin{bmatrix} e & d \\ d & c \end{bmatrix}.$$

There are m-class schemes for $m > 2$ which differ only by a permutation of associate classes, although the term "complement" is not appropriate in such cases.

Most known two-class PBIB designs have been classified by Bose and Shimamoto [6] into five types, distinguished primarily by the structure of their association schemes. The simplest type is <u>group divisible</u>, in which the $v = mn$ objects are arranged into m disjoint groups of n objects, and objects are first associates if and only if they are in the same group. For a group divisible scheme,

$$(1.13) \qquad v = mn,$$

$$\begin{matrix} n_1 = n-1, \\ n_2 = n(m-1), \end{matrix} \qquad P_1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}.$$

<u>Cyclic</u> type schemes are defined in terms of certain combinatorial properties and have parameters which may be expressed as follows in terms of an integer t.

$$(1.14) \qquad v = 4t + 1,$$

$$P_1 = \begin{bmatrix} t-1 & t \\ t & t \end{bmatrix}, \quad P_2 = \begin{bmatrix} t & t \\ t & t-1 \end{bmatrix},$$

$$n_1 = n_2 = \alpha_1 = \alpha_2 = 2t.$$

Association schemes with parameters (1.12), whatever their combinatorial structure, will be called pseudo-cyclic. In the next section we take up Latin square association schemes, another type in the Bose-Shimamoto classification, then introduce the new family of designs, negative Latin square, which are the principal topic of this paper.

2. Negative Latin square designs. An association scheme of Latin square type with $v = n^2$ objects and g constraints, which we denote as an $L_g(n)$ scheme, is defined by an n x n square array of the objects and a set of g-2 pairwise orthogonal Latin squares of order n. Two objects are first associates if and only if they occur in the same row or column of the array or in positions occupied by the same letter in any of the Latin squares. If to the set of Latin squares we adjoin two more n x n arrays of n letters, one in which the i-th letter occupies all positions in the i-th row and another in which the i-th letter occupies all positions in the i-th column, we have g pairwise orthogonal squares (not all Latin) and may define first associates somewhat more symmetrically as objects which occur in positions occupied by the same letter in any of the squares. Finite nets [7, 9] and orthogonal arrays [11] may be used as the basis for equivalent definitions. $L_g(n)$ parameters are given by

(2.1) $$v = n^2,$$

$$n_1 = g(n-1), \qquad P_1 = \begin{bmatrix} (g-1)(g-2)+n-2 & (n-g+1)(g-1) \\ (n-g+1)(g-1) & (n-g+1)(n-g) \end{bmatrix},$$

$$n_2 = (n-g+1)(n-1), \qquad P_2 = \begin{bmatrix} g(g-1) & g(n-g) \\ g(n-g) & (n-g)(n-g-1)+n-2 \end{bmatrix}.$$

These lead to further parameters

(2.2) $\sigma = g-1, \quad \tau = n-g, \quad \Delta = n^2 ,$

$$\alpha_1 = g(n-1), \quad \alpha_2 = (n-g+1)(n-1) .$$

Association schemes with parameters (2.1), whatever their combinatorial structure, will be called <u>pseudo-Latin square</u>.

Since there can be at most n-1 pairwise orthogonal Latin squares of order n, g cannot exceed n+1; moreover, if g = n+1, all pairs of objects are first associates and the design reduces to a BIB design. The result is the same with g = 0.

A Latin square association scheme with g = 1 constraint is a special case of a group divisible scheme, while it is easy to show that its complement has this structure in the case g = n. We may therefore assume

(2.3) $2 \leq g \leq n-1 .$

We observe that the complement of a Latin square association scheme with g constraints is a pseudo-Latin square scheme with n+1-g constraints. This was illustrated in the preceding paragraph and becomes obvious if we use the brief notation f = n+1-g and note pairs of symmetric expressions such as

$$n_1 = g(n-1), \quad n_2 = f(n-1),$$

and

$$p^1_{22} = f(f-1), \quad p^2_{11} = g(g-1) .$$

As a result, any pseudo-Latin square association scheme may be reduced by choice of notation to a pseudo-Latin square scheme with

(2.4) $2 \leq g \leq (n+1)/2 .$

These are simply the schemes of this family for which $n_1 \leq n_2$. An example will show that not all pseudo-Latin square schemes have Latin square structure. An $L_3(6)$ scheme can be constructed from any 6x6 Latin square. Its complement then has $L_4(6)$ parameters but cannot have Latin square structure since no set of 4-2 = 2 orthogonal 6x6 Latin squares exists. On the other hand, it is known [21, 9, 16] for a wide range of values of n and g that an association scheme with parameters (2.1) necessarily corresponds to a set of g-2 pairwise orthogonal Latin squares of order n.

While minor infringements of inequality (2.3) lead only to trivial special cases, we now obtain something interesting by committing a major violation. Negative values of n and g lead in many cases to parameters (2.1) which are non-negative integers. These parameters satisfy conditions (1.4) and (1.5), which reduce to algebraic identities in n and g, but differ from the parameters of any of the types of association schemes in the Bose-Shimamoto classification. This suggests the existence of a new series of 2-class PBIB designs, based on association schemes with such parameters. The name negative Latin square will be used for designs and association schemes in the new series. The simplest case is n = -4, g = -1, giving the following, which could be termed $L_{-1}(-4)$ parameters.

$$v = 16,$$

$$n_1 = 5, \qquad P_1 = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix},$$

$$n_2 = 10,$$

Designs are known with these parameters, showing that the negative Latin square family of designs is not vacuous.

Instead of using (2.1) with negative arguments for negative Latin square parameters it is convenient to have expressions in terms of positive arguments, which we shall still denote, however, by the same letters n and g. Then using the negative integers -n and -g in (2.1) we arrive at

(2.5)
$$v = n^2, \qquad P_1 = \begin{bmatrix} (g+1)(g+2)-n-2 & (n-g-1)(g+1) \\ (n-g-1)(g+1) & (n-g-1)(n-g) \end{bmatrix},$$
$$n_1 = g(n+1), \qquad P_2 = \begin{bmatrix} g(g+1) & g(n-g) \\ g(n-g) & (n-g)(n-g+1)-n-2 \end{bmatrix}.$$
$$n_2 = (n-g-1)(n+1),$$

In terms of the positive integers n and g, we denote these as $NL_g(n)$ parameters. Using (2.5) in (1.8) and (1.10),

(2.6)
$$\sigma = n-g-1, \quad \tau = g, \quad \Delta = n^2,$$
$$\alpha_1 = (n-g-1)(n+1), \quad \alpha_2 = g(n+1).$$

Alternatively, values of σ, τ, α_i could be obtained by using the negative integers -n and -g in (2.2). This amounts to using the negative square root of Δ in (1.8) and leads to negative values of σ and τ, finally giving values of θ_i and α_i which differ from those of (1.9) and (2.6) by an interchange of indices 1 and 2. In adopting expressions (2.6) we are following the customary [13] notation for θ_i and α_i.

The abbreviations $L_g(n)$ and $NL_g(n)$ will sometimes be shortened to L_g and NL_g when it is not necessary to specify the value of n.

Like the pseudo-Latin square family, which is also defined in terms of the form of its parameters, the negative Latin square family of association schemes contains the complement of each of its members; specifically, the complement of an $NL_g(n)$ scheme is an $NL_{n-g-1}(n)$ scheme. As a result, any negative Latin square scheme may be reduced by choice of notation to one for which

$$g \leq \frac{1}{2}(n-1),$$

or equivalently

$$n_1 \leq n_2 \ .$$

The requirement that p^1_{11} is non-negative places a lower bound on g.

If n is odd, we note that $L_{\frac{1}{2}(n+1)}(n)$ parameters are identical with $NL_{\frac{1}{2}(n-1)}(n)$ parameters and that both agree with pseudo-cyclic parameters (1.14) with argument $t = (n^2-1)/4$. These are the only L_g or NL_g schemes for which $n_1 = n_2$ and the only pseudo-cyclic schemes for which v is a square. No other schemes are common to any two of these three families.

All $NL_g(n)$ parameters satisfying $n_1 \leq n_2$ are listed in the following table for the range $n \leq 10$. The six schemes which were previously known and the four which are constructed for the first time in the present paper are identified in the "Remarks" column, together with references to publications, to known schemes in the Latin square family, and to sections of this paper in which constructions are presented. Parameters of designs with $NL_g(n)$ association schemes will be tabulated in Section 9.

TABLE 2.1

PARAMETERS OF $NL_g(n)$ ASSOCIATION SCHEMES

Scheme	V	n_1	n_2	p_{11}^1	p_{11}^2	Remarks
$NL_1(3)$	9	4	4	1	2	Known, $L_2(3)$; Sec. 5
$NL_1(4)$	16	5	10	0	2	Known, [12]; Secs. 4, 5, 7, 8
$NL_2(5)$	25	12	12	5	6	Known, $L_3(5)$; Sec. 5
$NL_2(6)$	36	14	21	4	6	
$NL_2(7)$	49	16	32	3	6	
$NL_3(7)$	49	24	24	11	12	Known, $L_4(7)$; Sec. 5
$NL_2(8)$	64	18	45	2	6	Known, [19]
$NL_3(8)$	64	27	36	10	12	New, Secs. 5, 7
$NL_2(9)$	81	20	60	1	6	New, Secs. 5, 7
$NL_3(9)$	81	30	50	9	12	New, Sec. 5
$NL_4(9)$	81	40	40	19	20	Known, $L_5(9)$; Sec. 5
$NL_2(10)$	100	22	77	0	6	New, Sec. 8
$NL_3(10)$	100	33	66	8	12	
$NL_4(10)$	100	44	55	18	20	

3. <u>A characterizing property</u>. We observe that for association schemes of pseudo-cyclic, pseudo-Latin square and negative Latin square types, the multiplicities α_1, α_2 of the characteristic roots of NN^T are equal in some order to the numbers n_1, n_2 of objects in the two associate classes. This proves sufficiency in the following theorem. The necessity statement shows that this property characterizes these three types of association schemes.

Theorem 3.1. In order for the parameters α_1, α_2 in a two-class association scheme to be equal in some order to the parameters n_1, n_2, it is necessary and sufficient that the scheme be of pseudo-cyclic, pseudo-Latin square or negative Latin square type.

Before completing the proof of this theorem, we state a simple lemma.

Lemma 3.1. The parameters α_1, α_2 in a two-class association scheme are equal in some order to the parameters n_1, n_2 if and only if $v = \Delta$.

Proof of lemma. From (1.4) and (1.7),

$$n_1 + n_2 = \alpha_1 + \alpha_2 \ .$$

The lemma follows from this and (1.11).

Proof of theorem (necessity). If the scheme is of pseudo-cyclic type we are finished. If not, then by Theorems 5.3 and 5.5 of [13], Δ is the square of an integer n, and using the lemma,

$$n^2 = \Delta = v \ .$$

Then

$$n_2 = n^2 - 1 - n_1 \ . \quad (3.1)$$

Using (1.8),

(3.2) $$\sigma + \tau + 1 = \Delta^{\frac{1}{2}} = n,$$

partially identifying σ and τ.

Case I. Suppose $n_1 = \alpha_1$. Then from (1.10) and (3.1),

$$n_1 = [\sigma\, n_1 + (\sigma+1)(n^2-1-n_1)]/n ,$$

reducing to

$$n_1 = (\sigma+1)(n-1) .$$

This identifies σ and τ completely. If we set $\sigma+1 = g$ we have

$$n_1 = g(n-1) ,$$

$$n_2 = (n-g+1)(n-1) ,$$

and from (1.12),

$$p_{12}^1 = (g-1)(n-g+1) ,$$

$$p_{12}^2 = g(n-g) .$$

The parameters v, n_i, p_{12}^i are of the form of (2.1), and it follows from (1.4) that the same is true of the remaining p_{jk}^i. Therefore the scheme is of pseudo-Latin square type.

Case II. Suppose $n_1 = \alpha_2$. Then using (1.10) and (3.1) as in Case I we find

$$n_1 = \tau(n+1)$$

and setting $\tau = g$ we again use (1.12), this time arriving at parameters of the form of (2.5). Therefore, the scheme is of negative Latin square type and the proof is complete.

It is clear from Lemma 3.1 that the condition on n_i and α_i in Theorem 3.1 could be replaced by the condition $v = \Delta$. The fact that v is a square is a distinctive property of the Latin square and negative Latin square schemes but is not peculiar to them, as shown, for example, by numerous group divisible schemes and by the triangular scheme with $v = 36$. However, inspection of a list of arithmetically possible parameters for two-class association schemes leads to the interesting conjecture that when v is a square, a high proportion of these parameters fall in the group divisible, L_g and NL_g series. As an illustration, in the range $v \leq 100$, v a square, $n_1 \leq n_2$, there are at most 65 sets of two-class parameters, of which 59 are in these three series.

4. Some Preliminary Theorems. Several results, most of them from other sources, which will be needed in Sections 5 and 7 for the construction of association schemes, are collected in this section for convenient reference.

In an association scheme with classes 1, ..., m, we may introduce a zero-th associate class by letting each object be the zero-th associate of itself and of no other object. We define additional parameters

$$n_0 = 1,$$

$$p_{ij}^{0} = n_i \text{ if } i = j,$$
$$= 0 \text{ otherwise},$$

$$p_{0k}^{i} = p_{k0}^{i} = 1 \text{ if } i = k,$$
$$= 0 \text{ otherwise}.$$

This convention increases the conciseness and symmetry of many statements about association schemes and their parameters. We shall retain the term "m-class" for a scheme with classes 0, 1, ..., m.

The method of differences was introduced for construction of incomplete block designs in the module theorem of Bose and Nair [5] and later stated in somewhat greater generality by Sprott [23]. The following is the portion of the theorem which applies to association schemes, using the terminology of the zeroth associate class.

Theorem 4.1. Module theorem. Let the elements of an additive Abelian group G of finite order v be partitioned into disjoint sets $\mathcal{A}_0 = \{0\}, \mathcal{A}_1, \ldots, \mathcal{A}_m$. Let $\mathcal{A}_i$ contain n_i elements, denoted by

$$\mathcal{A}_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}\}.$$

We set up an association relation among the elements of G by taking y as an i-th associate of x if and only if the difference y-x is in $\mathcal{A}_i$. Then each element has n_i i-th associates, and the relation is an m-class association scheme with parameters v, n_i, p^i_{jk} if and only if

(i) each group element is in the same $\mathcal{A}_i$ as its inverse;

(ii) for any i, j, k in the range 0, 1, ..., m, and for any fixed element $x \in \mathcal{A}_i$, there are exactly p^i_{jk} ordered pairs u, w, where $1 \le u \le n_j$, $1 \le w \le n_k$, such that

$$a_{j,u} - a_{k,w} = x \,. \tag{4.1}$$

Our application of the module theorem will be to the additive group of a finite field, using the multiplicative group in the construction of the sets $\mathcal{A}_i$. Our procedure is similar to that of Sprott, but a self-contained account will be given here.

Theorem 4.2. In a finite field of order v with additive group G and multiplicative group G', let m be a divisor of the order v-1 of G' such that $N = (v-1)/m$ is even if v is odd, and let ξ be a generator of G'. Let $G_0 = \{0\}$, let G_1 be the multiplicative subgroup of order N generated by ξ^m, and let G_i, i = 2, ..., m, be the coset of G_1 which contains ξ^{i-1}. Define an association relation $\mathfrak{Z}(v, m)$ in which two elements x, y of G are i-th associates if and only if $y-x \in G_i$, i = 0, 1, ..., m. Then for i, j, k in the range i, ..., m and interpreted modulo m where necessary,

(4.2) $\mathfrak{Z}(v,m)$ is an m-class partially balanced association scheme with parameters v, $n_i = N$, p^i_{jk},

(4.3) $p^i_{jk} = p^i_{j+1,\ k+1} = p^1_{j-i+1,\ k-i+1}$,

(4.4) p^i_{jk} is equal to the number of elements of G_{j-i+1} which occur in the set obtained by adding the unit element 1 to each element of G_{k-i+1}.

Proof. To prove (4.2) we shall verify that the sets G_i satisfy conditions (i) and (ii) of Theorem 4.1. The first of these conditions is automatic if v is even, since in this case every nonzero element is of order 2 and is its own additive inverse. If v is odd, the unit element 1 is given by

$$1 = \xi^{mN}$$

and its additive inverse is given by

$$-1 = (\xi^m)^{N/2},$$

where N/2 is an integer by hypothesis. Therefore -1 is an element of the subgroup G_1 generated by ξ^m. It follows that for every element y of any G_i, $-y = y(\xi^m)^{N/2}$ is also in G_i, verifying condition (i) of Theorem 4.1.

An element $x = a_{i,t} \in G_i$ may be expressed

$$a_{i,t} = \xi^{mt + i-1},$$

and (4.1) may be written

$$(4.5) \qquad \xi^{mu + j-1} - \xi^{mw + k-1} = \xi^{mt + i-1} .$$

This equation is equivalent to

$$(4.6) \qquad \xi^{m(u-t) + j-1} - \xi^{m(w-t) + k-1} = \xi^{i-1} .$$

As u and w range independently over the residue classes 1, 2, ..., N modulo N, the same is true of u-t and w-t. Then each of $\xi^{m(u-t)}$ and $\xi^{m(w-t)}$ ranges over G_1, and the two terms in the left hand side of (4.6) range independently over G_j and G_k. The number of solutions u,w of (4.1) and of (4.5) is thus equal to the number of solutions of

$$(4.7) \qquad a_{j,u} - a_{k,w} = \xi^{i-1}.$$

But this is independent of t and hence of the particular element x chosen from G_i. Denoting the number of solutions by p^i_{jk}, we have verified condition (ii) of Theorem 4.1, completing the proof of (4.2).

Multiplying (4.5) by ξ^d gives the equivalent equation

$$(4.8) \qquad \xi^{mu + j + d-1} - \xi^{mw + k + d-1} = \xi^{mt + i + d-1} ,$$

which has the same number p^i_{jk} of solutions for fixed i, j, k, t. But this number of solutions may also be interpreted as $p^{i+d}_{j+d,\ k+d}$, where indices are reduced modulo m to fall in the range 1, 2, ..., m. ξ^d and ξ^{d-m} are in the same coset of G_1, and reducing modulo m merely means that the cosets are still designated by the representatives named in the theorem. With this interpretation of indices, we have

$$p^i_{jk} = p^{i+d}_{j+d,\ k+d} ,$$

and two special cases give (4.3).

From (4.7), p^1_{jk} is the number of solutions u,w of

$$(4.9) \qquad a_{j,u} = a_{k,w} + 1,$$

that is, the number of elements of G_j in the set obtained by adding the unit element 1 to each element of G_k. Together with (4.3) this gives (4.4) and completes the proof of Theorem 4.2.

Determining the m^3 parameters p^i_{jk} of an m-class association scheme is considerably simplified for the $\mathcal{J}(v,m)$ schemes by (4.3), which says that matrices P_2, ..., P_m may be obtained from P_1 by cyclic permutation of rows and columns. The standard relations $p^i_{jk} = p^i_{kj}$ and $n_i p^i_{jk} = n_j p^j_{ik}$, the latter of which may be simplified because $n_i = n_j$, reduce the m^2 parameters p^1_{jk} to a subset of approximately $m^2/6$ of them. The analogous number of independent parameters in the absence of (4.3) is $m^3/6$. Enumerating solutions of (4.9) to find the independent p^1_{jk} values is still a non-trivial problem. It can be reduced to finding the number of ordered pairs u,w, where $0 \leq u < m$, $0 \leq w < m$, such that the equation

(4.10) $$\xi^{j+mu} + \xi^{k+mw} + 1 = 0$$

holds in GF(v). This problem in finite fields has been extensively studied, especially in the case of prime v, but not solved completely. A survey is given in [8].

For given v, m, the association scheme $\mathfrak{J}(v,m)$ is determined uniquely up to a certain permutation of associate classes 2, 3, ..., m. Since G' is a cyclic group, the subgroup G_1 of a given order N is unique, and with it the first associate class. The partition of G' into cosets is also unique. However, there are $\Phi(v-1)$ choices of the generator ξ, where Φ is the Euler totient function, and different choices may result in different assignments of the indices 2, ..., m to the cosets. The indexing of elements within cosets will also be affected, but this is irrelevant for the association schemes. If s is a positive integer less than and prime to m, and if instead of the generator ξ we use a generator η such that

$$\xi = \eta^t,$$

$$t \equiv s \pmod{m},$$

then the coset representative ξ^{i-1} will be expressed

$$\xi^{i-1} = \eta^{s(i-1)}$$

and the i-th associate class in our original formulation will receive a new index congruent modulo m to $1 + s(i-1)$. The number of different permutations of associate classes that can arise for a given v and m will thus be $\Phi(m)$, the number of possible values of s.

The <u>associationmatrices</u> $A_0, A_1, \ldots, A_m$ of an m-class association scheme are matrices of order v defined by

$$(4.11) \qquad A_0 = I, \; A_i = (a_{\mu\nu}^{(i)}) \; ,$$

where $a_{\mu\nu}^{(i)} = 1$ if objects μ and ν are i-th associates,

$= 0$ otherwise.

Clearly A_i is a symmetric matrix with all row and column sums equal to n_i. We may prove [24, 4]

Theorem 4.3. Matrices $A_0 = I, A_1, \ldots, A_m$ are association matrices of an m-class partially balanced association scheme with parameters v, $n_i = p_{ii}^0$, p_{jk}^i if and only if

(4.12) each A_i is a symmetric v x v matrix of 0's and 1's,

$$(4.13) \qquad \sum_{i=0}^{m} A_i = J, \text{ the v x v matrix of 1's,}$$

$$(4.14) \qquad A_j A_k = \sum_{i=0}^{m} p_{jk}^i A_i, \; j, k = 0, 1, \ldots, m.$$

This theorem can be simplified as follows in the case m = 2.

Theorem 4.4. A_1 is the first association matrix of a 2-class partially balanced association scheme with parameters v, n_i, p_{jk}^i if and only if, defining $A_2 = J - I - A_1$,

(4.15) A_1 and A_2 are symmetric v x v matrices of 0's and 1's,

$$(4.16) \qquad A_1^2 = n_1 I + p_{11}^1 A_1 + p_{11}^2 A_2 \; .$$

Useful information can be obtained from certain submatrices of association matrices. We partition matrices A_1 and A_2 for a 2-class scheme into submatrices whose sets of rows and columns correspond to an initial object α, the set of n_1 first associates of α, and the set of n_2 second associates of α. For convenience, we may choose notation so that α is in leading position with its first associates in the next n_1 positions. The following illustrates the partition and defines notation for the submatrices.

(4.17)
$$A_1 = \left[\begin{array}{c|c|c} 0 & 1 \ldots 1 & 0 \ldots 0 \\ \hline 1 & & \\ \cdot & & \\ \cdot & B_1 & C_1 \\ \cdot & & \\ 1 & & \\ \hline 0 & & \\ \cdot & C_1^T & D_1 \\ \cdot & & \\ 0 & & \end{array}\right], \quad A_2 = \left[\begin{array}{c|c|c} 0 & 0 \ldots 0 & 1 \ldots 1 \\ \hline 0 & & \\ \cdot & & \\ \cdot & B_2 & C_2 \\ \cdot & & \\ 0 & & \\ \hline 1 & & \\ \cdot & C_2^T & D_2 \\ \cdot & & \\ 1 & & \end{array}\right].$$

B_1 and B_2 are symmetric $n_1 \times n_1$ matrices; D_1 and D_2 are symmetric $n_2 \times n_2$ matrices.

Lemma 4.5. Submatrices B_1, C_1, C_1^T, D_1 have uniform row totals p_{11}^1, p_{12}^1, p_{11}^2, p_{12}^2 respectively.

Proof. The inner product of rows θ and φ of A_1 is equal to the number of first associates common to objects θ and φ. The results for B_1 and C_1^T are obtained by setting $\theta = 1$ while φ ranges over the remaining rows. The results for C_1 and D_1 follow by subtraction from the uniform row totals n_1 of A_1.

Theorem 4.6. A partitioned matrix A_1 of the form (4.17) is the first association matrix of a two-class partially balanced association scheme with parameters v, n_i, p_{jk}^i if and only if, defining $A_2 = J - I - A_1$, and defining submatrices of A_2 by (4.17),

(4.18) $\qquad A_1$ and A_2 are matrices of 0's and 1's, (4.19) B_1 and D_1 are symmetric matrices of order n_1 and n_2 respectively,

(4.20) B_1 has row sums p_{11}^1 and C_1^T has row sums p_{11}^2,

(4.21) $J + B_1^2 + C_1C_1^T = n_1I + p_{11}^1B_1 + p_{11}^2B_2$,

(4.22) $B_1C_1 + C_1D_1 = p_{11}^1C_1 + p_{11}^2C_2$,

(4.23) $C_1^TC_1 + D_1^2 = n_1I + p_{11}^1D_1 + p_{11}^2D_2$.

Proof. If A_1 is any partitioned matrix of the form given in (4.17), the square of A_1 is given in part by

(4.24)
$$A_1^2 = \left[\begin{array}{c|c|c} & & \\ \hline & J + B_1^2 + C_1C_1^T & B_1C_1 + C_1D_1 \\ \hline & C_1^TB_1 + D_1C_1^T & C_1^TC_1 + D_1^2 \end{array}\right] .$$

Now suppose A_1 is the first association matrix of the specified two-class scheme. Then (4.18) to (4.20) hold, (4.16) holds and implies

(4.25)
$$A_1^2 = \left[\begin{array}{c|c|c} n_1 & p_{11}^1 \cdot \cdot \cdot \cdot p_{11}^1 & p_{11}^2 \cdot \cdot \cdot p_{11}^2 \\ \hline p_{11}^1 & & \\ \cdot & & \\ \cdot & n_1I + p_{11}^1B_1 + p_{11}^2B_2 & p_{11}^1C_1 + p_{11}^2C_2 \\ \cdot & & \\ p_{11}^1 & & \\ \hline p_{11}^2 & & \\ \cdot & & \\ \cdot & p_{11}^1C_1^T + p_{11}^2C_2^T & n_1I + p_{11}^1D_1 + p_{11}^2D_2 \\ \cdot & & \\ p_{11}^2 & & \end{array}\right] ;$$

and comparison of (4.24) and (4.25) gives (4.21) to (4.23).

Conversely, suppose A_1 satisfies (4.18) to (4.23). (4.18) and (4.19) imply (4.15). (4.24) holds, and with the aid of (4.17) to (4.20) the first row and column of A_1^2 may be shown to be as in (4.25), while (4.21) to (4.23) give the rest of (4.25). But (4.25) implies (4.16), and by Theorem 4.4, A_1 is the first association matrix of a two-class scheme with the specified parameters.

While our primary concern in this paper is with association schemes, some balanced and partially balanced designs will be discussed in Sections 7 and 8. The next theorem and corollary are well-known characterizations of these designs in terms of the incidence matrix N of objects and blocks.

Theorem 4.7. N is the incidence matrix of objects and blocks in a PBIB design with parameters v, b, r, k, λ_i, n_i, p^i_{jk} if and only if

(i) N is a v x b matrix of 0's and 1's,

(ii) every column of N contains exactly k 1's,

(iii) $NN^T = rI + \lambda_1 A_1 + \dots + \lambda_m A_m$, where

(iv) $A_1, \dots, A_m$ satisfy the conditions of Theorem 4.3 or Theorem 4.4.

Corollary 4.7.1. N is the incidence matrix of objects and blocks in a BIB design with parameters v, b, r, k, λ, if and only if

(i) N is a v x b matrix of 0's and 1's

(ii) Every column of N contains exactly k 1's,

(iii) $NN^T = rI + \lambda(J-I)$.

A matrix of the form $rI + \lambda_1 A_1 + \dots + \lambda_m A_m$, whether or not it arises as a product NN^T, has characteristic roots whose values and multiplicities are of some interest and have been discussed, for example, in [4]. In the case m = 2 they are given by expressions (1.6) to (1.8) in this paper.

It is sometimes possible to construct an m-class association scheme $\mathcal{G}$ by the device of combining associate classes in an $\hat{m}$-class scheme $\hat{\mathcal{G}}$ with the same v, $\hat{m} > m$. In terms of association matrices, this means that the matrices $\hat{A}_1, \ldots, \hat{A}_{\hat{m}}$ of scheme $\hat{\mathcal{G}}$ are arranged into m disjoint non-empty sets, and the i-th association matrix of $\mathcal{G}$ is taken as the sum of all matrices in the i-th set. The resulting association relation does not in general meet the conditions of partial balance; necessary and sufficient conditions that it will do so are derived in Theorem 5.1 of [4] and are stated in Theorem 4.8 below for the case m=2. We continue to use the notion of the zero-th associate class.

Theorem 4.8. Given a partially balanced association scheme $\hat{\mathcal{G}}$ with more than two classes and with parameters v, $\hat{n}_\alpha$, $\hat{p}^\alpha_{\beta\gamma}$, let the indices of the associate classes be partitioned into disjoint sets $S_0 = \{0\}$, S_1, S_2. Define a two-class association relation $\mathcal{G}$ in which two objects are taken as i-th associates if and only if their associate class in $\hat{\mathcal{G}}$ has its index in S_1. Then $\mathcal{G}$ satisfies the conditions of partial balance if and only if, for i = 0, 1, 2 and for some integers p^i_{11},

(4.26) $$\sum_{\beta,\nu\in S_1} \hat{p}^\alpha_{\beta\nu} = p^i_{11}, \text{ uniformly for } \alpha \in S_i .$$

If (4.26) is satisfied, $\mathcal{G}$ has parameters v, p^i_{11}, where $n_1 = p^0_{11}$ and the other parameters n_2, p^i_{jk} are defined by the standard identities (1.3).

We note that for i = 0, (4.26) reduces to the statement $\sum_{\alpha\in S_1} \hat{n}_\alpha = n_1$, which is equivalent to $\sum_{\alpha\in S_2} \hat{n}_\alpha = n_2$. For i = 1, 2, the left hand side of (4.26) represents p^i_{11} as the sum of the elements of a submatrix of $\hat{P}_\alpha$. Somewhat more generally, if (4.26) is satisfied, then for i, j, k = 1, 2, and for any $\alpha \in S_i$, p^i_{jk} is the sum of the elements of the submatrix of $\hat{P}_\alpha$ which has S_j as its set of row indices and S_k as its set of column indices.

Example. As the objects in an association relation $\hat{\mathcal{G}}$ take the 16 ordered quadruples

0 0 0 0	0 1 0 0	1 0 0 0	1 1 0 0
0 0 0 1	0 1 0 1	1 0 0 1	1 1 0 1
0 0 1 0	0 1 1 0	1 0 1 0	1 1 1 0
0 0 1 1	0 1 1 1	1 0 1 1	1 1 1 1

and take two objects as i-th associates if they differ in exactly i positions. By interpreting the quadruples as rectangular coordinates we may interpret the objects as the 16 vertices of the 4-dimensional unit cube, two vertices being i-th associates if their distance is $\sqrt{i}$. $\hat{\mathcal{G}}$ is found to be a 4-class association scheme with parameters

$$v = 16,\quad \hat{n}_1 = 4,\quad \hat{n}_2 = 6,\quad \hat{n}_3 = 4,\quad \hat{n}_4 = 1,$$

$$\hat{P}_1 = \begin{bmatrix} 0&3&0&0 \\ 3&0&3&0 \\ 0&3&0&1 \\ 0&0&1&0 \end{bmatrix}, \quad \hat{P}_2 = \begin{bmatrix} 2&0&2&0 \\ 0&4&0&1 \\ 2&0&2&0 \\ 0&1&0&0 \end{bmatrix}, \quad \hat{P}_3 = \begin{bmatrix} 0&3&0&1 \\ 3&0&3&0 \\ 0&3&0&0 \\ 1&0&0&0 \end{bmatrix}, \quad \hat{P}_4 = \begin{bmatrix} 0&0&4&0 \\ 0&6&0&0 \\ 4&0&0&0 \\ 0&0&0&0 \end{bmatrix}.$$

The matrices $\hat{P}_\alpha$ are repeated below, with $\hat{P}_0$ included and with zero-th row and column adjoined to each to display the $\hat{p}^\alpha_{\beta\nu}$ with zero indices. We now combine associate classes, taking $S_0 = \{0\}$, $S_1 = \{3, 4\}$, $S_2 = \{1, 2\}$. The matrices below are partitioned into the submatrices described in the preceding paragraph.

$$\hat{P}_0 = \left[\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad \hat{P}_1 = \left[\begin{array}{c|cc|cc} 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 \\ \hline 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad \hat{P}_2 = \left[\begin{array}{c|cc|cc} 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 4 & 0 & 1 \\ \hline 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right],$$

$$\hat{P}_3 = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 3 & 0 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ \hline 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right], \quad \hat{P}_4 = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

The submatrices over which the sum in (4.26) is taken are, respectively,

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the sums are 5 for $\alpha \in S_0$, 0 for $\alpha \in S_1$, 2 for $\alpha \in S_2$, showing that the new association relation $\mathcal{G}$ is the two-class scheme with parameters

$$\begin{aligned} &v = 16, \\ &n_1 = 5, \quad P_1 = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}. \\ &n_2 = 10, \end{aligned}$$

This is the Negative Latin Square type scheme $NL_1(4)$. An isomorphic scheme is obtained by taking $S_1 = \{1, 4\}$.

In Section 5, Theorem 4.8 will be applied to some schemes $\mathcal{G}$ of the $\mathcal{C}(v, \hat{m})$ family. Sums of elements of $\hat{P}_\alpha$ in these schemes, $\alpha = 2, \ldots, \hat{m}$, may be expressed as sums of elements of $\hat{P}_1$ by suitable application of (4.3). A rather simple consequence of (4.3), stated as a lemma for later reference, is that the form of the set S_1 can be partially specified without loss of generality.

Lemma 4.9. In applying Theorem 4.8 to a scheme $\mathcal{C}(v, \hat{m})$ any set S_1 satisfying (4.26) is related by a cyclic permutation of the class indices

1, 2, ..., $\hat{m}$ to a set $\{i_1, i_2, \ldots, i_m\}$ which satisfies (4.26) with the same p_{11}^i and satisfies

$$i_1 = 1 < i_2 < \ldots < i_m ,$$

$$i_2 - 1 \leq i_{t+1} - i_t, \ t = 2, \ldots, m-1,$$

$$i_2 - 1 \leq \hat{m}+1 - i_m.$$

5. Construction of NL_g schemes from finite fields. This section will make use of some special cases of the association schemes $\mathfrak{g}(v,M)$ of Theorem 4.2. For n^2 a prime power, $\mathfrak{g}(n^2, n+1)$ will first be described for reference, then $\mathfrak{g}(n^2, n-1)$ will be used in the construction of some $NL_g(n)$ schemes.

The multiplicative subgroup used in constructing $\mathfrak{g}(n^2, n+1)$ from $GF(n^2)$ has n-1 elements, which along with the zero element are readily shown to form a subfield of order n. Among the special features which follow from this are the following simple expressions for the parameters p_{jk}^i of $\mathfrak{g}(n^2, n+1)$.

$$(5.1) \qquad p_{ii}^i = n-2, \ p_{jj}^i = p_{ij}^j = p_{ji}^j = 0, \ p_{jk}^i = 1,$$

$$i, j, k \text{ distinct}, \ 1 \leq i, j, k \leq n+1.$$

Because of the uniformity of these values, this scheme lends itself exceptionally well to the formation of 2-class schemes by combination of associate classes. If Theorem 4.8 is applied with an arbitrary set S_1 of g of the indices 1, ..., n+1, condition (4.26) is satisfied and the resulting two-class scheme, in which $n_1 = g(n-1)$, is of $L_g(n)$ type. This turns out to be only a familiar construction in slightly disguised form. It can be shown that the n+1 associate classes in the scheme $\mathfrak{g}(n^2, n+1)$ are equivalent to the n + 1 constraints of a complete set of pairwise orthogonal Latin squares of order n, a fact which will be taken up in Section 6 from the point of view of finite geometry. Consideration of Latin squares gives (5.1) at once and reduces the combination of associate classes to a simple application of the definition of the $L_g(n)$ scheme.

The scheme $\mathfrak{S}(n^2, n-1)$ uses a multiplicative subgroup of order n+1 and has less regularity than $\mathfrak{S}(n^2, n+1)$. In particular, the writer is unable to give general expressions for p^i_{jk}, though he conjectures that $0 \leq p^i_{jk} \leq 2$ for $1 \leq i, j, k \leq n-1$. However, there are analogies with the $\mathfrak{S}(n^2, n+1)$ scheme and our success in combining g associate classes of size n-1 to give a two-class scheme with $n_1 = g(n-1)$ suggests an attempt, with the $\mathfrak{S}(n^2, n-1)$ scheme, to combine g associate classes of size n+1 to give a two-class scheme with $n_1 = g(n+1)$, hopefully of negative Latin square type. It is not obvious that a set S_1 of g indices can be found which meets condition (4.26) of Theorem 4.8, or that a two-class scheme if obtained will be of NL_g type.

However, in the range n < 10 it is easy to write down the association schemes $\mathfrak{S}(n^2, n-1)$ in sufficient detail that p^i_{jk} values can be computed explicitly, and then to search empirically for suitable sets S_1. The results of this computation are given in tables which follow, and fortunately several schemes of NL_g type are obtained, including the three new schemes $NL_3(8)$, $NL_2(9)$, $NL_3(9)$. It is not known whether the same method yields any $NL_g(n)$ schemes for n > 10.

The objects in all association schemes discussed in this section may be taken as elements of finite fields and will be represented in a notation which is convenient for field operations. The elements of GF(p) for a prime p will be denoted by the residues 0, 1, ..., p-1, and a polynomial of degree at most q-1 with coefficients in GF(p) will be denoted briefly by the q-tuple of its coefficients:

$$\sum_{i=0}^{q-1} a_i x^i \equiv (a_{q-1}, \ldots, a_0) \equiv a_{q-1} \ldots a_0 .$$

Under addition and multiplication modulo a polynomial Q(x) of degree q, irreducible over GF(p), the polynomials $(a_{q-1}, \ldots, a_0)$ represent the field $GF(p^q)$. The polynomial Q will be chosen here so that a root ξ of Q(x) = 0 is a primitive element of $GF(p^q)$. This will in general be possible for more than one choice of the polynomial Q and the primitive element ξ, and while different choices lead to fields, and hence association schemes, which are abstractly identical, the association schemes will differ by a permutation of associate classes, as remarked in Section 4. For definiteness, the table for each $\mathcal{G}(n^2, n-1)$ will list the equation $Q(\xi) = 0$ used in its construction. Each table of powers of ξ will be arranged so that row i contains the set

$$\mathcal{G}_i = \{\xi^{(n-1)u + i-1}, \; u = 0, 1, \ldots, n\}$$

of i-th associates of the zero element in $\mathcal{G}(n^2, n-1)$. The i-th associates of an element θ are obtained by adding θ to each of the i-th associates of zero. The matrix P_1 exhibits the parameters p^1_{jk}, which are calculated by means of (4.4); it follows from (4.3) that the matrices $P_2, \ldots, P_{n-1}$ may be obtained from P_1 by cyclic permutation of rows and columns.

It may be verified by straightforward calculation that each set $S_1 = \{i_1, i_2, \ldots, i_g\}$ listed for an $NL_g(n)$ scheme meets condition (4.26) with the appropriate values of p^i_{11}. In the $NL_g(n)$ scheme, the first associates of the zero object are the elements in rows $i_1, i_2, \ldots, i_g$ of the table of powers of ξ, and the first associates of θ are obtained by adding θ to the first associates of zero. The search by trial and error for the sets S_1 was the only part of the construction method which was tentative as well as tedious. It was expedited by restricting S_1 to the form described in Lemma 4.9. The search was exhaustive and the author can report . for each $\mathcal{G}(n^2, n-1)$ scheme considered, that the sets S_1 listed, other sets obtained from them by cyclic permutation of the indices 1, 2, ... n, and the complements of these sets, are the only sets of associate classes which can be combined to give two-class association schemes.

The methods of this section base on finite fields thus fail to provide constructions for the schemes $NL_2(7)$ and $NL_2(8)$, or to give any new schemes not of the NL_g family in the range $v \leq 100$. The attempt to construct $NL_2(7)$ from $\mathcal{G}(49, 6)$ was supplemented by attempts with other $\mathcal{G}(v,m)$ schemes, such as $\mathcal{G}(49, 12)$, in which combination of classes could give two associate classes of sizes $n_1 = 16$, $n_2 = 32$, but condition (4.26) was not satisfied in any case.

TABLE 5.1. $\mathfrak{Z}(3^2, 2)$

Elements of $GF(3^2)$ represented as polynomials

$a_1 a_0 \equiv a_1 \xi + a_0,\ a_i \in GF(3)$, where $Q(\xi) \equiv \xi^2 + 2\xi + 2 = 0$

Table of powers ξ^{2u+i-1} and of i-th associates of 00

i \ 2u	0	2	4	6
1	01	11	02	22
2	10	21	20	12

$$P_1 = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

This is the known two-class scheme $NL_1(3)$.

TABLE 5.2. $\mathfrak{Z}(4^2, 3)$

Elements of $GF(2^4)$ represented as polynomials

$a_3 a_2 a_1 a_0 \equiv a_3 \xi^3 + a_2 \xi^2 + a_1 \xi + a_0,\ a_i \in GF(2)$, where

$Q(\xi) \equiv \xi^4 + \xi + 1 = 0$

Table of powers $\xi^{3u + i-1}$ and of i-th associates of 0000

i \ 3u	0	3	6	9	12
1	0001	1000	1100	1010	1111
2	0010	0011	1011	0111	1101
3	0100	0110	0101	1110	1001

$$P_1 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} .$$

The following two-class scheme may be obtained by combining associate classes in $\mathfrak{g}(4^2, 3)$.

$NL_1(4)$, a known scheme. $S_1 = \{1\}$.

TABLE 5.3. $\mathfrak{g}(5^2, 4)$

Elements of $GF(5^2)$ represented as polynomials

$a_1a_0 \equiv a_1 \xi + a_0$, $a_i \in GF(5)$, where $Q(\xi) \equiv \xi^2 + 4 \xi + 2 = 0$.

Table of powers $\xi^{4u + i-1}$ and
table of i-th associates of 00

i \ 4u	0	4	8	12	16	20
1	01	22	21	04	33	34
2	10	41	31	40	14	24
3	13	02	44	42	03	11
4	43	20	32	12	30	23

$$P_1 = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix} .$$

The following two-class scheme may be obtained by combining associate classes in $\mathfrak{g}(5^2, 4)$

$NL_2(5)$, a known scheme. $S_1 = \{1, 3\}$.

TABLE 5.4. $\mathfrak{z}(7^2, 6)$

Elements of $GF(7^2)$ represented as polynomials

$$a_1 a_0 \equiv a_1 \xi + a_0, \; a_i \in GF(7), \text{ where } Q(\xi) \equiv \xi^2 + 6\xi + 3 = 0.$$

Table of powers $\xi^{6u + i-1}$ and
table of i-th associates of 00

i \ 6u	0	6	12	18	24	30	36	42
1	01	24	64	21	06	53	13	56
2	10	61	33	31	60	16	44	46
3	14	03	65	45	63	04	12	32
4	54	30	43	22	23	40	34	55
5	26	35	02	41	51	42	05	36
6	11	15	20	52	66	62	50	25

$$P_1 = \begin{bmatrix} 0 & 0 & 2 & 2 & 1 & 2 \\ 0 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 2 & 2 & 0 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 & 0 \end{bmatrix} .$$

The following two-class scheme may be obtained by combining associate classes in $\mathfrak{z}(7^2, 6)$.

$NL_3(7)$, a known scheme. $S_1 = \{1, 3, 5\}$.

TABLE 5.5. $\mathfrak{Z}(8^2, 7)$

Elements of $GF(2^6)$ represented by polynomials

$$a_5a_4a_3a_2a_1a_0 \equiv a_5 \xi^5 + \ldots + a_0,\ a_i \in GF(2), \text{ where } Q(\xi) \equiv \xi^6 + \xi + 1 = 0.$$

Table of powers $\xi^{7u + i-1}$ and
table of i-th associates of 000000

i \ 7u	0	7	14	21	28	35	42	49	56
1	000001	000110	010100	111011	011100	001011	111010	011010	011111
2	000010	001100	101000	110101	111000	010110	110111	110100	111110
3	000100	011000	010011	101001	110011	101100	101101	101011	111111
4	001000	110000	100110	010001	100101	011011	011001	010101	111101
5	010000	100011	001111	100010	001001	110110	110010	101010	111001
6	100000	000101	011110	000111	010010	101111	100111	010111	110001
7	000011	001010	111100	001110	100100	011101	001101	101110	100001

$$P_1 = \begin{bmatrix} 2&0&0&2&0&2&2 \\ 0&2&2&0&2&1&2 \\ 0&2&2&1&2&2&0 \\ 2&0&1&0&2&2&2 \\ 0&2&2&2&2&0&1 \\ 2&1&2&2&0&0&2 \\ 2&2&0&2&1&2&0 \end{bmatrix}.$$

The following two-class scheme may be obtained by combining associate classes in $\mathfrak{Z}(8^2, 7)$.

$NL_3(8)$, a new scheme. $S_1 = \{1, 2, 6\}$.

TABLE 5.6. $\mathfrak{F}(9^2, 8)$

Elements of $GF(3^4)$ represented as polynomials

$$a_3a_2a_1a_0 \equiv a_3\,\xi^3 + \dots + a_0,\ a_i \in GF(3),\ \text{where } Q(\xi) \equiv \xi^4 + 2\,\xi^3 + 2 = 0.$$

Table of powers $\xi^{8u + i-1}$ and of i-th associates of 0000

i \ 8u	0	8	16	24	32	40	48	56	64	72
1	0001	0112	0212	2110	2012	0002	0221	0121	1220	1021
2	0010	1120	2120	0102	2122	0020	2210	1210	0201	1211
3	0100	2201	0202	1020	0222	0200	1102	0101	2010	0111
4	1000	1012	2020	1201	2220	2000	2021	1010	2102	1110
5	1001	1121	2202	0011	1202	2002	2212	1101	0022	2101
6	1011	2211	1022	0110	0021	2022	1122	2011	0220	0012
7	1111	1112	1221	1100	0210	2222	2221	2112	2200	0120
8	2111	2121	0211	2001	2100	1222	1212	0122	1002	1200

$$P_1 = \begin{bmatrix} 1&0&2&0&0&2&2&2\\ 0&2&2&0&2&2&0&2\\ 2&2&2&0&1&2&1&0\\ 0&0&0&2&2&2&2&2\\ 0&2&1&2&0&2&1&2\\ 2&2&2&2&2&0&0&0\\ 2&0&1&2&1&0&2&2\\ 2&2&0&2&2&0&2&0 \end{bmatrix}.$$

The following two-class schemes may be obtained by combining associate classes in $\mathfrak{F}(9^2, 8)$.

$NL_2(9)$, a new scheme. $S_1 = \{1, 5\}$.

$NL_3(9)$, a new scheme. $S_1 = \{1, 2, 7\}$.

$NL_4(9)$, a known scheme. $S_1 = \{1, 3, 5, 7\}$ or $\{1, 2, 5, 6\}$.

To illustrate the computation of the matrix P_1 for $\mathfrak{F}(v, m)$ schemes, we evaluate some of the values p^1_{jk} for $\mathfrak{F}(5^2, 4)$. We recall that if the unit element, 01 in this example, is added to each element of $G_k = \{a_{k,u}$, $u = 0, 1, \ldots, n\}$, the resulting set contains exactly p^1_{jk} elements of G_j. The following working table is so arranged that p^1_{jk} is given by the number of columns which contain k in the first row and j in the last row.

k	1	1	1	1	1	1	2	2	2	2	2	2
u	0	1	2	3	4	5	0	1	2	3	4	5
$a_{k,u}$	01	22	21	04	33	34	10	41	31	40	14	24
$a_{k,u}$ + 01	02	23	22	00	34	30	11	42	32	41	10	20
$j \ni a_{k,u} + 01 \in G_j$	3	4	1	-	1	4	3	3	4	2	2	4

Tabulation of the results gives

k	1	1	1	1	2	2	2	2
j	1	2	3	4	1	2	3	4
p^1_{jk}	2	0	1	2	0	2	2	2

6. Geometric interpretations of $\mathfrak{F}(v, m)$ association schemes. The finite field scheme $\mathfrak{F}(n^2, n+1)$ furnishes an easy construction of a finite Euclidean plane geometry of order n. The n^2 objects of the association scheme are interpreted as points of this geometry. It is easy to show that lines satisfying the incidence postulates of the geometry are obtained by defining as a line every set consisting of a point and its n-1 i-th associates, i = 1, ..., n+1. In particular, there are $n^2 + n$ lines, each containing n points. For fixed i, the points fall into n pairwise disjoint lines, which comprise a parallel class. The geometry obtained for any n

(of prime power form) is the unique Desarguesian plane.

We remark that if $v = n^k$, t is a divisor of k, and $N = n^t-1$, so that $m = (v-1)/N = n^{k-t} + n^{k-2t} + \ldots + 1$, then the association scheme $\mathfrak{G}(v, m)$ may be used to generate some of the t-dimensional subspaces in EG(k, n), the Euclidean geometry of dimension k and order n, giving all such subspaces (lines) in the case t = 1.

Conversely, in the case t = 1, a finite Euclidean geometry may be used to construct association schemes $\mathfrak{G}(n^k, m)$, where $n_i = N = (n^k-1)/m = n-1$. Associate classes are identified with the m parallel classes of lines and the i-th associates of a point are the points which occur with it on a line of the i-th parallel class. In the case k = 2 of a plane geometry, the Desarguesian plane gives an $\mathfrak{G}(n^2, n+1)$ scheme while a non-Desarguesian plane gives a pseudo-$\mathfrak{G}(n^2, n+1)$ scheme, which has the same parameters n_i, p^i_{jk} but whose elements do not correspond to those of $\mathfrak{G}(n^2, n+1)$ under any one-one mapping which preserves the association relation.

In the $L_g(n)$ scheme which is obtained from $\mathfrak{G}(n^2, n+1)$ by class combination, the set of first associates of an object can now be interpreted as a simple geometric figure. Some g of the parallel classes are chosen--speaking informally, g of the directions on the plane. The first associates of a point θ are the remaining points on the lines through θ in the g chosen directions.

Retaining the identification of the elements of $GF(n^2)$ with the points of EG(2,n), we turn to the association scheme $\mathfrak{G}(n^2, n-1)$ and ask what geometric figure is formed by the n+1 i-th associates of a point θ.

As i ranges over the values 1, ..., n-1, a collection of n-1 disjoint figures is obtained which exhausts the n^2-1 points of the plane other than θ.

In an $NL_g(n)$ scheme formed by combining g associate classes, the set of first associates of θ will be the union of g of these geometric figures.

DEFINITION. For $\theta \in GF(n^2)$, $\mathcal{C}_i(\theta)$ will denote the set of i-th associates of θ in $\mathcal{J}(n^2, n+1)$ and $\mathcal{C}_i(\theta)$ will denote the set of i-th associates of θ in $\mathcal{J}(n^2, n-1)$.

Theorem 6.1. The number of elements in $\mathcal{C}_i(\theta) \cap C_j(\theta)$ is

1 if n is even,

2 if n is odd and $i \equiv j \pmod 2$,

0 if n is odd and $i \not\equiv j \pmod 2$.

Proof. $\mathcal{C}_i(\theta) = \{\theta + \xi^{(n+1)u+ i-1}, \quad u = 0, \ldots, n-2\}$.

$C_j(\theta) = \{\theta + \xi^{(n-1)w + j-1}, \quad w = 0, \ldots, n\}$.

The number of elements common to these sets is the number of pairs u, w of integers in the specified ranges for which

$$\theta + \xi^{(n+1) u + i-1} = \theta + \xi^{(n-1)w + j-1} .$$

This equation is equivalent to

(6.1) $\qquad (n + 1)u + i-1 = (n-1)w + j-1 = y-1$, say,

which in turn is equivalent to

(6.2) $\qquad 0 < y \leq n^2 - 1,$

$y \equiv i \pmod{(n+1)},$

$y \equiv j \pmod{(n-1)}.$

Methods of elementary number theory applied either to the Diophantine equation (6.1) or the congruences (6.2) show that if $d = (n+1, n-1)$, the greatest common divisor of n+1 and n-1, then the number of solutions is d if $i \equiv j \pmod d$ and is zero otherwise. If n is even, $d = 1$; if n is odd, $d = 2$; the conclusion of the theorem follows at once.

This theorem shows that if θ is a point of EG(2,n) and if n is even, then the points of each set $\mathcal{C}_j(\theta)$ are distributed one each over the lines on θ. If n is odd, the points of $\mathcal{C}_j(\theta)$ are distributed two each over half the lines on θ. The next theorem gives a deeper insight.

Theorem 6.2. For any $\theta \in GF(n^2)$, and for any i, j, $1 \leq i \leq n-1$, $1 \leq j \leq n+1$, no three elements of $\mathcal{C}_i(\theta)$ are pairwise j-th associates in $\mathfrak{F}(n^2, n+1)$.

Remarks. In terms of the EG(2,n) induced by the scheme $\mathfrak{F}(n^2, n+1)$, this theorem says that no three points of a set $\mathcal{C}_i(\theta)$ are collinear. We are in the fortunate position of having three methods of proof of this theorem, of which all are instructive and two will be given here. These two proofs use the following well-known facts on finite fields. For $x \in GF(n^2)$, $x^{n^2} = x$. Also, the mapping $x \longrightarrow x^n$ is an automorphism of $GF(n^2)$ which reduces to the identify $x^n = x$ if and only if $x \in GF(n)$.

Proof I. Three distinct elements of $\mathcal{C}_i(\theta)$ may be represented

$$(6.3) \qquad \varphi_t = \theta + \xi^{u_t(n-1) + i-1}, \quad t = 1, 2, 3,$$

where ξ is a primitive element of $GF(n^2)$ and u_1, u_2, u_3 are distinct modulo n + 1. It will be convenient to use the abbreviation

$$(6.4) \qquad \eta_t = \xi^{u_t(n-1)} .$$

η_1, η_2, η_3 are nonzero elements which are distinct since φ_1, φ_2, φ_3 are distinct.

The representation we are using for the geometry has the property that φ_1, φ_2, φ_3, regarded as points, are collinear if and only if, regarded as elements of $GF(n^2)$, they satisfy an equation

$$(6.5) \qquad a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 = 0 ,$$

where a_1, a_2, a_3, not all zero, are elements of the subfield $GF(n)$, and

$$(6.6) \qquad a_1 + a_2 + a_3 = 0 .$$

Using (6.3), (6.4) and (6.6) and simplifying, we find that (6.5) is equivalent to

$$(6.7) \qquad a_1\eta_1 + a_2\eta_2 + a_3\eta_3 = 0 .$$

Since the mapping $x \longrightarrow x^n$ is an automorphism, a valid equation is obtained if it is applied to all the field elements in equation (6.7). Under the mapping, each of a_1, a_2, a_3, 0 maps into itself, ξ^{n-1} maps into $\xi^{n^2-n} = \xi^{1-n}$, and η_t accordingly maps into $\xi^{u_t(1-n)} = \eta_t^{-1}$. The new equation is

$$(6.8) \qquad a_1\eta_1^{-1} + a_2\eta_2^{-1} + a_3\eta_3^{-1} = 0 .$$

The system of equations (6.6), (6.7), (6.8) in a_1, a_2, a_3 has determinant of coefficients

$$\begin{vmatrix} 1 & 1 & 1 \\ \eta_1 & \eta_2 & \eta_3 \\ \eta_1^{-1} & \eta_2^{-1} & \eta_3^{-1} \end{vmatrix} = \eta_1^{-1}\,\eta_2^{-1}\,\eta_3^{-1} \begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_1^2 & \eta_2^2 & \eta_3^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \eta_1^{-1}\,\eta_2^{-1}\,\eta_3^{-1}\,(\eta_1-\eta_2)(\eta_2-\eta_3)(\eta_3-\eta_1).$$

Since η_1, η_2, η_3 are distinct, the determinant is nonzero and the system has no nontrivial solution a_1, a_2, a_3. Then $\varphi_1, \varphi_2, \varphi_3$ are not collinear. Proof I is complete.

Proof II. We need the fact that for $z \in GF(n^2)$, $z^n + z$ is an element of the subfield $GF(n)$. The proof is that under the automorphism $x \longrightarrow x^n$, this element maps into itself.

$$z^n + z \longrightarrow z^{n^2} + z^n = z + z^n.$$

The elements $\varphi \equiv \varphi(w)$ of $C_i(\theta)$ may be expressed

$$(6.9) \qquad \varphi = \xi^{i-1} \beta^w, \; w = 0, \ldots, n,$$

where $\beta = \xi^{n-1}$. Let $\varphi_0 = \theta + \xi^{i-1}\beta^{w_o}$ and φ be distinct elements of $C_i(\theta)$, where (6.9) determines a value $w \neq w_o$ corresponding to φ. Then φ and φ_0 are j-th associates in scheme $\mathfrak{F}(n^2, n+1)$ for some value of j, which we now determine. That is, we find j so that $\varphi - \varphi_0$ can be expressed in the form $\xi^{j-1} \alpha^u$, where $\alpha = \xi^{n+1}$.

$$\varphi - \varphi_0 = (\varphi - \theta) - (\varphi_0 - \theta) = \xi^{i-1} \beta^w - \xi^{i-1} \beta^{w_o}.$$

Now $-1 = \beta^c$, where

$$c = \tfrac{1}{2}(n + 1) \text{ if n is odd,}$$

$$c = 0 \text{ if n is even.}$$

$$\begin{aligned} \varphi - \varphi_0 &= \xi^{i-1} \beta^w + \xi^{i-1} \beta^{w_o+c} \\ &= \xi^{i-1 + nw-w} + \xi^{i-1 + n(w_o+c) - (w_o+c)} \\ &= \xi^{i-1 + n(w_o+c)-w} [\xi^{n(w-w_o-c)} + \xi^{w-w_o-c}]. \end{aligned}$$

The factor in brackets is an element of GF(n) since it is of the form $z^n + z$, and is nonzero since $\varphi \neq \varphi_0$. It may therefore be expressed α^u for some u. Then

$$\varphi - \varphi_0 = \xi^{i-1 + n(w_0+c)-w} \alpha^u ,$$

determining that φ and φ_0 are jth associates in $\Sigma(n^2, n+1)$ for $1 \leq j \leq n+1$,

$$(6.10) \qquad j \equiv i + n(w_0 + c) - w \pmod{n + 1}.$$

As φ ranges over $C_i(\theta)$, $\varphi \neq \varphi_0$, w ranges over distinct values 0, 1, ..., n, $w \neq w_0$. Clearly the corresponding values of j are distinct. Thus no two elements of $C_i(\theta)$ are common jth associates of φ_0 for any j, where φ_0 is an arbitrary element of $C_i(\theta)$. This implies the conclusion of the theorem and completes Proof II.

An oval in a finite plane of order n is defined as a set of n + 1 points, n odd, or n + 2 points, n even, with the property that no three points of the set are collinear. It is known that an oval is a maximal set with this property. In the EG(2,n) generated by $\Sigma(n^2, n-1)$, let θ be any point and let i = 1, 2, ..., n-1. Theorem 6.2 shows for n odd that $C_i(\theta)$ is an oval. Theorems 6.2 and 6.1 show for n even that $\{\theta\} \cup C_i(\theta)$ is an oval.

The n + 1 points of a non-degenerate conic in PG(2,n) furnish an example of an oval when n is odd.

A non-degenerate conic in PG(2,n), $n = 2^t$, has the property that its n + 1 tangent lines are concurrent in an (n + 2)nd point which together with the points of the conic makes up an oval. It has been shown by Segre [20] that in the Desarguesian projective plane of odd order n, every oval is a conic. In particular, this holds for every oval in a Desarguesian Euclidean plane of odd order n. The EG(2,n) constructed from $\mathfrak{S}(n^2, n+1)$ is Desarguesian. Therefore for n odd, the n-1 sets $C_i(\theta)$ for any element θ are conics, pairwise disjoint, all confined to the Euclidean plane (and thus disjoint from the "line at infinity"), all disjoint from the point θ and with it exhausting the n^2 points of the plane.

It is a plausible conjecture that the scheme $\mathfrak{S}(n^2, n-1)$ for $n = 2^t$ also leads to sets $C_i(\theta)$ which are conics, and this has been verified for n = 4 and n = 8. Segre's proof does not investigate ovals for even n.

Without giving details, we state that these conics can be exploited to give information on $\mathfrak{S}(n^2, n-1)$. The algebraic statements that $C_1(\theta)$ is a multiplicative subgroup in $GF(n^2)$, each $C_i(0)$ is a coset, and each $C_i(\theta)$ is obtained by addition, all have implications for the equations of the conics. Each p^i_{jk} can be interpreted as the number of points of intersection of two conics. We conjecture that this will lead to a proof that $0 \leq p^i_{jk} \leq 2$.

After the author conjectured that the sets $C_i(\theta)$ were conics, the first proof of Theorem 6.2 was found by R. C. Bose. Using some ideas from this proof, the author then devised the second proof. While these were the first premeditated proofs, a third method became available when R. H. Bruck noticed that the configuration of n-1 disjoint conics in the Euclidean plane of order n could be obtained in many ways by taking suitable plane sections of a configuration he had already discovered in the projective 3-space of order n, consisting of two lines and n-1 ruled quadrics, all disjoint and exhausting the points of the space. The details, which will not be given here, are part of the theory of spreads in projective space [10].

The $NL_g(n)$ schemes obtained in Section 5 by combination of associate classes now inherit a geometric interpretation: the set of first associates of a point θ is the union of g "concentric" conics about θ. Unfortunately, neither the algebraic construction nor the geometric representation has enabled the author to determine in general which $NL_g(n)$ schemes can be formed from $\mathfrak{Z}(n^2, n-1)$ schemes. The case $n = 2m + 1$, $g = m$ is rather special and is discussed in the following paragraph.

As noted in Section 2, the $NL_m(2m + 1)$ scheme, the $L_{m+1}(2m+1)$ scheme, and the pseudo-cyclic scheme with $v = (2m + 1)^2$ all have the same parameters. The scheme $\mathfrak{Z}((2m + 1)^2, 2))$ is pseudo-cyclic; it is identical with the $L_{m+1}(2m+1)$ scheme obtained from $\mathfrak{Z}((2m + 1)^2, 2m + 2)$ by combining the set $S_1 = \{1, 3, \ldots, 2m + 1\}$ of associate classes; it is also identical with the $NL_m(2m + 1)$ scheme obtained from $\mathfrak{Z}((2m + 1)^2, 2m)$ by combining the set $S_1 \{1, 3, \ldots, (2m-1\}$ of associate classes. Geometrically, the set of first associates of a point θ appears first as the union of half of the $2m + 2$ lines through θ (with θ deleted), and second as the union of half of the $2m$ conics $C_i(\theta)$. Even in this special case there are association schemes with the same parameters but with less geometric regularity. The $L_m(2m + 1)$ scheme can be constructed using an arbitrary set S_1 of $m + 1$ associate classes, giving each point θ a set of first associates which is a union of lines through θ but not in general a union of conics $C_i(\theta)$. At least one negative Latin square construction, $NL_4(9)$ using $S_1 = \{1, 2, 5, 6\}$, gives θ a set of first associates which is a union of conics $C_i(\theta)$ but not of lines. Two solutions of the pseudo-cyclic scheme for a given n may be identical as association schemes in spite of differences in geometric structure; that is, they may be related by a one-one correspondence of objects which preserves the association relation without preserving algebraic or geometric relationships.

Thus even in simple cases our geometric interpretation of negative Latin square association schemes needs some clarification. It will probably be of interest to make a geometrical investigation of $\mathfrak{z}(v,m)$ schemes other than those that have been employed here.

7. Direct construction of NL_g designs from finite geometries. A design with the negative Latin square association scheme $NL_3(8)$ occurs as the case $s = 4$ of a family of two-class designs with parameters

$$(7.1) \qquad \begin{aligned} v &= s^3, \quad s = 2^t, \\ n_1 &= (s+2)(s-1), \\ p_{11}^1 &= s-2, \\ p_{11}^2 &= s+2, \\ r &= s+2, \\ k &= s, \\ \lambda_1 &= 1, \\ \lambda_2 &= 0, \end{aligned}$$

constructed by Ray-Chaudhuri [19]. The construction uses PG(3,s), the projective 3-space of order s, $s = 2^t$, in which there are $s + 1$ points on each line, $s^2 + s + 1$ points on each plane, and $s^3 + s^2 + s + 1$ points in all. In one plane γ a non-degenerate conic Q is chosen. Each of the $s + 1$ points of Q is on one line of γ which contains no other points of Q and is called a tangent line. A special property of planes of even order is that the tangents of a conic are all concurrent in a point P. Let R be the set $Q \cup \{P\}$, containing $s + 2$ points of γ. The s^3 points not on γ are taken as objects in the design, and two of these points are first associates if and only if the line containing them also contains a point of R. All such lines, with the points of R deleted, are the blocks of the design. The parameters are immediate with the exception of p_{11}^1 and p_{11}^2, which follow from certain properties of conics.

There is no value of $s = 2^t$ other than 4 for which the design of the Ray-Chaudhuri family is of NL_g type, but we shall describe a generalization which leads to infinitely many NL_g designs, among others. This generalization seems to have gone unnoticed until now.

Various known theorems and formulas in finite geometry, which have been adapted from [25] and from Chapter 2 of [2], will be stated as needed without further reference. The number of elements in a finite set S will be denoted by $|S|$.

Let $\Sigma = PG(n,s)$ be a projective space of dimension n and order s, where s is a prime power. Let $\Gamma = PG(n-1,s)$ be a fixed subspace of dimension n - 1, and let Δ be the complement of Γ in Σ. A set R of points of Γ is chosen. $\bar{R}$ denotes the complement of R in Γ. Δ contains s^n points, which are taken as the objects in a two-class design $\mathfrak{g}(R)$. Each line not entirely in Γ contains s points of Δ and one point of Γ. Two points of Δ are taken as first associates if and only if the line joining them contains a point of R. All such lines, with the points of R deleted, are the blocks of $\mathfrak{g}(R)$. Clearly, (7.2) $$v = s^n ,$$

$$n_1 = (s - 1)\ |R| ,$$

$$r = |R| ,$$

$$k = s ,$$

$$b = s^{n-1}\ |R| ,$$

$$\lambda_1 = 1 ,$$

$$\lambda_2 = 0 .$$

$\mathcal{S}(R)$ will be partially balanced if and only if condition (iii) of (1.3) is satisfied. We proceed to interpret this as a condition on the set R.

If A and B are two points of Δ which are i-th associates, $i = 1, 2$, we denote by $p^i_{11}(A,B)$ the number of points C which are common first associates of A and B. The required points C are of two types which will be enumerated separately.

DEFINITION. D is the point of Γ on line AB. Let C be a common first associate of A and B. Then we define C to be a collinear common first associate of A and B (c-point of A and B) if C is on line AB, and a diagonal common first associate of A and B (d-point of A and B) if C is not on line AB.

Obviously there are $s - 2$ c-points of A and B if $D \in R$ and none if $D \in \bar{R}$.

If C is a d-point of A and B, then lines AC and BC respectively must meet Γ in points D' and D'' of R. Plane ABC meets Γ in a line m on D which also contains D' and D''. Suppose that m contains ν points of R. Then the ordered pair of points D', D'' can be chosen in $(\nu - 1)(\nu - 2)$ ways if $D \in R$, and in $\nu(\nu-1)$ ways if $D \in \bar{R}$, and the plane determined by m, A and B contains a like number of d-points of A and B. The total number of d-points of A and B can be obtained by summing over the lines m which are in Γ and contain D.

DEFINITION. $T_\nu(R)$ is the set of lines of Γ which contain exactly ν points of R, $\nu = 0, 1, \ldots, s+1$.

DEFINITION. $x_\nu(D)$, $\nu = 0, 1, \ldots, s+1$, is the number of lines of $T_\nu(R)$ which contain D.

Now if A and B are first associates, so that $D \in R$,

$$p_{11}^1(A,B) = s - 2 + \sum_{\nu=0}^{s+1} (\nu - 1)(\nu - 2)\, x_\nu(D), \tag{7.3}$$

and if A and B are second associates, so that $D \in \bar{R}$,

$$p_{11}^2(A,B) = \sum_{\nu=0}^{s+1} \nu(\nu - 1)\, x_\nu(D). \tag{7.4}$$

This is enough to prove

LEMMA 7.1. $\mathcal{S}(R)$ is a two-class PBIB design if and only if the right hand side of (7.3) has the same value for all points $D \in R$ and the right hand side of (7.4) has the same value for all points $D \in \bar{R}$. In this case $\mathcal{S}(R)$ will have parameters (7.2), along with

$$p_{11}^1 = p_{11}^1(A,B),\ p_{11}^2 = p_{11}^2(A,B).$$

REMARK. The condition of Lemma 7.1 will be recognized as essentially a condition on the variance of the numbers ν. It is implied by the condition the following lemma places on their frequency distribution.

LEMMA 7.2. $\mathcal{L}(R)$ is a two-class PBIB design if for fixed $\nu = 0, 1, \ldots s+1$, the frequencies $x_\nu(D)$ are equal for all $D \in R$ and are equal for all $D \in \bar{R}$. In this case $\mathcal{L}(R)$ will have parameters as stated in Lemma 7.3.

In our first application of these lemmas we take $R = Q$, a non-degenerate quadric in $\Gamma = PG(n-1,s)$, denoting $\bar{Q} = \bar{R}$. All lines of Γ fall into the following four sets $T_\nu(Q)$.

(7.5) $T_0(Q)$: non-intersectors, containing no points of Q,

$T_1(Q)$: tangents, each containing 1 point of Q,

$T_2(Q)$: secants, each containing 2 points of Q,

$T_{s+1}(Q)$: rulings, each containing s+1 points of Q.

Thus nonzero frequencies $x_\nu(D)$ can occur only for $\nu = 0, 1, 2, s+1$, and (7.3) and (7.4) reduce to

$$p^1_{11}(A,B) = s - 2 + s(s - 1)\, x_{s+1}(D), \quad D \in Q, \tag{7.6}$$

$$p^2_{11}(A,B) = 2\, x_2(D), \quad D \in \bar{Q}\ . \tag{7.7}$$

In a particular non-degenerate quadric Q in PG(n-1,s), the number $x_{s+1}(D)$ of rulings on D is the same for all points $D \in Q$, so that $p^1_{11}(A,B)$ has a uniform value p^1_{11} for all pairs A, B of first associates in $\mathcal{L}(Q)$.

We must specify the dimension n before proceeding further. If n = 2t, so that Γ has odd dimension 2t-1, the number $x_2(D)$ of secant lines on D is the same for all points $D \in \bar{Q}$, so that $p^2_{11}(A,B)$ has a uniform value p^2_{11} for all pairs A, B of second associates in $\mathcal{L}(Q)$. If n is odd, so that Q is a non-degenerate quadric in a space Γ of even dimension, the points $D \in \bar{Q}$ are of different types which are contained in different numbers $x_2(D)$ of secant lines. In this case $p_{11}^{2}(A,B)$ does not have the same value for all pairs A, B of second associates.

We conclude that if Q is a non-degenerate quadric in Γ, the design $\mathcal{L}(Q)$ is a two-class partially balanced design if and only if the dimension n of Σ is even.

Let n = 2t. In $\Gamma = PG(2t-1,s)$ there are two types of non-degenerate quadrics, which we shall call hyperbolic and elliptic, differing in the number of points, ruling lines, and secants. In the following formulas, the upper signs hold for hyperbolic quadrics and the lower signs hold for elliptic quadrics.

$$(7.8) \qquad |Q| = (s^{t-1} \pm 1)(s^{t} \mp 1)/(s-1),$$

$$x_{s+1}(D) = (s^{t-2} \pm 1)(s^{t-1} \mp 1)/(s-1), \quad D \in Q,$$

$$x_{2}(D) = s^{t-1}(s^{t-1} \pm 1)/2, \quad D \in \overline{Q}\ .$$

The parameters of $\mathcal{L}(Q)$ can now be computed in both cases and compared with (2.1) and (2.5) to complete the proof of

THEOREM 7.1. If n = 2t and Q is a non-degenerate quadric in Γ, the design $\mathcal{L}(Q)$ is a two-class PBIB design with association scheme parameters

$$(7.9) \qquad v = s^{2t},$$

$$n_{1} = (s^{t-1} \pm 1)(s^{t} \mp 1),$$

$$p_{11}^{1} = s^{t-1}(s^{t-1} \mp 1) \pm s^{t} - 2,$$

$$p_{11}^{2} = s^{t-1}(s^{t-1} \pm 1),$$

and design parameters

$$(7.10) \qquad r = (s^{t-1} \pm 1)(s^t \mp 1)/(s - 1) ,$$

$$k = s,$$

$$b = s^{2t-1} r,$$

$$\lambda_1 = 1,$$

$$\lambda_2 = 0.$$

If Q is hyperbolic the upper signs hold and $\mathcal{L}(Q)$ is of pseudo-Latin square type $L_g(s^t)$, $g = s^{t-1} + 1$. If Q is elliptic the lower signs hold and $\mathcal{L}(Q)$ is of negative Latin square type $NL_g(s^t)$, $g = s^{t-1} - 1$.

Since the required projective spaces and quadrics exist for every s which is a prime or a power of a prime and for every positive integer t, our construction gives a doubly infinite family of designs having NL_g association schemes. The following schemes with $v \leq 100$ are included.

$$s = 2,\ t = 2,\ NL_1(4),$$

$$s = 2,\ t = 3,\ NL_3(8),$$

$$s = 3,\ t = 2,\ NL_2(9).$$

The spaces Σ, Γ, and Δ and the quadric Q may be used to construct other designs which have the same association scheme as $\mathcal{L}(Q)$.

We note that each block of $\mathcal{L}(Q)$ is the intersection $\ell \cap \Delta$ of Δ with a line ℓ of Σ, where ℓ intersects Γ in a point of Q. We define a more general design $\mathcal{L}_\nu(Q)$, $\nu = 0, 1$, with sets of blocks constructed as follows from the set of all lines ℓ which are in Σ but not in Γ.

Design	Blocks
$\mathcal{L}_0(Q)$	$\{\ell \cap \Delta \mid \ell \cap \Gamma \in \bar{Q}\}$
$\mathcal{L}_1(Q)$	$\{\ell \cap \Delta \mid \ell \cap \Gamma \in Q\}$

The subscript ν may be interpreted as the number of points of Q contained in ℓ. The following theorem is now obvious.

THEOREM 7.2. If $n = 2t$ and Q is a non-degenerate quadric in Γ, then $\mathcal{L}_\nu(Q)$, $\nu = 0, 1$, is partially balanced with the same association scheme as $\mathcal{L}(Q)$ described in Theorem 7.1. $\mathcal{L}_\nu(Q)$ has association scheme parameters (7.9). $\mathcal{L}_1(Q)$ is identical with $\mathcal{L}(Q)$. $\mathcal{L}_0(Q)$ has design parameters

(7.11) $r = |\bar{Q}|$,

$k = s$,

$b = s^{2t-1} |\bar{Q}|$,

$\lambda_1 = 0$,

$\lambda_2 = 1$.

Let π be a plane in Σ but not in Γ. π intersects Δ in a set of s^2 points which we shall use as a block of a design, and intersects Γ in a line which falls in one of the classes $T_\nu(Q)$. We define designs $\mathcal{P}_\nu(Q)$ with sets of blocks constructed as follows from the set of all planes π which are in Σ but not in Γ.

Design	Blocks
$\mathcal{P}_\nu(Q)$	$\{\pi \cap \Delta \mid \pi \cap \Gamma \in T_\nu(Q)\}$, $\nu = 0, 1, 2, s+1$.

The subscript ν may be interpreted as the number of points of Q contained in π.

If A is a point of Δ, planes containing A are determined by the lines of $T_\nu(Q)$, and these planes lead to the blocks of $\mathcal{P}_\nu(Q)$ which contain A. Therefore A is contained in $|T_\nu(Q)|$ blocks.

If A and B are two points of Δ and D is the intersection of line AB with Γ, planes containing A and B are determined by the lines of $T_\nu(Q)$ which contain D, and these planes lead to the blocks of $\mathcal{P}_\nu(Q)$ which contain both A and B. Therefore A and B occur together in $x_\nu(D)$ blocks. We now use the fact, stated in part in (7.8), that for a non-degenerate Q in Γ of odd dimension, all of the frequencies $x_\nu(D)$ satisfy the uniformity condition of Lemma 7.2. This gives us the following theorem.

THEOREM 7.3. If $n = 2t$ and Q is non-degenerate, then $\mathcal{P}_\nu(Q)$, $\nu = 0, 1, 2, s+1$, is partially balanced with the same association scheme as $\mathcal{C}(Q)$, described in Theorem 7.1. $\mathcal{P}_\nu(Q)$ has association scheme parameters (7.9) and design parameters

(7.12)
$$r = T_\nu(Q) ,$$
$$k = s^2,$$
$$b = s^{2t-2} |T_\nu(Q)|,$$
$$\lambda_1 = x_\nu(D),\ D \in Q,$$
$$\lambda_2 = x_\nu(D),\ D \in \overline{Q}.$$

Formulas for $|T_\nu(Q)|$ and $x_\nu(D)$ are listed below. In each case Q is understood to be a non-degenerate quadric in $\Gamma = PG(2t-1,s)$. The upper signs hold if Q is hyperbolic and the lower signs hold if Q is elliptic.

TABLE 7.1

ν	$\lvert T_\nu(Q)\rvert$	$x_\nu(D),\ D \in Q$	$x_\nu(D),\ D \in \bar{Q}$
0	$\frac{s^{2t-2}(s^t \mp 1)(s^{t-1} \mp 1)}{2(s+1)}$	0	$\frac{s^{t-1}(s^{t-1} \mp 1)}{2}$
1	$\frac{s^{t-2}(s^{2t-2} - 2)(s^t \mp 1)}{s-1}$	$s^{t-2}(s^{t-1} \mp 1)$	$\frac{s^{2t-2} - 1}{s - 1}$
2	$\frac{s^{2t-2}(s^t \mp 1)(s^{t-1} \pm 1)}{2(s - 1)}$	s^{2t-2}	$\frac{s^{t-1}(s^{t-1} \pm 1)}{2}$
s+1	$\frac{(s^{2t-2} - 1)(s^t \mp 1)(s^{t-2} \pm 1)}{(s^2 - 1)(s - 1)}$	$\frac{(s^{t-1} + 1)(s^{t-2} \pm 1)}{s - 1}$	0

There are sets R other than quadrics for which the design $\mathfrak{L}(R)$ is partially balanced, as illustrated by the Ray-Chaudhuri family of designs described at the beginning of the section. Our final construction uses an interesting set whose properties have been investigated by Bose [1].

Take $\Sigma = PG(2,q)$, $\Gamma = PG(2,q)$, where $q = s^2$, and represent the points of Γ by homogeneous coordinates (y_1, y_2, y_3), $y_i \in GF(q)$. Take $R = W$, where W is the set of points of Γ for which the equation

(7.13) $$y_1^{s+1} + y_2^{s+1} + y_3^{s+1} = 0$$

is satisfied. Bose shows that

(7.14) $$\lvert W\rvert = s^3 + 1;$$

$$x_1(D) = 1 \text{ and } x_{s+1}(D) = s^2,\ D \in W;$$

$x_1(D) = s + 1$ and $x_{s+1}(D) = s^2 - s$, $D \in \overline{W}$;

otherwise, $x_\nu(D) = 0$.

We prove the following theorem by applying Lemma 7.2 and comparing parameters with (2.5).

THEOREM 7.4. $\mathcal{S}(W)$ is a two-class PBIB design with parameters

$$(7.15) \qquad v = s^6,$$

$$n_1 = (s^2 - 1)(s^3 + 1),$$

$$p_{11}^1 = s^2(s^2 + 1) + s^3 - 2,$$

$$p_{11}^2 = s^2(s^2 - 1),$$

$$r = s^3 + 1,$$

$$k = s^2,$$

$$b = s^4(s^3 + 1),$$

$$\lambda_1 = 1,$$

$$\lambda_2 = 0.$$

This design is of negative Latin square type $NL_g(s^3)$, $g = s^2 - 1$.

Three other designs $\mathcal{S}_0(W)$, $\rho_1(W)$, and $\rho_{s+1}(W)$ with the same association scheme can be constructed from W by the methods of Theorems 7.2 and 7.3. These designs have the same association scheme parameters as the designs $\mathcal{S}_\nu(Q)$ and $\rho_\nu(Q)$, $t = 3$, but have different design parameters r, k, b, λ_i.

The following tables give the parameters of the designs constructed in this section which have NL_g association schemes with $v \leq 100$. Additional designs exist, of course, for these association schemes.

TABLE 7.2

Parameters of Designs $\mathcal{L}_\nu(Q)$ and $\mathcal{P}_\nu(Q)$, Q Elliptic,

$s = 2$, $t = 2$. $NL_1(4)$ Designs

Design	ν	r	k	b	λ_1	λ_2	
$\mathcal{L}_1(Q)$	16	5	2	40	1	0	
$\mathcal{L}_0(Q)$	16	10	2	80	0	1	
$\mathcal{P}_0(Q)$	16	10	4	40	0	3	
$\mathcal{P}_1(Q)$	16	15	4	60	3	3	(Balanced design)
$\mathcal{P}_2(Q)$	16	10	4	40	4	1	
$\mathcal{P}_3(Q)$	Null design; $r = b = 0$.						

TABLE 7.3

Parameters of Designs $\mathcal{L}_\nu(Q)$ and $\mathcal{P}_\nu(Q)$, Q Elliptic

$s = 2$, $t = 3$. $NL_3(8)$ Designs

Design	v	r	k	b	λ_1	λ_2
$\mathcal{L}_1(Q)$	64	27	2	864	1	0
$\mathcal{L}_0(Q)$	64	36	2	1152	0	1
$\mathcal{P}_0(Q)$	64	120	4	1920	0	10
$\mathcal{P}_1(Q)$	64	270	4	4320	10	15
$\mathcal{P}_2(Q)$	64	216	4	3456	16	6
$\mathcal{P}_3(Q)$	64	45	4	720	5	0

TABLE 7.4

Parameters of Designs $\mathcal{L}_\nu(Q)$ and $\mathcal{P}_\nu(Q)$, Q Elliptic,

$s = 3$, $t = 2$. $NL_2(9)$ Designs

Design	v	r	k	b	λ_1	λ_2	
$\mathcal{L}_1(Q)$	81	10	3	270	1	0	
$\mathcal{L}_0(Q)$	81	30	3	810	0	1	
$\mathcal{P}_0(Q)$	81	45	9	405	0	6	
$\mathcal{P}_1(Q)$	81	40	9	360	4	4	(Balanced Design)
$\mathcal{P}_2(Q)$	81	45	9	405	9	3	
$\mathcal{P}_4(Q)$	Null design; $r = b = 0$						

TABLE 7.5

Parameters of Designs $\mathcal{L}_\nu(W)$ and $\mathcal{P}_\nu(W)$, $s = 2$.

$NL_3(8)$ Designs

Design	v	r	k	b	λ_1	λ_2
$\mathcal{L}_1(W)$	64	9	4	144	1	0
$\mathcal{L}_0(W)$	64	12	4	192	0	1
$\mathcal{P}_1(W)$	64	9	16	36	1	3
$\mathcal{P}_3(W)$	64	12	16	48	4	2

8. Association schemes with $p_{11}^1 = 0$.

We begin with a theorem which holds for any two-class association scheme $\mathcal{G}$ with $p_{11}^1 = 0$, then derive stronger results for NL_g schemes with this property.

We define the following sets for a two-class association scheme.

$$S_0 = \{\text{an initial object } \alpha\},$$
$$S_1 = \{\text{the } n_1 \text{ first associates of } \alpha\}, \tag{8.1}$$
$$S_2 = \{\text{the } n_2 \text{ second associates of } \alpha\}.$$

Interpreted as sets of rows and columns, these sets define the following partition of A_1, the first association matrix of $\mathcal{G}$. This is the same partition used in (4.17). Submatrix B_1 reduces to a zero matrix because $p_{11}^1 = 0$.

$$A_1 = \left[\begin{array}{c|c|c} 0 & 1 \;\ldots\; 1 & 0 \;\ldots\; 0 \\ \hline 1 & & \\ \vdots & 0 & C_1 \\ 1 & & \\ \hline 0 & & \\ \vdots & C_1^T & D_1 \\ 0 & & \end{array}\right] \tag{8.2}$$

Theorem 8.1. If $p_{11}^1 = 0$ in a two-class association scheme with parameters v, n_i, p_{jk}^i, then

(i) C_1 as defined in (8.2) is the incidence matrix of a BIB design $\mathcal{C}$ with parameters

$$\hat{v} = n_1,\ \hat{b} = n_2,\ \hat{r} = p_{12}^1,\ \hat{k} = p_{11}^2,\ \hat{\lambda} = p_{11}^2 - 1;$$

(ii) any block of $\mathcal{C}$ is disjoint from at least p^2_{12} other blocks.

Proof. C_1 is a matrix of 0's and 1's with $n_1 = \hat{v}$ rows and $n_2 = \hat{b}$ columns. The column totals of C_1 are equal to the row totals of C_1^T, which by Lemma 4.5 are uniformly equal to $p^2_{11} = \hat{k}$. Statement (4.21) of Theorem 4.6, with $B_1 = 0$, $B_2 = J - I$, $p^1_{12} = n_1 - p^1_{11} - 1 = n_1 - 1$, reduces to

$$C_1 C_1^T = p^1_{12}\, I + (p^2_{11} - 1)(J - I).$$

By Corollary 4.7.1 this proves (i).

By (4.23), the matrix $C_1^T\, C_1 + D_1^2$ must have elements $p^1_{11} = 0$ in all positions occupied by 1's in D_1, and by Lemma 4.5 there are exactly p^2_{12} such positions in each row. Since D_1^2 has non-negative elements, any row of $C_1^T\, C_1$ must contain at least p^2_{12} zero elements. But the i-th element of a particular row β of $C_1^T\, C_1$ can be interpreted as the number of objects which the i-th block of $\mathcal{C}$ has in common with block β, proving (ii).

For the rest of this section, $\mathcal{G}$ will be taken as an $NL_g(n)$ scheme with

$$p^1_{11} = g^2 + 3g - n = 0.$$

A_1 will denote the first association matrix of this scheme. Expressing

$$n = g^2 + 3g,$$

$\mathcal{C}$ has parameters

(8.3) $$v = n^2 = g^2(g + 3)^2,$$

$$n_1 = g(g^2 + 3\,g + 1),$$

$$n_2 = (g^2 + 2\,g - 1)(g^2 + 3\,g + 1),$$

$$p_{11}^{1} = 0,$$

$$p_{12}^{1} = g^3 + 3g^2 + g - 1 = (g+1)(g^2 + 2g - 1),$$

$$p_{22}^{1} = g^4 + 4g^3 + 3g^2 - 2g = g(g + 2)(g^2 + 2g - 1),$$

$$p_{11}^{2} = g^2 + g = g(g + 1),$$

$$p_{12}^{2} = g^3 + 2g^2 = g^2(g + 2),$$

$$p_{22}^{2} = g^4 + 4g^3 + 4g^2 - g - 2 = (g + 1)(g + 2)(g^2 + g-1),$$

$$\sigma = g^2 + 2g - 1,$$

$$\tau = g.$$

Theorem 8.1 shows in this case that C_1 is the incidence matrix of a BIB design which has parameters

(8.4) $$\hat{v} = n_1 = g(g^2 + 3g+1),$$

$$\hat{b} = n_2 = (g^2 + 2g-1)(g^2 + 3g-1),$$

$$\hat{r} = p_{12}^{1} = (g+1)(g^2 + 2g-1),$$

$$\hat{k} = p_{11}^{2} = g(g + 1),$$

$$\hat{\lambda} = p_{11}^{2} - 1 = g^2 + g-1,$$

and which has the property that

(8.5) each block is disjoint from at least $p_{12}^{2} = g^2(g+2)$ other blocks.

The existence of such a design is thus a necessary condition for the existence of the $NL_g(g^2 + 3g)$ association scheme. The next lemma and two theorems will show that it is sufficient as well and that the design can be used to construct scheme [illegible] . The proof of sufficiency must be based on a design which is not assumed to arise from an association scheme. Let $\mathcal{X}$ be a BIB design which has parameters (8.4) and property (8.5) but is otherwise arbitrary. Let X_1 be the incidence matrix of this design, so that

$$(8.6) \qquad X_1X_1^T = \hat{r}\ I + \hat{\lambda}\ (J - I)$$
$$= p_{12}^1\ I + (p_{11}^2 - 1)(J - I),$$

and let

$$(8.7) \qquad X_2 = J - X_1.$$

Matrix C_1 may be regarded as a special case of X_1.

We may regard sets S_1 and S_2 respectively as the set of rows and the set of columns of matrix X_1, or we may regard them as the set of objects and the set of blocks in the corresponding design. The latter interpretation is convenient for the definition of the following sets.

(8.8) S_{11} = {the p_{11}^2 objects in an intial block γ},

S_{12} = {the remaining p_{12}^2 objects of S_1},

S_{20} = {block γ},

S_{21} = {a set of p_{12}^2 blocks disjoint from γ},

S_{22} = {the remaining p_{22}^2 blocks of S_2}.

Lemma 8.2. Let a BIB design $\mathcal{X}$ have parameters (8.4) and property (8.5). Then each block of $\mathcal{X}$ is disjoint from exactly p_{12}^2 blocks and intersects each remaining block in exactly g objects.

Proof. In the terminology of (8.7), let f_i denote the number of objects common to block γ and the i-th remaining block. Since $f_i = 0$ for all blocks of S_{21}, the well known formulas due to Hussain [15],

$$\sum_i f_i = k\,(r - 1),$$

$$\sum_i f_i^2 = k(r - 1) + k(k - 1)(\lambda - 1),$$

remain valid if the summation is restricted to the blocks of S_{22}. Straightforward computation shows that the f_i's for this subset have mean g and satisfy

$$\Sigma\,(f_i - g)^2 = 0\ ,$$

showing that γ intersects each block of S_{22} in exactly g objects. Finally, γ, which is an arbitrary block, is disjoint from precisely the p_{12}^2 blocks of S_{21}. This proves the lemma.

This lemma depends on the parameters (8.3) and fails in general for the BIB design described in Theorem 8.1. It appears therefore that the construction method of this section for $NL_g(g^2 + 3g)$ schemes will not be applicable to other two-class schemes with $p_{11}^1 = 0$.

In each row of the symmetric matrix $X_1^T X_1$, the element in diagonal position is equal to $k = p_{11}^2$, p_{12}^2 elements are equal to 0, and the p_{22}^2 other elements are equal to g. Thus we may express

$$(8.9) \qquad X_1^T X_1 = p_{11}^2 I + 0\ .\ Y_1 + gY_2,$$

where Y_1 and Y_2 are matrices of 0's and 1's, each row of Y_1 contains p_{12}^2 1's, each row of Y_2 contains p_{22}^2 1's, and

$$(8.10) \qquad I + Y_1 + Y_2 = J,$$

where J is a b x b matrix of 1's. Y_1 and Y_2 are symmetric because $X_1^T X_1$ is symmetric.

Theorem 8.3. If A_1, the first association matrix of an $NL_g(g^2 + 3g)$ scheme with parameters v, n_i, p^i_{jk} given by (8.3), then C_1^T and D_1 as defined in (8.2) are respectively the incidence matrix and first association matrix of a two-class PBIB design $\mathcal{C}'$ and association scheme $\mathcal{S}$ with the following parameters.

$$(8.11) \qquad \tilde{v} = n_2,\ \tilde{b} = n_1,\ \tilde{\gamma} = p^2_{11},\ \tilde{k} = p^1_{12},\ \tilde{\lambda}_1 = 0,\ \tilde{\lambda}_2 = g,$$

$$\tilde{n}_1 = p^2_{12},$$

$$\tilde{n}_2 = p^2_{22},$$

$$\tilde{p}^1_{11} = 0,$$

$$\tilde{p}^1_{12} = g^3 + 2g^2 - 1 = (g+1)(g^2+g-1),$$

$$\tilde{p}^1_{22} = g^4 + 3g^3 + 2g^2 - g - 1 = (g+1)^2(g^2+g-1),$$

$$\tilde{p}^2_{11} = g^2,$$

$$\tilde{p}^2_{12} = g^3 + g^2 = g^2(g+1),$$

$$\tilde{p}^2_{22} = g^4 + 3g^3 + 3g^2 - g - 3.$$

Proof. The values $\tilde{v}$, $\tilde{b}$, $\tilde{\gamma}$ and $\tilde{k}$ follow from Theorem 8.1. From (4.23),

$$(8.12) \qquad C_1^T C_1 + D_1^2 = n_1 I + 0.D_1 + p^2_{11} D_2.$$

From (8.9), denoting by $\tilde{Y}_i$, $i = 1, 2$, the form taken by Y_i in the case $X_1 = C_1$,

$$(8.13) \qquad C_1^T C_1 = p^2_{11} I + 0.\tilde{Y}_1 + g\,\tilde{Y}_2$$

By subtraction,

$$D_1^2 = (n_1 - p_{11}^2)\, I + p_{11}^2\, D_2 - g\, \tilde{Y}_2 .$$

$\tilde{Y}_2$ and D_2 are matrices of 0's and 1's, each with 0's on the main diagonal and each with exactly p_{22}^2 1's in each row. If $\tilde{Y}_2 \neq D_2$, negative elements will occur in off-diagonal positions in the difference $p_{11}^2\, D_2 - g\, \tilde{Y}_2$ and hence in D_1^2. But this is impossible, since D_1 has non-negative elements. Therefore $\tilde{Y}_2 = D_2$, $\tilde{Y}_1 = D_1$, and we compute

(8.14) $$C_1^T\, C_1 = p_{11}^2 I + 0.D_1 + g\, D_2,$$

(8.15) $$D_1^2 = p_{12}^2\, I + 0.D_1 + g^2 D_2.$$

Now D_1 and D_2 are symmetric matrices of 0's and 1's satisfying $I + D_1 + D_2 = J$, and (8.15) is sufficient to show by Theorem 4.4 that D_1 is the first association matrix of a two-class association scheme with parameters $\tilde{n}_1 = p_{12}^2$, $\tilde{p}_{11}^1 = 0$, $\tilde{p}_{11}^2 = g^2$. (8.14) is then sufficient to show by Corollary 4.7.1 that C_1^T is the incidence matrix of a PBIB design with this association scheme and with parameters including $\tilde{\lambda}_1 = 0$, $\tilde{\lambda}_2 = g$. This implies the rest of (8.11) and the proof is complete.

All of the nontrivial submatrices of A_1 have now been identified with the BIB design $\mathcal{C}$ or its dual. This motivates the next theorem in which matrices furnished by the design $\mathcal{X}$ are used in the definition of an association matrix.

Theorem 8.4. Let X_1 be the incidence matrix of a BIB design with parameters (8.4) and property (8.5), and define X_2, Y_1, Y_2 by (8.6), (8.9), (8.10). Let A_1^* be the matrix

$$(8.16)\qquad A_1^* = \left[\begin{array}{c|c|c} 0 & 1 \ldots 1 & 0 \ldots 0 \\ \hline 1 & & \\ \vdots & 0 & X_1 \\ 1 & & \\ \hline 0 & & \\ \vdots & X_1^T & Y_1 \\ 0 & & \end{array}\right].$$

Then A_1^* is the first association matrix of an $NL_g(g^2+3g)$ scheme with parameters (8.3).

Proof. In Theorem 4.6, take $A_1 = A_1^*$, $B_1 = 0$, $C_1 = X_1$, $D_1 = Y_1$. A_1^* is a matrix of 0's and 1's; the same is true of $A_2^* = J - I - A_1^*$, since the diagonal elements of Y_1 are 0's. Y_1 is symmetric. From (8.9), X_1^T has uniform row sums $\tilde{r} = p_{11}^2$. We have verified (4.18) to (4.20). In order to prove by Theorem 4.6 that A_1^* is the specified association matrix it now suffices to verify (4.21) to (4.23), which reduce to

$$(8.17)\qquad X_1 X_1^T = p_{12}^1 I + (p_{11}^2 - 1)(J - I),$$

$$(8.18)\qquad X_1 Y_1 = p_{11}^2 X_2,$$

$$(8.19)\qquad X_1^T X_1 + Y_1^2 = n_1 I + p_{11}^2 Y_2.$$

(8.17) is equivalent to (8.6). The known row and column sums of X_1 and Y_1 give the following relations, in which matrices of 1's of various orders are all denoted by the same letter J.

$$(8.20)\qquad X_1 J = p_{12}^1 J,$$

$$(8.21)\qquad J X_1 = p_{11}^2 J,$$

(8.22) $\quad X_1^T J = p_{11}^2 J,$

(8.23) $\quad J Y_1 = p_{12}^2 J.$

Solving (8.9) and (8.10) for Y_1,

(8.24) $\quad Y_1 = g I + J - g^{-1} X_1^T X_1.$

We next multiply on the left by X_1 and apply (8.20) and (8.6).

$$X_1 Y_1 = g X_1 + X_1 J - g^{-1} X_1 X_1^T X_1$$

$$= g X_1 + p_{12}^1 J - g^{-1} [(p_{12}^1 - p_{11}^2 + 1) I + (p_{11}^2 - 1) J] X_1 .$$

(8.21) is applied and the result is simplified with the aid of (8.3) to give

(8.25) $\quad X_1 Y_1 = p_{11}^2 (J - X_1) ,$

proving (8.18). We now multiply (8.24) on the right by Y_1 and apply (8.23) and (8.25).

$$Y_1^2 = g Y_1 + J Y_1 - g^{-1} X_1^T X_1 Y_1$$

$$= g Y_1 + p_{12}^2 J - g^{-1} p_{11}^2 X_1^T (J - X_1).$$

(8.22) and (8.9) are applied and the result is simplified with the aid of (8.3) and (8.10) to give

(8.26) $\quad Y_1^2 = p_{12}^2 I + g^2 Y_2 .$

Adding (8.9) and (8.26) gives (8.19), completing the proof.

We are now assured that if matrix X_1 exists, then even though it was not assumed to occur, as C_1 does, as a submatrix of an association matrix, it does in fact play exactly this role in A_1^*. Moreover, A_1^* is unique for a given X_1. The design $\mathcal{X}$ and matrices X_1, X_2 will hereafter be denoted by $\mathcal{C}$, C_1, C_2. We also drop the distinction between A_1 and A_1^*.

The following corollary paraphrases Theorem 7.4 in terms of objects and blocks of $\mathcal{C}$, without use of the matrices A_1 or C_1.

Corollary 8.4.1. Let $\mathcal{C}$ be a BIB design with parameters (8.4) and property (8.5). Define a two-class association relation $\mathcal{G}$ on the set of objects $S_0 \cup S_1 \cup S_2$, where

$$S_0 = \{\text{an initial object } \alpha\}, \tag{8.27}$$

$$S_1 = \{\text{objects } \beta_i \text{ of } \mathcal{C}\},$$

$$S_2 = \{\text{blocks } \gamma_j \text{ of } \mathcal{C}\},$$

and sets of first associates are defined as follows.

Object	Set of first associates
α	S_1
β_i	$S_0 \cup \{\text{blocks of } S_2 \text{ containing } \beta_i\}$
γ_j	$\{\text{objects of } S_1 \text{ contained in } \gamma_j\} \cup \{\text{blocks of } S_2 \text{ disjoint from } \gamma_j\}$

Then $\mathcal{G}$ is an $NL_g(g^2 + 3g)$ scheme, with parameters (8.3).

The following table lists the important parameters of $\mathcal{C}$ and $\mathcal{G}$ for a few values of g. D J denotes the number of blocks of $\mathcal{C}$ which are disjoint from any given block.

TABLE 8.1

Parameters of BIB design $\mathcal{C}$				$\hat{v}$	$\hat{b}$	$\hat{r}$	$\hat{k}$	$\hat{\lambda}$	DJ
Parameters of NL_g scheme $\mathcal{G}$	g	n	v	n_1	n_2	p^1_{12}	p^2_{11}		p^2_{12}
	1	4	16	5	10	4	2	1	3
	2	10	100	22	77	21	6	5	16
	3	18	324	57	266	56	12	11	45
	4	28	784	116	667	115	20	19	96

Let A_1 be partitioned into submatrices whose sets of rows and columns are S_0, S_{11}, S_{12}, S_{20}, S_{21}, S_{22}. This is a refinement of the partition (8.2). The submatrix, say $\hat{0}$, with S_{11} as its row set and S_{21} as its column set falls in $\mathcal{C}_1$, the incidence matrix of BIB design $\mathcal{C}$. $\hat{0}$ is a zero submatrix, since by (8.8) no object of S_{11} is in any block of S_{21}. Define notation as follows for other submatrices of $_1$.

(8.28)
$$\mathcal{C}_1 = \left[\begin{array}{c|c|c} 1 & & \\ \vdots & \hat{0} & E_1 \\ 1 & & \\ \hline 0 & & \\ \vdots & F_1 & G_1 \\ 0 & & \end{array}\right]$$

The submatrix, say $\tilde{0}$, with S_{21} as its row and column set falls in D_1, the first association matrix of association scheme $\mathcal{D}$. Since S_{21} is the set of first associates in $\mathcal{D}$ of initial object β, and $\tilde{p}^1_{11} = 0$, it follows from Lemma 4.5 that $\tilde{0}$ is a zero submatrix. Define notation as follows for other submatrices of $\mathcal{D}_1$.

(8.29)

$$D_1 = \left[\begin{array}{c|c|c} 0 & 1 \; . \; . \; . \; . \; 1 & 0 \; . \; . \; . \; . \; 0 \\ \hline 1 & & \\ \vdots & \tilde{0} & H_1 \\ 1 & & \\ \hline 0 & & \\ \vdots & H_1^T & K_1 \\ 0 & & \end{array}\right]$$

A_1 may now be written as follows. The table below indicates the column sets, and also applies to the row sets.

(8.30)

$$A_1 = \left[\begin{array}{c|c|c|c|c|c} 0 & 1 \ldots\ldots 1 & 1 \ldots\ldots 1 & 0 & 0 \ldots\ldots 0 & 0 \ldots\ldots 0 \\ \hline 1 & & & 1 & & \\ \vdots & 0 & 0 & \vdots & \hat{0} & E_1 \\ 1 & & & 1 & & \\ \hline 1 & & & 0 & & \\ \vdots & 0 & 0 & \vdots & F_1 & G_1 \\ 1 & & & 0 & & \\ \hline 0 & 1 \ldots\ldots 1 & 0 \ldots\ldots 0 & 0 & 1 \ldots\ldots 1 & 0 \ldots\ldots 0 \\ 0 & & & & & \\ \vdots & \hat{0}^T & F_1^T & & \tilde{0} & H_1 \\ 0 & & & & & \\ \hline 0 & & & 0 & & \\ \vdots & E_1^T & G_1^T & \vdots & H_1^T & K_1 \\ 0 & & & 0 & & \end{array}\right]$$

Set of columns	S_0	S_{11}	S_{12}	S_{20}	S_{21}	S_{22}
Number of Columns	1	p_{11}^2	p_{12}^2	1	p_{12}^2	p_{22}^2

Theorem 8.5. If A_1, the first association matrix of an $NL_g(g^2+3g)$ scheme, is partitioned as in (8.30), submatrices E_1, F_1, G_1, H_1 are incidence matrices of BIB designs, say $\mathcal{E}$, [illegible], [illegible], [illegible], with parameters as follows.

Matrix	v	b	r	k	λ
E_1	$g(g+1)$	$(g+1)(g+2)(g^2+g-1)$	$(g+2)(g^2+g-1)$	g	$(g-1)(g+2)$
F_1	$g^2(g+2)$	$g^2(g+2)$	$g(g+1)$	$g(g+1)$	g
G_1	$g^2(g+2)$	$(g+1)(g+2)(g^2+g-1)$	$(g+1)(g^2+g-1)$	g^2	$(g-1)(g+1)$
H_1	$g^2(g+2)$	$(g+1)(g+2)(g^2+g-1)$	$(g+1)(g^2+g-1)$	g^2	$(g-1)(g+1)$

Each of designs $\mathcal{E}$, $\mathcal{G}$, $\mathcal{H}$ has the property that each block is disjoint from at least $g^2(g+1)$ other blocks.

Proof. The results for H_1 follow from Theorem 8.1, applied to association scheme $\mathcal{D}$.

The proof for E_1, F_1, G_1 will use Corollary 4.7.1. It is easily verified, say from (8.30) and (8.3), that each of these matrices has the size v x b stated for it in the theorem, and that F_1^T has the same size as F_1. We next show that each of E_1, G_1, F_1^T has uniform column totals which are equal respectively to g, g^2, $g(g+1)$. This is proved for E_1 by applying Lemma 8.2 to design $\mathcal{C}$, making use of the block γ corresponding to the first column of C_1. The result for G_1 then follows by subtraction from g^2+g, the uniform column total of C_1. Let η_0 be any column of F_1^T and let η be the column of A_1 which includes η_0. It is clear from (8.30) that the sum of the 1's of η_0 is equal to the inner product of columns η and γ, where γ is the column in S_{20}. But for any $\eta \in S_{12}$, η and γ are second associates and have inner product $p_{11}^2 = g(g+1)$, proving that F_1^T has uniform column totals $g(g+1)$.

We now need, in part, the products $C_1 C_1^T$ and $C_1^T C_1$, in partitioned form, computed in two ways. By multiplication of the partitioned matrix C_1,

$$(8.31) \qquad C_1 C_1^T = \left[\begin{array}{c|c} J + E_1 E_1^T & \\ \hline & F_1F_1^T + G_1G_1^T \end{array}\right],$$

$$(8.32) \qquad C_1^T C_1 = \left[\begin{array}{c|c} F_1^T F_1 & \\ \hline & E_1^T E_1 + G_1^T G_1 \end{array}\right].$$

From (8.6) and (8.14),

$$(8.33) \qquad C_1 C_1^T = \left[\begin{array}{c|c} p_{12}^1 I + (p_{11}^2 - 1)(J-I) & \\ \hline & p_{12}^1 I + (p_{11}^2 - 1)(J-I) \end{array}\right],$$

$$(8.34) \qquad C_1^T C_1 = \left[\begin{array}{c|c} g(g+1)I + g(J-I) & \\ \hline & g(g+1)I + 0.K_1 + gK_2 \end{array}\right],$$

where $K_2 = J - I - K_1$ is a submatrix of D_2.

If (8.31) and (8.33) are solved for $E_1E_1^T$, Corollary 4.7.1 now shows that $\mathcal{E}$ is a BIB design with the parameters stated in the theorem.

If (8.32) and (8.34) are solved for $F_1^TF_1$, Corollary 4.7.1 now shows that F_1^T is the incidence matrix of a BIB design with $v = b = g^2(g+2)$, $r = k = g(g+1)$, $\lambda = g$. Since this is a symmetric design, the dual design $\mathcal{F}$ is balanced and has the same parameters.

The product $F_1F_1^T$ is now known, making it possible to solve (8.31) and (8.33) for $G_1G_1^T$, and one more application of Corollary 4.7.1 shows that $\mathcal{G}$ is a BIB design with the parameters stated in the theorem.

Submatrix K_1 in (8.29) has the same relation to D_1 as submatrix D_1 in (4.17) has to A_1, and Lemma 4.5 shows that K_1 has uniform row totals

$\tilde{p}_{12}^2 = g^2(g+1)$. There must therefore be at least this number of zero elements in each row of the matrix $g(g+1)\,I + 0.K_1 + gK_2$, and comparison of (8.32) with (8.34) shows that the same is true of $E_1^T E_1 + G_1^T G_1$. This means that any column of matrix $\left[\frac{E_1}{G_1}\right]$ has inner product zero with at least $g^2(g+1)$ other columns. In particular, this must hold for the inner product of any column of E_1 with other columns of E_1, and similarly for G_1. This proves that designs $\mathcal{E}$ and $\mathcal{G}$ have the property stated in the last sentence of the theorem. The proof is now complete.

Design $\mathcal{C}$ is trivial in the case $g = 1$, giving a fourth method of construction of the $NL_1(4)$ scheme. This construction gives an easy proof of the uniqueness of the scheme.

Design $\mathcal{C}$ is far from trivial for $g \geq 2$, as suggested by the rapidly growing parameters in Table 8.1. Theorems 8.3 and 8.5 give useful information on the structure of the design but have not yet led to proofs of existence or of nonexistence in any of the cases $g \geq 3$. In the case $g = 2$ just enough of the structure of $\mathcal{C}$ is determined that an empirical study is feasible. The author conjectured that the design did not exist in this case, undertook an empirical search in hopes of proving its nonexistence, and in the course of the search inadvertently constructed it.

Design $\mathcal{C}$ inthe case $g = 2$ has parameters

(8.35) $\quad v = 22,\ b = 77,\ r = 21,\ k = 6,\ \lambda = 5$.

Denoting objects by 1, 2, ..., 22, a solution of this design is given by the blocks in the following table.

TABLE 8.2

1	2	3	4	5	6

1	2	7	8	9	10
1	2	11	12	13	14
1	2	15	16	17	18
1	2	19	20	21	22

3	4	7	8	11	12	3	5	7	9	15	17	4	5	7	10	19	22
3	4	9	10	13	14	3	5	8	10	16	18	4	5	8	9	20	21
3	4	15	16	19	20	3	5	11	13	19	21	4	5	11	14	15	18
3	4	17	18	21	22	3	5	12	14	20	22	4	5	12	13	16	17

5	6	7	8	13	14	4	6	7	9	16	18	3	6	7	10	20	21
5	6	9	10	11	12	4	6	8	10	15	17	3	6	8	9	19	22
5	6	15	16	21	22	4	6	11	13	20	22	3	6	11	14	16	17
5	6	17	18	19	20	4	6	12	14	19	21	3	6	12	13	15	18

1	3	7	13	16	22	2	3	7	14	18	19
1	3	8	14	15	21	2	3	8	13	17	20
1	3	9	11	18	20	2	3	9	12	16	21
1	3	10	12	17	19	2	3	10	11	15	22

1	4	7	14	17	20	2	4	7	13	15	21
1	4	8	13	18	19	2	4	8	14	16	22
1	4	9	12	15	22	2	4	9	11	17	19
1	4	10	11	16	21	2	4	10	12	18	20

1	5	7	12	18	21	2	5	7	11	16	20
1	5	8	11	17	22	2	5	8	12	15	19
1	5	9	14	16	19	2	5	9	13	18	22
1	5	10	13	15	20	2	5	10	14	17	21

1	6	7	11	15	19	2	6	7	12	17	22
1	6	8	12	16	20	2	6	8	11	18	21
1	6	9	13	17	21	2	6	9	14	15	20
1	6	10	14	18	22	2	6	10	13	16	19

7	8	15	18	20	22	8	10	11	14	19	20
7	8	16	17	19	21	8	10	12	13	21	22
7	9	11	14	21	22	9	10	15	18	19	21
7	9	12	13	19	20	9	10	16	17	20	22
7	10	11	13	17	18	11	12	15	17	20	21
7	10	12	14	15	16	11	12	16	18	19	22
8	9	11	13	15	16	13	14	15	17	19	22
8	9	12	14	17	18	13	14	16	18	20	21

It is possible to verify by inspection that the foregoing design has property (8.5), that each block is disjoint from 16 other blocks.

Theorem 8.6. The $NL_2(10)$ association scheme exists.

Proof. Apply Corollary 8.4.1 to the foregoing BIB design.

The table of first associates requires 100 rows and 22 columns and will be omitted to save space.

By listing the 77 blocks of our BIB design we have been able to prove its existence without describing the empirical construction. This method of proof is reminiscent of Bhaskhara whose 1150 A.D. treatise on mathematics presented a sketch of a particularly lucid construction for the Pythagorean theorem, accompanied by the brief written proof, "Behold!" The situation is different when a claim of nonexistence or uniqueness is based on empirical search. A valid proof must show that the search was exhaustive, and this may involve a description which in written form is more tedious than the search itself. In the next theorem we have attempted to steer between tedium and nonproof by giving enough details that the interested or suspicious reader can fill in the rest.

Theorem 8.7. The $NL_2(10)$ association scheme is unique up to permutation of objects.

Proof. Let $\mathcal{G}$ be an $NL_g(n)$ association scheme, $n = g^2+3g$, with association matrix A_1. An initial object α may be chosen in n^2 different ways. For a given α, (8.1) determines sets S_1 and S_2, and (8.2) determines submatrix C_1 up to permutation of rows and independent permutation of columns. Theorem 8.1 shows that C_1 is the incidence matrix of a BIB design $\mathcal{C}$ with parameters (8.4) and property (8.5); for a particular α, $\mathcal{C}$ is determined up to permutation of objects and permutation of blocks.

Theorem 8.4 shows that any such matrix C_1 can be obtained by the same construction from some $NL_g(g^2+3g)$ association scheme $\mathcal{G}$, and that a particular C_1 determines A_1 and $\mathcal{G}$ uniquely. Two designs $\mathcal{C}$ which differ only in permutation of objects and of blocks lead to schemes $\mathcal{G}$ which differ only by a permutation of the objects of S_1 and a permutation of the objects of S_2. Then association schemes $\mathcal{G}$ which are inequivalent under permutation of objects must surely lead to designs $\mathcal{C}$ which are inequivalent under permutation of objects and blocks. The number of inequivalent schemes $\mathcal{G}$ is less than or equal to the number of inequivalent designs $\mathcal{C}$, and $\mathcal{G}$ is unique if $\mathcal{C}$ is unique.

The theorem will be proved by showing that $\mathcal{C}$ is unique in the case $g = 2$. It has parameters (8.35),

$$v = 22,\ b = 77,\ r = 21,\ k = 6,\ \lambda = 5,$$

and the property that any block is disjoint from 16 other blocks. By Lemma 8.2, each of the remaining 60 blocks intersects the given block in exactly two objects.

Without loss of generality we may assume that the initial block γ is 1 2 3 4 5 6,

$$S_{11} = \{1, 2, 3, 4, 5, 6\},$$

$$S_{12} = \{7, 8, \ldots, 22\}.$$

S_{21} is a set of $p^2_{12} = 16$ blocks, each containing six objects of S_{12}, comprising the blocks of a symmetric BIB design $\mathcal{E}$ with $\lambda = 2$. Each of the $p^2_{22} = 60$ blocks of S_{22} is the union of a block of design $\mathcal{E}$, containing two objects of S_{11}, and a block of design $\mathcal{G}$, containing four objects of S_{12}. Design $\mathcal{E}$ is uniquely determined by its parameters

$$v = 6,\ b = 60,\ r = 20,\ k = 2,\ \lambda = 4$$

to have as its blocks the 15 pairs $(i, j) = (j, i)$ of distinct objects of S_{11}, each pair repeated four times.

$\mathcal{C}(i, j)$ will denote the set of four blocks of $\mathcal{C}$ which contain a pair i, j of distinct objects of S_{11}, and $\mathcal{G}(i, j)$ will denote the set of four blocks of $\mathcal{G}$ which are contained in the blocks of $\mathcal{C}(i, j)$. Any two blocks of $\mathcal{C}$ which are not disjoint must intersect in exactly two objects, showing that

(8.36) the four blocks of $\mathcal{G}(i, j)$ are pairwise disjoint.

Thus for each (i, j), the blocks of $\mathcal{G}(i, j)$ contain all 16 of the objects of S_{12}. Also,

(8.37) a block of $\mathcal{G}(i, j)$ and $\mathcal{G}(i, k)$, $j \neq k$, must have exactly one object in common.

Thus the four objects of any block of $\mathcal{G}(i, j)$ are distributed one each over the four blocks of $\mathcal{G}'(i, k)$. Also if i, j, k, ℓ are pairwise distinct, the objects of any block of $\mathcal{G}(i, j)$ occur two each in two blocks of $\mathcal{G}(k, \ell)$.

Using the remarks of the preceding paragraph it is easy to choose notation for the objects of S_{12} and permute blocks within sets $\mathcal{C}(i, j)$ so that $\mathcal{C}(1, 2)$, $\mathcal{C}(3, 4)$, $\mathcal{C}(3, 5)$ and $\mathcal{C}(4, 5)$ are determined to the following extent

1	2	7	8	9	10
1	2	11	12	13	14
1	2	15	16	17	18
1	2	19	20	21	22

3	4	7	8	11	12
3	4	9	10	x	y
3	4	-	-	-	-
3	4	-	-	-	-
3	5	7	9	-	-
3	5	8	10	-	-
3	5	11	13	z	-
3	5	12	14	-	-
4	5	7	10	-	-
4	5	8	9	-	-
4	5	11	14	-	-
4	5	12	13	-	-

We must now decide whether to assign the pair 13, 14 in $\mathcal{C}(3, 4)$ to positions x, y or to some other block. In the latter case we may assume x, y = 15, 16, which requires z = 15 or 16; then object 15 cannot occur anywhere in $\mathcal{C}(4, 5)$ without violating (8.37). This contradiction shows that x, y = 13, 14: if one block of $\mathcal{G}(3, 4)$ is 7 8 11 12, then another block must be 9 10 13 14. This reasoning was used to assign the objects of two blocks of $\mathcal{G}(1, 2)$ to the blocks of $\mathcal{G}(3, 4)$ but it applies more generally to show that

(8.38) if i, j, k, ℓ are pairwise distinct, the objects of any block of $\mathcal{G}(i, j)$ occur two each in two blocks of $\mathcal{G}(k, \ell)$, and the remaining four objects in these two blocks occur together in another block of $\mathcal{G}(i, j)$.

Choice of notation will now give $\mathcal{C}(3, 4)$ and $\mathcal{C}(3, 5)$ the form listed in Table 8.2. After various applications of (8.36) to (8.38), $\mathcal{C}(4, 6)$ and $\mathcal{C}(5, 6)$ are determined uniquely and the other $\mathcal{C}(i, j)$ are determined in part.

$\mathcal{C}(1, 3)$ and $\mathcal{C}(2, 3)$ have the form

1	3	7	x	-	-		2	3	7	y	-	-
1	3	8	-	-	-		2	3	8	-	-	-
1	3	9	-	-	-		2	3	9	-	-	-
1	3	10	-	-	-		2	3	10	-	-	-

where x and y are equal in some order to 13 and 14. Objects 1 and 2 have played symmetrical rolesup to this point and may be exchanged if necessary so that x = 13, y = 14.

$\mathcal{C}(4, 5)$ reduces to one of these two cases.

Case I	4	5	7	10	19	22	Case II	4	5	7	10	20	21
	4	5	8	9	20	21 ,		4	5	8	9	19	22 .
	4	5	11	14	-	-		4	5	11	14	-	-
	4	5	12	13	-	-		4	5	12	13	-	-

Once either of these cases is assumed, the remaining sets of blocks $\mathcal{C}(i, j)$ are determined uniquely by arguments based on (8.36) to (8.38). This may first be established for the sixteen blocks of $\mathcal{C}(4, 5)$, $\mathcal{C}(3, 6)$, $\mathcal{C}(1, 3)$ and $\mathcal{C}(1, 4)$ by comparing them to each other and to blocks that have already been determined. The remaining $\mathcal{C}(i, j)$ are easily determined, completing the 60 blocks of S_{22}.

Case I leads to the blocks listed in Table 8.2. The 60 blocks of S_{22} in Case II are, apart from order, the blocks obtained from those of Case I by a permutation of objects which in cycle form may be expressed

$$(1\ 2)\quad (11\ 12)\quad (13\ 14)\quad (19\ 20)\quad (21\ 22)\ .$$

Therefore Case I and Case II are equivalent under permutation of objects and blocks. Case I will be assumed.

It remains to show that the 16 blocks of S_{21} are uniquely determined by the 61 blocks already fixed. The argument, which will be illustrated for one block, repeatedly uses the fact that two blocks of C intersect either in no objects or in two objects. Since these 16 blocks comprise a BIB design with objects 7, 8, ..., 22, and $\lambda = 2$, we may assume that two blocks have the form

(i) 7 8 x - - -

(ii) 7 8 - - - - .

The remaining objects in (i) and (ii) must be distinct.

Because of blocks

1 2 7 8 9 10 ,
3 4 7 8 11 12 ,
5 6 7 8 13 14 ,

(i) and (ii) cannot contain any of objects 9, 10, 11, 12, 13, 14 and must therefore contain all of objects 15, 16, ..., 22. Let x = 15. Then (i) contains the pair 7, 15 and because of blocks

3 5 7 9 15 17 ,
1 6 7 11 15 19 ,
2 4 7 13 15 21 ,

(i) cannot contain any of objects 17, 19, 21 and we have

(i) 7 8 15 - - - ,
(ii) 7 8 17 19 21 - .

Block (ii) contains the pair 7, 19 and because of block

4 5 7 10 19 22

cannot contain object 22; it contains the pair 17, 19 and because of block

5 6 17 18 19 20

cannot contain object 18 or 20. We now have

(i) 7 8 15 18 20 22

(ii) 7 8 17 19 21 - .

The remaining blocks of S_{21} may be completed by similar arguments, or by easier arguments toward the end.

This completes the proof of Theorem 8.7.

We are finished with the contributions of this section to the theory of negative Latin square association schemes, but we shall mention some by-products.

Association scheme $\mathcal{D}$ in the case g = 2 is another scheme outside the Bose-Shimomoto classification and appears to be new. It has parameters

$$(8.39) \qquad v = 77, \quad n_1 = 16, \quad n_2 = 60, \qquad P_1 = \begin{bmatrix} 0 & 15 \\ 15 & 45 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4 & 12 \\ 12 & 47 \end{bmatrix}.$$

It is readily constructed by identifying the 77 objects with the blocks of the BIB design $\mathcal{C}$, and taking two objects as first associates if and only if the corresponding blocks are disjoint.

We have already noted that the matrix A_1 has many submatrices of the form of C_1, C_1^T, D_1, as the partition (8.2) can be carried out for $v = n^2$ different choices of the initial object α. Submatrices of the form discussed in Theorem 8.5 are even more numerous, as the refinement of (8.2) to the partition (8.30) can be carried out for $n_2 = (g^2 + 2g-1)(g^2 + 3g+1)$ different choices of object γ. The number of partitions is thus equal to the number vn_2 of ordered pairs α, γ of second associates. Each submatrix E_1, F_1, F_1^T, G_1, H_1 is counted twice in this total, since the ordered pairs α, γ and γ, α lead to partitioned matrices which differ only by an interchange of F_1 and F_1^T, an interchange of G_1 and H_1, and an interchange of G_1^T and H_1^T. In the case $g = 2$, one detail of this is that A_1 contains 3850 pairs of 16 x 16 submatrices F_1 and F_1^T for a total of 7700 incidence matrices of the symmetric design with $r = 6$, $\lambda = 2$. These are all equivalent under permutation of rows and columns. We remark that Hussain [14] has shown that there are just three solutions of this design which are inequivalent, and that the type arising here is his type I.

In the case $g = 2$, the 15 sets $\mathcal{G}(i, j) = \mathcal{G}(j, i)$, $1 \leq i < j \leq 6$, have a curious interpretation. Each set is an arrangement of the 16 objects 7, 8, ..., 22 into four blocks of four objects. We assign four letters, say A, B, C, D, one each to the four blocks of each $\mathcal{G}(i, j)$. We then arrange the 16 objects in a 4 x 4 array, say

$$M = \begin{matrix} 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 \end{matrix}$$

and use M with each $\mathcal{G}(i, j)$ to define a 4 x 4 array $M(i, j)$ of the letters A, B, C, D by the rule: for each object $\theta \in \{7, 8, \ldots, 22\}$ assign the same letter to the position of M containing θ as has been assigned to the block of $\mathcal{G}(i, j)$ containing θ.

Then each letter occurs in four positions in $M(i, j)$ and it follows from (8.37) that these positions are occupied by four distinct letters in

M(i, k), $j \neq k$. This means that M(i, j) and M(i, k) are orthogonal 4 x 4 squares, and for fixed i, the five distinct squares M(i, j), $1 \leq j \leq 6$, $j \neq i$, are a complete orthogonal set of 4 x 4 squares. The simplest assignment of letters A, B, C, D to the blocks of the (i, j) in our solution of the design $\mathcal{S}$ leads to the squares M(i, j) displayed in the following table.

TABLE 8.3

i \ j	1	2	3	4	5	6
1		A A A A B B B B C C C C D D D D	A B C D C D A B B A D C D C B A	A B C D D C B A C D A B B A D C	A B C D B A D C D C B A C D A B	A B C D A B C D A B C D A B C D
2	A A A A B B B B C C C C D D D D		A B C D D C B A D C B A A B C D	A B C D C D A B A B C D C D A B	A B C D A B C D B A D C B A D C	A B C D B A D C C D A B D C B A
3	A B C D C D A B B A D C D C B A	A B C D D C B A D C B A A B C D		A A B B A A B B C C D D C C D D	A B A B C D C D A B A B C D C D	A B B A C D D C B A A B D C C D
4	A B C D D C B A C D A B B A D C	A B C D C D A B A B C D C D A B	A A B B A A B B C C D D C C D D		A B B A C D D C C D D C A B B A	A B A B C D C D B A B A D C D C
5	A B C D B A D C D C B A C D A B	A B C D A B C D B A D C B A D C	A B A B C D C D A B A B C D C D	A B B A C D D C C D D C A B B A		A A B B B B A A C C D D D D C C
6	A B C D A B C D A B C D A B C D	A B C D B A D C C D A B D C B A	A B B A C D D C B A A B D C C D	A B A B C D C D B A B A D C D C	A A B B B B A A C C D D D D C C	

Each square M(i, j) is also listed as M(j, i) in this table. With this duplication, each complete set of five pairwise orthogonal squares is simply the set of squares in one of the six rows. It will be noted that the only Latin squares are M(1, 3), M(1, 4), M(1, 5) and M(2, 6), the first three forming a complete set of three pairwise orthogonal Latin squares. For a different assignment of objects 7, 8, ..., 22 to array M, the M(i, j) will have the same orthogonality properties but need not include any Latin squares.

9. Parameters of designs. The emphasis in most of this paper is on PBIB association schemes rather than actual designs. This section, however, presents and briefly discusses tables of arithmetically possible sets of parameters for designs based on $NL_g(n)$ association schemes. Under "Remarks", the tables include information which has come to the author's attention on existence and non-existence of these designs, with references to published literature or to some results of this section. A more systematic study of detailed properties, construction methods, and nonexistence proofs for NL_g designs will be deferred to a later time. The present discussion and tables are a preliminary report and are intended to facilitate such a study, not take its place. In particular, the author has not made a recent search of the literature on particular designs and may have omitted some published results from the tables or duplicated them in the following paragraphs.

Given a set of parameters v, n_i, p^i_{jk}, σ, τ, α_i for a two-class scheme, we define a set of design parameters b, r, k, λ_i, θ_i to be arithmetically possible if the following well-known necessary conditions are satisfied.

(9.1) $$r\,v = b\,k\,,$$

$$n_1\lambda_1 + n_2\lambda_2 = r(k-1).$$

$\theta_i \geq 0$; $i = 1$ and 2;

if $\theta_i > 0$, $i = 1$ and 2, then $b \geq v$;

if $\theta_i = 0$, $i = 1$ or 2, then $b \geq v - \alpha_i$.

The special case $\lambda_1 = \lambda_2$ reduces to a balanced design and will be omitted. These conditions apply to all two-class designs but our tables are restricted to the negative Latin square family. We assume $n_1 < n_2$ (in particular omitting NL_g parameters with $n_1 = n_2$, which are already available in tables of Latin square parameters.) The tables include NL_g parameters for

all designs with $r \leq 10$, $k \leq 10$, all v,

all designs with $r \leq 15$, $k \leq 15$, $v \leq 100$,

and selected designs with $r > 15$ or $k > 15$, $v \leq 100$.

This range was determined by the desire to include a representative sample of designs while keeping the tables at a reasonable length. It will be observed that only a small proportion of the parameter sets are in the range $r \leq 10$, $k \leq 10$ which has traditionally been called practical.

Given an association scheme with parameters v, n_i, p^k_{jk}, the following are some simple methods by which designs can be generated from the association scheme itself.

In this and the two following paragraphs, i and j represent 1 and 2 in some order. A design with parameters

(9.2) $\quad v,\ b = vn_i/2,\ r = n_i,\ k = 2,\ \lambda_i = 1,\ \lambda_j = 0$

can be constructed by taking as blocks all pairs of ith associates.

A design with parameters

$$(9.3) \qquad v = b,\ r = k = n_i,\ \lambda_i = p^i_{ii},\ \lambda_j = p^j_{ii}$$

can be constructed by taking block θ as the set {ith associates of θ}.

A design with parameters

$$(9.4) \qquad v = b,\ r = k = n_i+1,\ \lambda_i = p^i_{ii} + 2,\ \lambda_j = p^j_{ii}$$

can be constructed by taking block θ as the set

$$\{\theta\} \cup \{i\text{-th associates of } \theta\}.$$

In the Remarks column of the following tables, the notation R(a, b, ..., c) for a design indicates that it can be obtained by replication from the designs for the same association scheme with serial numbers a, b, ..., c. Its blocks may be listed by merging the lists of blocks of those designs.

Each block in a particular design, regarded as a set of k objects, uniquely determines the complementary set of v-k objects. The complements of the b blocks of a design $\mathcal{A}$ comprise a second design, the complement of $\mathcal{A}$. If $\mathcal{A}$ has incidence matrix N, its complement has incidence matrix J-N and properties which follow readily from (4.7) if $\mathcal{A}$ is partially balanced. If a two-class design has parameters v, b, r, k, λ_i, the complement has the same association scheme and parameters

$$v,\ b,\ r' = b-r,\ k' = v-k,\ \lambda_i' = b-2r + \lambda_i\ .$$

Designs (9.3) and (9.4) for opposite choices of i are complements. A design exists if and only if its complement exists, and a few designs in our tables are disposed of by first proving the existence or non-existence of the complement.

If $\lambda_j = 0$ in a 2-class design, each block must be a set of objects which are pairwise i-th associates. In particular, any two objects of the block must have the remaining k-2 objects among their p_{ii}^i common i-th associates. This gives us a known [3] necessary condition for a 2-class design, where i and j represent 1 and 2 in some order.

(9.5) If $\lambda_j = 0$, then $k \leq p_{ii}^i + 2$.

A number of designs for which constructions by various methods are known to the author are listed in the tables with the remark, "Constructed, to appear." The details, which are beyond the scope of the present section, will be presented in a later paper.

TABLE 9.1

Parameters of Designs with $NL_1(4)$ Association Schemes

$v = 16,$

$n_1 = 5, \quad P_1 = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}, P_2 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}, \quad \alpha_1 = 10, \quad \sigma = 2,$

$\qquad \alpha_2 = 5, \quad \tau = 1.$

$n_2 = 10,$

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	16	5	2	40	1	0	6	2	Constructed, [12], (9.2) and Sec. 7.
2	16	5	5	16	0	2	1	9	Constructed, [12], (9.3).
3	16	5	5	16	2	1	5	1	Constructed, to appear.
4	16	5	8	10	1	3	0	8	Constructed, to appear.
5	16	6	6	16	0	3	0	12	Impossible, (9.5).
6	16	9	6	24	1	4	2	14	
7	16	10	2	80	0	1	8	12	Constructed, [12], (9.2) and Sec. 7.
8	16	10	2	80	2	0	12	4	Constructed, R(1, 1).
9	16	10	4	40	0	3	4	16	Constructed, Sec. 7.
10	16	10	4	40	4	1	12	0	Constructed, Sec. 7.
11	16	10	5	32	0	4	2	18	Constructed, R(2, 2).
12	16	10	5	32	2	3	6	10	Constructed, R(2, 3).
13	16	10	5	32	4	2	10	2	Constructed, R(3, 3).
14	16	10	8	20	2	6	0	16	Constructed, R(4, 4).
15	16	10	8	20	4	5	4	8	
16	16	10	8	20	6	4	8	0	Constructed, to appear.
17	16	10	10	16	4	7	0	12	Impossible, complement of No. 5.
18	16	11	11	16	6	8	1	9	Constructed, (9.4).
19	16	11	11	16	8	7	5	1	Constructed, complement of No. 3.
20	16	12	6	32	0	6	0	24	Impossible, (9.5).
21	16	12	6	32	2	5	4	16	
22	16	12	6	32	6	3	12	0	

TABLE 9.1 (Continued)

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
23	16	15	2	120	3	0	18	6	Constructed, R(1, 1, 1).
24	16	15	3	80	0	3	9	21	Constructed, to appear.
25	16	15	3	80	4	1	17	5	
26	16	15	4	60	1	4	8	20	
27	16	15	4	60	5	2	16	4	
28	16	15	5	48	0	6	3	27	Constructed, R(2, 2, 2).
29	16	15	5	48	2	5	7	19	Constructed, R(2, 2, 3).
30	16	15	5	48	6	3	15	3	Constructed, R(3, 3, 3).
31	16	15	6	40	1	7	2	26	R(5, 6).
32	16	15	6	40	3	6	6	18	Constructed, to appear.
33	16	15	6	40	7	4	14	2	
34	16	15	8	30	3	9	0	24	Constructed, R(4, 4, 4).
35	16	15	8	30	5	8	4	16	R(4, 15).
36	16	15	10	24	7	10	2	14	Complement of No. 6.
37	16	15	12	20	9	12	0	12	Impossible, complement has $\lambda_1 = -1$.

TABLE 9.2

Parameters of Designs with $NL_2(6)$ Association Schemes

$v = 36,$

$n_1 = 14, \quad P_1 = \begin{bmatrix} 4 & 9 \\ 9 & 12 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6 & 8 \\ 8 & 12 \end{bmatrix}, \quad \alpha_1 = 21, \quad \sigma = 3,$ $\alpha_2 = 14, \quad \tau = 2 .$

$n_2 = 21,$

(This scheme is unknown.)

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	36	2	8	9	1	0	7	0	Impossible, (9.5).
2	36	7	3	84	1	0	12	5	
3	36	7	4	63	0	1	10	8	
4	36	7	7	36	3	0	22	1	Impossible, (9.5).
5	36	7	9	28	1	2	0	7	
6	36	7	12	21	4	1	21	0	
7	36	8	8	36	1	2	1	8	
8	36	12	8	54	3	2	15	8	
9	36	14	2	252	1	0	19	12	
10	36	14	3	168	2	0	24	10	
11	36	14	4	126	0	2	20	16	
12	36	14	4	126	3	0	29	8	
13	36	14	6	84	5	0	39	4	
14	36	14	7	72	3	2	17	10	
15	36	14	7	72	6	0	44	2	Impossible, (9.5).
16	36	14	8	63	4	2	22	8	
17	36	14	9	56	5	2	27	6	
18	36	14	12	42	5	4	15	8	
19	36	14	14	36	7	4	25	4	
20	36	15	15	36	9	4	36	1	

TABLE 9.3

Parameters of Designs with $NL_2(7)$ Association Schemes

$$v = 49, \quad n_1 = 16, \quad n_2 = 32, \quad P_1 = \begin{bmatrix} 3 & 12 \\ 12 & 20 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6 & 10 \\ 10 & 21 \end{bmatrix}, \quad \alpha_1 = 32, \quad \alpha_2 = 16, \quad \sigma = 4, \quad \tau = 2.$$

(This scheme is unknown.)

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
									(No sets of parameters in the range $r \leq 15$, $k \leq 15$)
1	49	16	2	392	1	0	18	11	(9.2).
2	49	16	4	196	3	0	22	1	
3	49	16	7	112	0	3	7	28	
4	49	16	7	112	4	1	21	0	
5	49	16	8	98	1	3	9	23	
6	49	16	8	98	3	2	16	9	
7	49	16	14	56	1	6	0	35	
8	49	16	14	56	3	5	7	21	
9	49	16	14	56	5	4	14	7	
10	49	16	16	49	3	6	4	25	(9.3).
11	49	17	17	49	3	7	2	30	
12	49	17	17	49	5	6	9	16	(9.4).
13	49	17	17	49	7	5	16	2	

TABLE 9.4

Parameters of Designs with $NL_2(8)$ Association Schemes

$$v = 64,\quad n_1 = 18,\quad n_2 = 45,\qquad P_1 = \begin{bmatrix} 2 & 15 \\ 15 & 30 \end{bmatrix},\quad P_2 = \begin{bmatrix} 6 & 12 \\ 12 & 32 \end{bmatrix},\quad \alpha_1 = 45,\ \sigma = 5,\ \alpha_2 = 18,\ \tau = 2.$$

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	64	6	4	96	1	0	8	0	Constructed, [19].
2	64	6	16	24	0	2	0	16	Constructed, to appear.
3	64	9	3	192	1	0	11	3	
4	64	9	6	96	0	1	6	14	
5	64	10	10	64	0	2	4	2	
6	64	12	4	192	2	0	16	0	Constructed, R(1, 1).
7	64	15	4	240	0	1	12	20	Constructed, to appear.
8	64	15	10	96	0	3	6	3	
9	64	15	16	60	5	3	16	0	Constructed, to appear.

TABLE 9.5

Parameters of Designs with $NL_3(8)$ Association Schemes

$$v = 64,\quad n_1 = 27,\quad n_2 = 36,\qquad P_1 = \begin{bmatrix} 10 & 16 \\ 16 & 20 \end{bmatrix},\quad P_2 = \begin{bmatrix} 12 & 15 \\ 15 & 20 \end{bmatrix},\quad \alpha_1 = 36,\ \sigma = 4,\ \alpha_2 = 27,\ \tau = 3.$$

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	64	9	4	144	1	0	12	4	Constructed, Sec. 7.
2	64	9	9	64	0	2	1	17	
3	64	9	16	36	1	3	0	16	Constructed, Sec. 7.
4	64	10	10	64	2	1	12	4	
5	64	12	4	192	0	1	8	16	Constructed, Sec. 7.
6	64	12	16	48	4	2	16	0	Constructed, Sec. 7.
7	64	15	10	96	1	3	6	22	

TABLE 9.6

Parameters of Designs with $NL_2(9)$ Association Schemes

$$v = 81, \quad n_1 = 20, \quad n_2 = 60, \quad P_1 = \begin{bmatrix} 1 & 18 \\ 18 & 42 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 6 & 14 \\ 14 & 45 \end{bmatrix}, \quad \alpha_1 = 60, \ \sigma = 6, \quad \alpha_2 = 20, \ \tau = 2.$$

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	81	10	3	270	1	0	12	3	Constructed, Sec. 7.
2	81	12	6	162	0	1	9	18	
3	81	15	5	243	0	1	12	21	
4	81	15	9	135	0	2	9	27	
5	81	15	9	135	3	1	18	0	
6	81	16	16	81	0	4	4	40	
7	81	20	2	810	1	0	22	13	Constructed, (9.2).
8	81	20	3	540	2	0	24	6	Constructed, R(1, 1).
9	81	20	4	405	0	1	17	26	
10	81	20	6	270	2	1	21	12	
11	81	20	10	162	0	3	11	38	
12	81	20	10	162	3	2	20	11	

TABLE 9.7

Parameters of Designs with $NL_3(9)$ Association Schemes

$$v = 81,\quad n_1 = 30,\quad n_2 = 50,\quad P_1 = \begin{bmatrix} 9 & 20 \\ 20 & 30 \end{bmatrix},\quad P_2 = \begin{bmatrix} 12 & 18 \\ 18 & 31 \end{bmatrix},\quad \alpha_1 = 50,\ \sigma = 5,\quad \alpha_2 = 30,\ \tau = 3.$$

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	81	6	6	81	1	0	9	0	
2	81	10	6	135	0	1	6	15	
3	81	11	11	81	2	1	13	4	
4	81	12	6	162	2	0	18	0	
5	81	15	3	405	1	0	18	9	
6	81	15	5	243	2	0	21	3	
7	81	15	15	81	2	3	9	18	
8	81	18	6	243	3	0	27	0	
9	81	20	4	405	2	0	26	8	
10	81	20	6	270	0	2	12	30	
11	81	20	10	162	1	3	11	29	

TABLE 9.8

Parameters of Designs with $NL_2(10)$ Association Schemes

$$v = 100,\quad n_1 = 22,\quad n_2 = 77,\quad P_1 = \begin{bmatrix} 0 & 21 \\ 21 & 56 \end{bmatrix},\quad P_2 = \begin{bmatrix} 6 & 16 \\ 16 & 60 \end{bmatrix},\quad \alpha_1 = 77,\ \sigma = 7,\quad \alpha_2 = 22,\ \tau = 2.$$

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
									No sets of parameters in the range $r \leq 20$, $k \leq 20$
1	100	21	12	175	0	3	12	42	
2	100	22	2	1100	1	0	24	14	Constructed, (9.2).
3	100	22	8	275	0	2	16	36	
4	100	22	11	200	3	2	22	12	
5	100	22	20	110	5	4	20	10	
6	100	22	22	100	0	6	4	64	Constructed, (9.3).
7	100	23	23	100	2	6	9	49	Constructed, (9.4).

TABLE 9.9

Parameters of Designs with $NL_3(10)$ Association Schemes

$$v = 100,\quad n_1 = 33,\quad n_2 = 66,\quad P_1 = \begin{bmatrix} 8 & 24 \\ 24 & 42 \end{bmatrix},\quad P_2 = \begin{bmatrix} 12 & 21 \\ 21 & 44 \end{bmatrix},\quad \alpha_1 = 66,\ \sigma = 6,\quad \alpha_2 = 33,\ \tau = 3.$$

(This scheme is unknown.)

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	100	11	4	275	1	0	14	4	
2	100	12	12	100	0	2	4	24	
3	100	12	12	100	2	1	14	4	
4	100	15	12	125	1	2	10	20	
5	100	15	12	125	3	1	20	0	
6	100	18	45	40	6	9	0	30	
7	100	21	12	175	1	3	12	32	
8	100	21	12	175	3	2	22	12	
9	100	22	4	550	0	1	18	28	
10	100	22	10	220	0	3	10	40	
11	100	22	22	100	2	6	4	44	
12	100	22	22	100	4	5	14	24	
13	100	22	22	100	6	4	24	4	

TABLE 9.10

Parameters of Designs with $NL_4(10)$ Association Schemes

$$v = 100,\quad n_1 = 44,\quad n_2 = 55,\quad P_1 = \begin{bmatrix} 18 & 25 \\ 25 & 30 \end{bmatrix},\quad P_2 = \begin{bmatrix} 20 & 24 \\ 24 & 30 \end{bmatrix},\quad \alpha_1 = 55,\ \sigma = 5,\quad \alpha_2 = 44,\ \tau = 4.$$

(This scheme is unknown.)

No.	v	r	k	b	λ_1	λ_2	θ_1	θ_2	Remarks
1	100	11	5	220	1	0	15	5	
2	100	11	11	100	0	2	1	21	
3	100	11	20	55	1	3	0	20	
4	100	15	12	125	0	3	0	30	
5	100	21	12	175	4	1	32	2	
6	100	22	8	275	1	2	16	26	
7	100	22	22	100	3	6	4	34	
8	100	22	25	88	7	4	30	0	
9	100	22	40	55	7	10	0	30	
10	100	23	23	100	4	6	9	29	

The following table lists all arithmetically possible parameter sets of NL_g type with $v > 100$ in the range $r \leq 10$, $k \leq 10$.

TABLE 9.11

Other NL_g Parameters in the Range $r \leq 10$, $k \leq 10$

(These schemes and designs are unknown.)

Scheme	v	n_1	n_2	p^1_{11}	p^2_{11}	r	k	b	λ_1	λ_2	θ_1	θ_2
$NL_4(11)$	121	48	72	17	20	9	9	121	0	1	4	15
$NL_4(14)$	196	60	135	14	20	10	7	280	1	0	14	0
$NL_6(14)$	196	90	105	40	42	10	10	196	1	0	16	2
$NL_5(15)$	225	80	144	25	30	10	9	250	1	0	15	0

10. Generalized L_g and NL_g designs with m associate classes. The Latin square family of two-class association schemes and designs can be generalized in a natural way to a larger number of associate classes. The three-class case has been discussed [22] by Singh and Shukla, who were aware of the full generalization. In this section we describe the family of m-class Latin square association schemes, then define an m-class negative Latin square scheme.

If there exists a complete set of n-1 pairwise orthogonal Latin squares of order n, we may obtain a set of n + 1 pairwise orthogonal squares (not all Latin) by adjoining a square in which the i-th letter occupies all positions in the i-th row and a square in which the i-th letter occupies all positions in the i-th column.

To define an m-class association scheme, $m \leq n+1$, we arrange the n+1 orthogonal squares into m disjoint subsets, where, denoting by g_i the number of squares in the i-th set,

$$(10.1) \qquad g_1 + \dots + g_m = n+1 .$$

We arrange the n^2 objects in an n x n array and take two objects as i-th associates if and only if their positions in the array are occupied by the same letter in an orthogonal square of the i-th subset. It can be shown that this association relation is a partially balanced m-class scheme with the following parameters.

$$(10.2) \qquad v = n^2,$$

$$n_i = g_i(n-1),$$

$$p^i_{ii} = (g_i-1)(g_i-2) + n-2,$$

$$p^i_{ij} = p^i_{ji} = g_j(g_i-1),$$

$$p^i_{jj} = g_j(g_j-1),$$

$$p^i_{jk} = g_j g_k,$$

i, j, k distinct, $1 \leq i, j, k \leq m$.

The above definition is more restrictive than necessary. Denoting

$$g = g_1 + g_2 + \dots + g_{m-1} = n + 1-g_m,$$

we may still construct the m-class Latin square scheme if a set of g pair-wise orthogonal squares exists (equivalently, g-2 such Latin squares). Associate classes 1, 2, ..., m-1 are defined as before and objects are taken as m-th associates if they are not associates of any other class. Expresssions (10.2) apply. It may be conjectured that association schemes with these parameters exist in still more cases, though it may be preferable to treat them as a generalized pseudo-Latin square family in any cases where the orthogonal squares are not actually used.

It is now completely straightforward to define a generalized negative Latin square family of association schemes by using negative integers n, $g_1, \dots, g_m$ in expressions (10.1) and (10.2). In terms of positive parameters n^*, g_i^*, we take $n = -n^*$, $g_i = -g_i^*$, and substitute in (10.1) and (10.2). Dropping the stars, we have

(10.3) $\quad g_1 + \dots + g_m = n - 1,$

(10.4) $\quad v = n^2,$

$$n_i = g_i(n+1),$$

$$p^i_{ii} = (g_i+1)(g_i+2) - n - 2,$$

$$p^i_{ij} = p^i_{ji} = g_j(g_i+1),$$

$$p^i_{jj} = g_j(g_j+1),$$

$$p^i_{jk} = g_j g_k,$$

$$i,\ j,\ k \text{ distinct},\ 1 < i,\ j,\ k \leq m.$$

These parameters are integers satisfying conditions (1.4) and (1.5) and all except possibly p^i_{ii} are non-negative. The requirement

$$p^i_{ii} \geq 0$$

places a lower bound on g_i for a given n, i = 1, ..., m, and (10.3) then places an upper bound on the number m of associate classes for a given n.

11. Acknowledgements. Portions of this work appeared in 1956 in the author's Ph.D. thesis at Michigan State University [16]. Section 9 includes portions of unpublished tables which were compiled by the author in 1961 in an investigation for which computer facilities were provided by Purdue University.

REFERENCES

[1] Bose, R. C. (1959). "On the application of finite projective geometry for deriving a certain series of balanced Kirkman arrangements." The Golden Jubilee Commemoration Volume. (1958-1959), Calcutta Mathematical Society, 341-354.

[2] Bose, R. C. (1963). Mimeographed notes on combinatorial mathematics, Department of Statistics, University of North Carolina, Chapel Hill, N. C.

[3] Bose, R. C. and W. H. Clatworthy, (1955). "Some classes of partially balanced designs." Ann. Math. Statist. 26 212-232.

[4] Bose, R. C. and D. M. Mesner, (1959). "On linear associative algebras corresponding to association schemes of partially balanced designs." Ann. Math. Statist. 30 21-38.

[5] Bose, R. C. and K. R. Nair, (1939). "Partially balanced incomplete block designs." Sankhya 4 337-372.

[6] Bose, R. C. and T. Shimamoto, (1951) "Classification and analysis of partially balanced incomplete block designs with two associate classes." J. Amer. Statist. Assoc. 47 151-184.

[7] Bruck, R. H. (1951). "Finite nets, I. Numerical invariants." Can. J. Math. 3 94-107.

[8] Bruck, R. H. (1956). "Computational aspects of certain combinatorial problems." Proc. Symp. in Appl. Math. 6 31-43.

[9] Bruck, R. H. (1963). "Finite nets, II. Uniqueness and imbedding." Pac. J. Math. 13 421-457.

[10] Bruck, R. H. and R. C. Bose, (1964). "The construction of translation planes from projective spaces." Jour. of Algebra 1; also University of North Carolina Inst. of Statist. Mimeo Series No. 378 (1963).

[11] Bush, K. A. (1952). "Orthogonal arrays of index unity." Ann. Math. Statist. 23 426-434.

[12] Clatworthy, W. H. (1955). "Partially balanced incomplete block designs with two associate classes and two treatments per block," J. Res. Natl. Bur. Stds. 54 177-190.

[13] Connor, W. S. and W. H. Clatworthy, (1954). "Some theorems for partially balanced designs." Ann. Math. Statist. 25 100-112.

[14] Hussain, Q. M. (1945). "On the totality of the solutions for the symmetrical incomplete block designs $\lambda = 2$, $k = 5$ or 6." Sankhya 7 204-206.

[15] Hussain, Q. M. (1948). "Structure of some incomplete block designs." Sankhya 8 381-383.

[16] Mesner, D. M. (1956). "An investigation of certain combinatorial properties of partially balanced incomplete block experimental designs and association schemes, with a detailed study of designs of Latin square and related types." Unpublished doctoral thesis, Michigan State University.

[17] Mesner, D. M. (1963). "A note on the parameters of PBIB association schemes." Univ. of N. Carolina Institute of Statistics Mimeograph Series No. 375; also Ann. Math. Statist. 36 331-336.

[18] Nair, K. R. and C. R. Rao, (1942). "A note on partially balanced incomplete block designs." Science and Culture 7 568-569.

[19] Ray-Chaudhuri, D. K. (1959). "On the application of the geometry of quadrics to the construction of partially balanced incomplete block designs and error correcting binary codes." Univ. of N. Carolina Institute of Statistics Mimeograph Series No. 230.

[20] Segre, B. (1955). "Ovals in a finite projective plane." Can. J. Math. 7 414-416.

[21] Shrikhande, S. S. (1959). "The uniqueness of the L_2 association scheme." Ann. Math. Statist. 30 781-798.

[22] Singh, N. K. and G. C. Shukla, (1963). "The non-existence of some partially balanced incomplete block designs with three associate classes." J. Ind. Statist. Assoc. 1 71-77.

[23] Sprott, D. A. (1955). "Some series of partially balanced incomplete block designs." Can. J. Math. 7 369-381.

[24] Thompson, W. A., Jr. (1958). "A note on PBIB design matrices." Ann. Math. Statist. 29 919-922.

[25] Ray-Chaudhuri, D. K. (1962). "Some results on quadrics in finite projective geometry based on Galois fields." Can. J. Math. 14 129-138.