The 4-vertex condition—extended abstract

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Recently, A. E. Brouwer, F. Ihringer & W. M. Kantor wrote a note called '*Strongly regular graphs satisfying the 4-vertex condition*'. Below we give the main results.

A graph Γ satisfies the *t-vertex condition* when for all triples (T, x_0, y_0) of a t-vertex graph T with two distinct distinguished vertices x_0, y_0, and all pairs of distinct vertices x, y of Γ, the number $n(x, y)$ of isomorphic copies of T in Γ, where the isomorphism maps x_0 to x and y_0 to y, does not depend on the choice of the pair x, y but only on whether they are adjacent or not.

This definition is due to Sims and Hestenes-Higman. Work was done by Ivanov, Klin, Reichard, Pech and others.

A graph satisfies the 3-vertex condition if and only if it is strongly regular (or complete or edgeless); it satisfies the v-vertex condition if and only if it has a rank 3 automorphism group. If a graph of order v satisfies the t-vertex condition for $t > 3$ then it also satisfies the $(t-1)$-vertex condition. Thus, we get a hierarchy of conditions of increasing strength between strongly regular and rank 3.

For $t = 4$, Sims gave a characterization and introduced the parameters α, β.

Proposition 0.1 (Sims) *A strongly regular graph Γ with parameters (v, k, λ, μ) satisfies the 4-vertex condition, with parameters (α, β), if and only if the number of edges in $\Gamma(x) \cap \Gamma(y)$ is α (resp. β) whenever the vertices x, y are adjacent (resp. nonadjacent). In this case, $k(\binom{\lambda}{2} - \alpha) = \beta(v - k - 1)$.*

The equality here follows by counting 4-cliques minus an edge.

Rank 4 examples

Below a small table with parameters of some edge-regular rank 4 graphs satisfying the 4-vertex condition.

v	k	λ	μ	α	β	group
144	55	22	20	87	90	$M_{12}.2$
280	36	8	4	1	4	$HJ.2$
300	104	28	40	78	160	$PGO_5(5)$
325	144	68	60	1153	900	$PGO_5(5)$
512	441	380	378	9780	9639	$2^9.\Gamma L_3(8)$
512	196	60	84	420	840	$2^9.\Gamma L_3(8)$
729	112	1	20	0	0	$3^6.2.L_3(4).2$
1120	729	468	486	69498	74358	$PSp_6(3).2$
1849	462	131	110	2980	1845	$43^2{:}(42 \times D_{22})$

Cyclotomic examples

Given (q, e, J), where $e \mid (q-1)/2$, and a fixed primitive element η of $\mathbb{F}_q$, consider the cyclotomic graph with vertex set $\mathbb{F}_q$, where two elements are adjacent when their difference is in $D = \{\eta^{ie+j} \mid 0 \le i < (q-1)/e, \ \ j \in J\}$. In some cases this yields a strongly regular graph that satisfies the 4-vertex condition. We give a few examples.

q	p^f	e	J	η	α	β	rk
1849	43^2	4	$\{0\}$	any	2980	1845	4
146689	383^2	4	$\{0\}$	any	11353825	10662960	4
121	11^2	6	$\{0,1,2\}$	any	200	206	5
625	5^4	6	$\{0,1,2\}$	any	5913	6022	5
5041	71^2	6	$\{0,1,2\}$	any	395641	396270	5
529	23^2	8	$\{0,1,2,3\}$	$\eta^2 = \eta + 4$	4215	4300	5

Disjoint t.i. planes in symplectic 6-space

Let V be a 6-dimensional vector space over $\mathbb{F}_q$ with q odd, provided with a nondegenerate symplectic form. Let Γ be the graph with as vertices the t.i. planes, adjacent when disjoint. Then Γ is strongly regular, with parameters $v = (q^3+1)(q^2+1)(q+1)$, $k = q^6$, $\lambda = q^2(q^3-1)(q-1)$, $\mu = (q-1)q^5$.

Its local graph Δ is the complement of the quadratic forms graph on $\mathbb{F}_q^3$ and is strongly regular with parameters $v' = q^6$, $k' = q^2(q^3-1)(q-1)$, $\lambda' = \mu' - q^2(q-2)$, $\mu' = q^2(q-1)(q^3-q^2-1)$. It follows that Γ satisfies the 4-vertex condition.

Nonsingular points joined by a tangent

Let V be a vector space of dimension $2m+1$ over $\mathbb{F}_q$ with q odd, and let Q be a nondegenerate quadratic form on V.

The graph $NO^{\varepsilon}_{2m+1}(q)$ that has as vertex set the set of nonsingular points of type ε, where two points are adjacent when the joining line is a tangent, is strongly regular with parameters $v = \frac{1}{2}q^m(q^m+\varepsilon)$, $k = (q^{m-1}+\varepsilon)(q^m-\varepsilon)$, $\lambda = 2(q^{2m-2}-1) + \varepsilon q^{m-1}(q-1)$, $\mu = 2q^{m-1}(q^{m-1}+\varepsilon)$.

These graphs satisfy the 4-vertex condition.

A prolific construction

Consider a nondegenerate polar space of rank d with maximal t.i. subspace M. For $x \notin M$, the set $x^\perp \cap M$ is a hyperplane in M, and the collection of such hyperplanes forms a 2-$(\frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}, \frac{q^{d-2}-1}{q-1})$ design on M. Let $(M, \mathcal{D})$ be an arbitrary 2-design on M with the same parameters, and let φ be an arbitrary bijection between the collection of hyperplanes and the set $\mathcal{D}$.

Let Γ be the collinearity graph of the polar space. Construct a graph Γ_φ by letting points x, y be adjacent in Γ_φ if either (i) both or neither is in M and x, y are adjacent in Γ, or (ii) one is in M, say x, and $x \in (y^\perp \cap M)^\varphi$.

(This construction is due to Kantor.)

Now Γ_φ is strongly regular with the same parameters as Γ. In the symplectic case it also satisfies the 4-vertex condition.