Association schemes.

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1. Introduction.

Association schemes are perhaps the most important unifying concept in algebraic combinatorics. They provide a common view point for the treatment of problems in several fields, such as coding theory, design theory, algebraic graph theory and finite group theory.

Roughly, an association scheme is a very regular partition of the edge-set of a complete graph. Each of the partition classes defines a graph, and the adjacency matrices of these graphs (together with the identity matrix) form the basis of an algebra (known as the Bose-Mesner algebra of the scheme). But since this basis consists of 0-1 matrices, we see that the algebra is not only closed under matrix multiplication, but also under componentwise (Hadamard, Schur) multiplication. The interplay between these two algebra structures on the Bose-Mesner algebra yields strong information on the structure and parameters of an association scheme.

The relation of the theory of association schemes to the various fields varies. Finite group theory mainly serves as a source of examples - any generously transitive permutation representation of a group yields an association scheme -, but also as a source of inspiration - many group-theoretic concepts and results can be mimicked in the (more general) setting of association schemes. On the other hand, in combinatorics association schemes form a tool. Using inequalities for the parameters of an association scheme, one finds upper bounds for the size of error-correcting codes, and lower bounds for the number of blocks of a t-design. Many uniqueness proofs for combinatorial structures depend crucially on the additional information one gets in case such inequalities hold with equality.

The graphs of the partition classes of an association scheme are very special. For instance distance-regular graphs and Kneser graphs are of this type. For these graphs the spectrum of the adjacency matrix can be computed from the parameters of the association scheme. Thus results about the eigenvalues of graphs (cf. Godsil's chapter) can be applied.

An association scheme with just two classes is the same as a pair of complementary strongly regular graphs. Because strongly regular graphs are themselves important combinatorial objects and because their treatment forms a good introduction to the more general theory, we shall start with a section on strongly regular graphs. In the subsequent section we treat the elements of the general theory of association schemes: the adjacency matrices, the Bose-Mesner algebra, the eigenvalues, the Krein parameters, the absolute bound and Delsarte's inequality on subsets of an association scheme. The last section is devoted to some special association schemes and the significance to other fields of combinatorics, as indicated above.

Association schemes were introduced by BOSE & SHIMAMOTO (1952) and BOSE & MESNER (1959) defined the algebra that bears their name. DELSARTE (1973) found the linear programming bound, studied P- and Q-polynomial schemes, and applied the resulting theory to codes and designs. HIGMAN (1974) introduced the more general concept of coherent configuration to study permutation representations of finite groups. There is much literature on association schemes and related topics. BANNAI & ITO (1984) is the first text book on association schemes. On the more special topic of distance-regular graphs a lot of material can be found in BIGGS (1974) and in BROUWER, COHEN & NEUMAIER (1989). Some introductory papers on association schemes are DELSARTE (1974), GOETHALS (1979), chapter 21 of MACWILLIAMS & SLOANE (1977), HAEMERS (1979), chapter 17 of CAMERON & VAN LINT (1980) and SEIDEL (1983).

2. Strongly regular graphs.

A simple graph of order v is *strongly regular* with parameters v, k, λ, μ whenever it is not complete or edgeless and

(i) each vertex is adjacent to k vertices,

(ii) for each pair of adjacent vertices there are λ vertices adjacent to both,

(iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

For example, the pentagon is strongly regular with parameters $(v,k,\lambda,\mu) = (5,2,0,1)$. One easily verifies that a graph G is strongly regular with parameters (v,k,λ,μ) if and only if its complement $\overline{G}$ is strongly regular with parameters $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$. The line graph of the complete graph of order m, known as the *triangular graph* $T(m)$, is strongly regular with parameters $(\frac{1}{2}m(m-1), 2(m-2), m-2, 4)$. The complement of $T(5)$ has parameters $(10,3,0,1)$. This is the Petersen graph.

A graph G satisfying condition (i) is called k-regular. It is well-known and easily seen that the adjacency matrix of a k-regular graph has an eigenvalue k with eigenvector $\mathbf{1}$ (the all-one vector), and that every other eigenvalue ρ satisfies $|\rho| \le k$ (see Godsil's chapter or Biggs (1974)). For convenience we call an eigenvalue *restricted* if it has an eigenvector perpendicular to $\mathbf{1}$. We let I and J denote the identity and all-one matrices, respectively.

2.1. Theorem. *For a simple graph G of order v, not complete or empty, with adjacency matrix A, the following are equivalent:*

(i) *G is strongly regular with parameters (v,k,λ,μ) for certain integers k, λ, μ,*

(ii) *$A^2 = (\lambda-\mu)A + (k-\mu)I + \mu J$ for certain reals k, λ, μ,*

(iii) *A has precisely two distinct restricted eigenvalues.*

Proof. The equation in (ii) can be rewritten as

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Now (i) $\Leftrightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii): Let ρ be a restricted eigenvalue, and u a corresponding eigenvector perpendicular to $\mathbf{1}$. Then $Ju = 0$. Multiplying the equation in (ii) on the right by u yields $\rho^2 = (\lambda-\mu)\rho + (k-\mu)$. This quadratic equation in ρ has two distinct solutions. (Indeed, $(\lambda-\mu)^2 = 4(\mu-k)$ is impossible since $\mu \le k$ and $\lambda \le k-1$.)
(iii) $\Rightarrow$ (ii): Let r and s be the restricted eigenvalues. Then $(A-rI)(A-sI) = \alpha J$ for some real number α. So A^2 is a linear combination of A, I and J. ■

As an application, we show that quasisymmetric block designs give rise to strongly regular graphs. A quasisymmetric design is a 2-(v,k,λ) design such that any two blocks meet in either x or y points, for certain fixed x, y. (Cf. the chapter on block designs.) Given this situation, we may define a graph G on the set of blocks, and call two blocks adjacent when they meet in x points. Then there exist coefficients $\alpha_1, \cdots, \alpha_7$ such that $NN^\top = \alpha_1 I + \alpha_2 J$, $NJ = \alpha_3 J$, $JN = \alpha_4 J$, $A = \alpha_5 N^\top N + \alpha_6 I + \alpha_7 J$, where A is the adjacency matrix of the graph G. (The α_i can be readily expressed in terms of v, k, λ, x, y.) Then G is strongly regular by (ii) of the previous theorem. (Indeed, from the equations just given it follows straightforwardly that A^2 can be expressed as a linear combination of A, I and J.) A large class of quasisymmetric block designs is provided by the 2-(v,k,λ) designs with $\lambda = 1$ (also known as Steiner systems $S(2,k,v)$) - such designs have only two intersection numbers since no two blocks can meet in more than one point. This leads to a substantial family of strongly regular graphs, including the triangular graphs $T(m)$ (derived from the trivial design consisting of all pairs out of an m-set).

Another connection between strongly regular graphs and designs is found as follows: Let A be the adjacency matrix of a strongly regular graph with parameters (v,k,λ,λ) (i.e., with $\lambda = \mu$; such a graph is sometimes called a (v,k,λ) graph). Then, by (2.1.ii)

$$AA^\top = A^2 = (k-\lambda)I + \lambda J,$$

which reflects that A is the incidence matrix of a square ('symmetric') 2-(v,k,λ) design. (And in this way one obtains precisely all square 2-designs possessing a polarity without absolute points.) For instance, the triangular graph $T(6)$ provides a square 2-(15,8,4) design, the complementary design of the design of points and planes in the projective space $PG(3,2)$. Similarly, if A is the adjacency matrix of a strongly regular graph with parameters $(v,k,\lambda,\lambda+2)$, then $A+I$ is the incidence matrix of a square 2-(v,k,λ) design (and in this way one obtains precisely all square 2-designs possessing a polarity with all points absolute).

2.2. Theorem. *Let G be a strongly regular graph with adjacency matrix A and parameters (v,k,λ,μ). Let r and s $(r > s)$ be the restricted eigenvalues of A and let f, g be their respective multiplicities. Then*

(i) $k(k-1-\lambda) = \mu(v-k-1)$,

(ii) $rs = \mu - k, \quad r+s = \lambda - \mu$,

(iii) $f,\ g = \frac{1}{2}\left(v-1 \mp \dfrac{(r+s)(v-1)+2k}{r-s}\right)$.

(iv) *r and s are integers, except perhaps when $f = g$, $(v,k,\lambda,\mu) = (4t+1, 2t, t-1, t)$ for some integer t.*

Proof. (i) Fix a vertex x of G. Let $\Gamma(x)$ and $\Delta(x)$ be the sets of vertices adjacent and non-adjacent to x, respectively. Counting in two ways the number of edges between $\Gamma(x)$ and $\Delta(x)$ yields (i). The equations (ii) are direct consequences of (2.1.ii), as we saw in the proof. Formula (iii) follows from $f+g = v-1$ and $0 = \mathrm{tr}\,A = k + fr + gs = k + \frac{1}{2}(r+s)(f+g) + \frac{1}{2}(r-s)(f-g)$. Finally, when $f \neq g$ then one can solve for r and s in (iii) (using (ii)) and find that r and s are rational, and hence integral. But $f = g$ implies $(\mu-\lambda)(v-1) = 2k$, which is possible only for $\mu - \lambda = 1$, $v = 2k+1$. ■

These relations imply restrictions for the possible values of the parameters. Clearly, the right hand sides of (iii) must be positive integers. These are the so-called *rationality conditions*. As an example of the application of the rationality conditions we can derive the following result due to HOFFMAN & SINGLETON (1960).

2.3. Theorem. *Suppose $(v,k,0,1)$ is the parameter set of a strongly regular graph. Then $(v,k) = (5,2)$, $(10,3)$, $(50,7)$ or $(3250,57)$.*

Proof. The rationality conditions imply that either $f = g$, which leads to $(v,k) = (5,2)$, or $r-s$ is an integer dividing $(r+s)(v-1)+2k$. By use of (2.2.i-ii) we have

$$s = -r-1, \quad k = r^2+r+1, \quad v = r^4+2r^3+3r^2+2r+2,$$

and thus we obtain $r = 1$, 2 or 7. ■

The first three possibilities are uniquely realized by the pentagon, the Petersen graph and the Hoffman-Singleton graph. For the last case existence is unknown (but see ASCHBACHER (1971)).

Conference matrices

Some important classes of strongly regular graphs are related to so-called conference matrices (the name stems from their application to networks for conference telephony). Since these matrices are not treated elsewhere in this Handbook, we shall elaborate a bit on them here. (See also WALLIS, STREET & WALLIS (1972).)

A *conference matrix* of order n is an $n \times n$ matrix C with zero diagonal and 1 or -1 elsewhere, such that $CC^\top = (n-1)I$. Two conference matrices are *equivalent* if one can be obtained from the other by a permutation of rows and columns and multiplication of some rows and columns by -1. DELSARTE, GOETHALS & SEIDEL (1971) proved that every conference matrix C of order n is equivalent to one of the form

$$\begin{pmatrix} 0 & \mathbf{1}^\top \\ \mathbf{1} & S \end{pmatrix},$$

where S is skew-symmetric (i.e., $S = -S^\top$) in case $n = 2$ or $n \equiv 0 \pmod 4$, and S is symmetric in case $n \equiv 2 \pmod 4$. We call S a *core* of C. If S is symmetric, then $\frac{1}{2}(J-S-I)$ is the adjacency matrix of a strongly regular graph with parameters $(n-1, \frac{1}{2}(n-2), \frac{1}{4}(n-6), \frac{1}{4}(n-2))$. We call these graphs *conference graphs*; they are characterized by $f = g$, so they are the only cases in which non-integral eigenvalues can occur. Conversely, one can reconstruct a conference matrix from a conference graph. BELEVITCH (1950) (see also VAN LINT & SEIDEL (1966)) proved that for a symmetric conference matrix $n-1$ must be the sum of two squares. Thus we have a necessary condition for the existence of conference graphs, called the conference matrix condition.

If a conference matrix C is symmetric and has constant column sums, then $\frac{1}{2}(J-C-I)$ is the adjacency matrix of a strongly regular graph with parameters $(n = 4r^2+4r+2,\ 2r^2+2,\ r^2-1,\ r^2)$, or the complement. (Here $n-1$ is a square, so the conference matrix condition is automatically fulfilled.) For many values of n conference matrices are known; see for instance GOETHALS & SEIDEL (1967) and MATHON (1978). PALEY (1933) defined a matrix S whose rows and columns are indexed by the elements of a finite field $\mathbb{F}_q$ (q odd), by

$$S_{xy} = \chi(x-y)$$

where χ is the quadratic residue character (i.e., $\chi(a) = 0$, 1 or -1 according to a being zero, a nonzero square, or a non-square, respectively). It follows that S is the core of a conference matrix (S is symmetric if $q \equiv 1 \pmod 4$ and skew-symmetric otherwise). If q is a square, then C is equivalent to one with constant row sum. Conference matrices are related to Hadamard matrices (see the chapter on Block Designs): if C is skew symmetric, then $C+I$ is a Hadamard matrix, and if C is symmetric, then

$$\begin{pmatrix} -C+I & C+I \\ C+I & C-I \end{pmatrix}$$

is one.

Further restrictions

Except for the rationality conditions and the conference matrix condition, a few other restrictions on the parameters are known. We mention two of them.
The Krein conditions, due to SCOTT (1973), can be stated as follows:

$$(r+1)(k+r+2rs) \le (k+r)(s+1)^2,$$
$$(s+1)(k+s+2rs) \le (k+s)(r+1)^2.$$

Seidel's absolute bound (see DELSARTE, GOETHALS & SEIDEL (1975), KOORNWINDER (1976), SEIDEL (1979)) reads

$$v \le f(f+3),$$
$$v \le g(g+3).$$

The Krein conditions and the absolute bound are special cases of general inequalities for association schemes - we'll meet them again in the next section; the conference matrix condition is the analogue of the Bruck-Chowla-Ryser theorem for square 2-designs. In BROUWER & VAN LINT (1984) one may find a list of all known restrictions; this paper gives a survey of the recent results on strongly regular graphs. It is a sequel to HUBAUT (1975)'s earlier survey of constructions. SEIDEL (1979) gives a good treatment of the theory. Some other references are BOSE (1963), CAMERON (1978) and CAMERON & VAN LINT (1980).

3. Association schemes.

An *association scheme with d classes* is a finite set X together with $d+1$ relations R_i on X such that

(i) $\{R_0, R_1, \cdots, R_d\}$ is a partition of $X \times X$;

(ii) $R_0 = \{(x,x) \mid x \in X\}$;

(iii) if $(x,y) \in R_i$, then also $(y,x) \in R_i$, for all $x,y \in X$ and $i \in \{0, \cdots, d\}$;

(iv) for any $(x,y) \in R_k$ the number p_{ij}^k of $z \in X$ with $(x,z) \in R_i$ and $(z,y) \in R_j$ depends only on i, j and k.

The numbers p_{ij}^k are called the *intersection numbers* of the association scheme. The above definition is the original definition of BOSE & SHIMAMOTO (1952); it is what DELSARTE (1973) calls a symmetric association scheme. In Delsarte's more general definition, (iii) is replaced by:

(iii′) for each $i \in \{0, \cdots, d\}$ there exists a $j \in \{0, \cdots, d\}$ such that $(x,y) \in R_i$ implies $(y,x) \in R_j$,

(iii″) $p_{ij}^k = p_{ji}^k$, for all $i,j,k \in \{0, \cdots, d\}$.

D.G. HIGMAN (1974 1975 1976) studied the even more general concept of a *coherent configuration*. He requires (i), (iii′), (iv) and

(ii′) $\{(x,x) \mid x \in X\}$ is a union of some R_j.

If (ii) holds, then the coherent configuration is called *homogeneous*. We shall not treat coherent configurations here, but content ourselves with the remark that a coherent configuration with at most 5 classes must be an association scheme (in the sense of Delsarte), see HIGMAN (1975).

Define $n := |X|$, and $n_i := p_{ii}^0$. Clearly, for each $i \in \{1, \cdots, d\}$, (X,R_i) is a simple graph which is regular of degree n_i.

3.1. Theorem. *The intersection numbers of an association scheme satisfy*

(i) $p_{0j}^k = \delta_{jk}$, $p_{ij}^0 = \delta_{ij} n_j$, $p_{ij}^k = p_{ji}^k$,

(ii) $\sum_i p_{ij}^k = n_j$, $\sum_j n_j = n$,

(iii) $p_{ij}^k n_k = p_{ik}^j n_j$,

(iv) $\sum_l p_{ij}^l p_{kl}^m = \sum_l p_{kj}^l p_{il}^m$.

Proof. (i)-(iii) are straightforward. The expressions at both sides of (iv) count quadruples (w,x,y,z) with $(w,x) \in R_i$, $(x,y) \in R_j$, $(y,z) \in R_k$, for a fixed pair $(w,z) \in R_m$. ∎

It is convenient to write the intersection numbers as entries of the so-called *intersection matrices* $L_0, \cdots, L_d$:

$$(L_i)_{kj} = p_{ij}^k.$$

Note that $L_0 = I$. From the definition it is clear that an association scheme with two classes is the same as a pair of complementary strongly regular graphs. (With Delsarte's more general definition one also gets the skew-symmetric conference matrices.) If (X,R_1) is strongly regular with parameters (v,k,λ,μ), then the intersection matrices of the scheme are

$$L_1 = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k-\lambda-1 \\ 0 & \mu & k-\mu \end{pmatrix} \quad L_2 = \begin{pmatrix} 0 & 0 & v-k-1 \\ 0 & k-\lambda-1 & v-2k+\lambda \\ 1 & k-\mu & v-2k+\mu-2 \end{pmatrix}.$$

We see that (iii) generalises (2.2.i).

The Bose-Mesner algebra.

The relations R_i of an association scheme are described by their adjacency matrices A_i of order n defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{whenever } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, A_i is the adjacency matrix of the graph (X,R_i). In terms of the adjacency matrices, the axioms (i)-(iv) become

(i) $\sum_{i=0}^{d} A_i = J$,

(ii) $A_0 = I$,

(iii) $A_i = A_i^\top$, for all $i \in \{0, \cdots, d\}$,

(iv) $A_i A_j = \sum_k p_{ij}^k A_k$, for all $i,j,k \in \{0, \cdots, d\}$.

From (i) we see that the 0-1 matrices A_i are linearly independent, and by use of (ii)-(iv) we see that they generate a commutative $(d+1)$-dimensional algebra **A** of symmetric matrices with constant diagonal. This algebra was first studied by BOSE & MESNER (1959) and is called the Bose-Mesner algebra of the association scheme.

Since the matrices A_i commute, they can be diagonalized simultaneously (see MARCUS & MINC

(1965)), that is, there exist a matrix S such that for each $A \in \mathbf{A}$, $S^{-1}AS$ is a diagonal matrix. Therefore $\mathbf{A}$ is semisimple and has a unique basis of minimal idempotents $E_0, \cdots, E_n$ (see BURROW (1965)). These are matrices satisfying

$$E_i E_j = \delta_{ij} E_i,$$
$$\sum_{i=0}^{d} E_i = I.$$

The matrix $\frac{1}{n}J$ is a minimal idempotent (idempotent is clear, and minimal follows since $\mathrm{rk}\, J = 1$). We shall take $E_0 = \frac{1}{n}J$. Let P and $\frac{1}{n}Q$ be the matrices relating our two bases for $\mathbf{A}$:

$$A_j = \sum_{i=0}^{d} P_{ij} E_i,$$
$$E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

Then clearly

$$PQ = QP = nI.$$

It also follows that

$$A_j E_i = P_{ij} E_i,$$

which shows that the P_{ij} are the eigenvalues of A_j and that the columns of E_i are the corresponding eigenvectors. Thus $\mu_i := \mathrm{rk}\, E_i$ is the multiplicity of the eigenvalue P_{ij} of A_j (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$). We see that $\mu_0 = 1$, $\sum_{i=0}^{d} \mu_i = n$, and $\mu_i = \mathrm{tr}\, E_i = n\,(E_i)_{jj}$ (indeed, E_i has only eigenvalues 0 and 1, so $\mathrm{rk}\, E_k$ equals the sum of the eigenvalues).

3.2. Theorem. *The numbers P_{ij} and Q_{ij} satisfy*

(i) $P_{i0} = Q_{i0} = 1$, $P_{0i} = n_i$, $Q_{0i} = \mu_i$,

(ii) $P_{ij} P_{ik} = \sum_{l=0}^{d} p_{jk}^{l} P_{il}$,

(iii) $\mu_i P_{ij} = n_j Q_{ji}$, $\sum_i \mu_i P_{ij} P_{ik} = n n_j \delta_{jk}$, $\sum_i n_i Q_{ij} Q_{ik} = n \mu_j \delta_{jk}$,

(iv) $|P_{ij}| \leq n_j$, $|Q_{ij}| \leq \mu_j$.

Proof. Part (i) follows easily from $\sum_i E_i = I = A_0$, $\sum_i A_i = J = nE_0$, $A_i J = n_i J$, and $\mathrm{tr}\, E_i = \mu_i$.

Part (ii) follows from $A_j A_k = \sum_l p_{jk}^{l} A_l$.

The first equality in (iii) follows from $\sum_i n_j Q_{ji} P_{ik} = n n_j \delta_{jk} = \mathrm{tr}\, A_j A_k = \sum_i \mu_i P_{ij} P_{ik}$, since P is nonsingular; this also proves the second equality, and the last one follows since $PQ = nI$.

The first inequality of (iv) holds because the P_{ij} are eigenvalues of the n_j-regular graphs (X, R_j). The second inequality then follows by use of (iii). ■

Relations (iii) are often referred to as the *orthogonality relations*, since they state that the rows (and columns) of P (and Q) are orthogonal with respect to a suitable weight function.

If $d = 2$, and (X, R_1) is strongly regular with parameters (v, k, λ, μ), the matrices P and Q are

$$P = \begin{pmatrix} 1 & k & v-k-1 \\ 1 & r & -r-1 \\ 1 & s & -s-1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & f & g \\ 1 & f\frac{r}{k} & g\frac{s}{k} \\ 1 & -f\frac{r+1}{v-k-1} & -g\frac{s+1}{v-k-1} \end{pmatrix},$$

where r, s, f and g can be expressed in terms of v, k, λ by use of (2.2).

In general the matrices P and Q can be computed from the intersection numbers of the scheme, as follows from the following

3.3. Theorem. *For $i = 0, \cdots, d$, the intersection matrix L_j has eigenvalues P_{ij} $(0 \le i \le d)$.*

Proof. (3.2.ii) yields

$$P_{ij} \sum_k P_{ik} (P^{-1})_{km} = \sum_{k,l} P_{il} (L_j)_{lk} (P^{-1})_{km},$$

hence $PL_j P^{-1} = \mathrm{diag}\,(P_{0j}, \cdots, P_{dj})$. ∎

Thanks to this theorem, it is relatively easy to compute P, Q $(= \frac{1}{n} P^{-1})$ and μ_i $(= Q_{0i})$. It is also possible to express P and Q in terms of the (common) eigenvectors of the L_j. Indeed, $PL_j P^{-1} = \mathrm{diag}\,(P_{0j}, \cdots, P_{dj})$ implies that the rows of P are left eigenvectors and the columns of Q are right eigenvectors. In particular, μ_i can be computed from the right eigenvector u_i and the left eigenvector $v_i^\top$, normalized such that $(u_i)_0 = (v_i)_0 = 1$, by use of $\mu_i u_i^\top v_i = n$. Clearly, each μ_i must be an integer. These are the *rationality conditions* for an association scheme. As we saw in the case of a strongly regular graph, these conditions can be very powerful. Godsil (see his chapter) puts the rationality conditions in a more general form, which is not restricted to association schemes.

The Krein parameters.

The Bose-Mesner algebra **A** is not only closed under ordinary matrix multiplication, but also under componentwise (Hadamard, Schur) multiplication (denoted $\circ$). Clearly $\{A_0, \cdots, A_d\}$ is the basis of minimal idempotents with respect to this multiplication.

Write

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{d} q_{ij}^k E_k.$$

The numbers q_{ij}^k thus defined are called the *Krein parameters*. (Our q_{ij}^k are those of Delsarte, but differ from SEIDEL (1979)'s by a factor n.) As expected, we now have the analogue of (3.1) and (3.2).

3.4. Theorem. *The Krein parameters of an association scheme satisfy*

(i) $q_{0j}^k = \delta_{jk}$, $q_{ij}^0 = \delta_{ij} \mu_j$, $q_{ij}^k = q_{ji}^k$,

(ii) $\sum_i q_{ij}^k = \mu_j$, $\sum_j \mu_j = n$,

(iii) $q_{ij}^k \mu_k = q_{ik}^j \mu_j$,

(iv) $\sum_l q_{ij}^l q_{kl}^m = \sum_l q_{kj}^l q_{il}^m$,

(v) $Q_{ij} Q_{ik} = \sum_{l=0}^{d} q_{jk}^l Q_{il}$,

(vi) $n \mu_k q_{ij}^k = \sum_l n_l Q_{li} Q_{lj} Q_{lk}$.

Proof. Let $\Sigma(A)$ denote the sum of all entries of the matrix A. Then $JAJ = \Sigma(A) J$, $\Sigma(A \circ B) = \mathrm{tr}\, AB^\top$ and $\Sigma(E_i) = 0$ if $i \ne 0$, since then $E_i J = n E_i E_0 = 0$. Now (i) follows by use of $E_i \circ E_0 = \frac{1}{n} E_i$, $q_{ij}^0 = \Sigma(E_i \circ E_j) = \mathrm{tr}\, E_i E_j = \delta_{ij} \mu_j$, and $E_i \circ E_j = E_j \circ E_i$, respectively. Equation (iv) follows by evaluating $E_i \circ E_j \circ E_k$ in two ways, and (iii) follows from (iv) by taking $m = 0$. Equation (v) follows from evaluating $A_i \circ E_j \circ E_k$ in two ways, and (vi) follows from (v), using the orthogonality relation $\sum_l n_l Q_{lm} Q_{lk} = \delta_{mk} \mu_k n$.

Finally, by use of (iii) we have

$$\mu_k \sum_j q_{ij}^k = \sum q_{ik}^j \mu_j = n.\mathrm{tr}\,(E_i \circ E_k) = n \sum_l (E_i)_{ll} (E_k)_{ll} = \mu_i \mu_k,$$

proving (ii). ∎

The above results illustrate a dual behaviour between ordinary multiplication, the numbers p_{ij}^k and the matrices A_i and P on the one hand, and Schur multiplication, the numbers q_{ij}^k and the matrices E_i and Q on the other hand. If two association schemes have the property that the intersection numbers of one are the Krein parameters of the other, then the converse is also true. Two such schemes are said to be (formally) dual to each other. One scheme may have several (formal) duals, or none at all (but when the scheme is invariant under a regular abelian group, there is a natural way to define a dual scheme, cf. DELSARTE (1973)). In fact usually the Krein parameters are not even integers. But they cannot be negative. These important restrictions, due to SCOTT (1973), are the so-called Krein conditions.

3.5. Theorem. *The Krein parameters of an association scheme satisfy* $q_{ij}^k \geq 0$ *for all* $i,j,k \in \{0, \cdots, d\}$.

Proof. The numbers $\frac{1}{n} q_{ij}^k$ $(0 \leq k \leq d)$ are the eigenvalues of $E_i \circ E_j$ (since $(E_i \circ E_j)E_k = \frac{1}{n} q_{ij}^k E_k$). On the other hand, the Kronecker product $E_i \otimes E_j$ is positive semidefinite, since each E_i is. But $E_i \circ E_j$ is a principal submatrix of $E_i \otimes E_j$, and therefore is positive semidefinite as well, i.e., has no negative eigenvalue. ■

The Krein parameters can be computed by use of equation (3.4.vi). This equation also shows that the Krein condition is equivalent to

$$\sum_l n_l Q_{li} Q_{lj} Q_{lk} \geq 0 \quad \text{for all } i,j,k \in \{0, \cdots, d\}.$$

In case of a strongly regular graph we obtain

$$q_{11}^1 = \frac{f^2}{v} \left(1 + \frac{r^3}{k^2} - \frac{(r+1)^3}{(v-k-1)^2} \right) \geq 0,$$

$$q_{22}^2 = \frac{g^2}{v} \left(1 + \frac{s^3}{k^2} - \frac{(s+1)^3}{(v-k-1)^2} \right) \geq 0$$

(the other Krein conditions are trivially satisfied in this case), which is equivalent to the result mentioned in the previous section.

NEUMAIER (1981), generalized Seidel's absolute bound to association schemes, and obtained

3.6. Theorem. *The multiplicities* μ_i $(0 \leq i \leq d)$ *of an association scheme with d classes satisfy*

$$\sum_{q_{ij}^k \neq 0} \mu_k \leq \begin{cases} \mu_i \mu_j & \text{if } i \neq j, \\ \frac{1}{2}\mu_i(\mu_i + 1) & \text{if } i = j. \end{cases}$$

Proof. The left hand side equals $\mathrm{rk}\,(E_i \circ E_j)$. But $\mathrm{rk}\,(E_i \circ E_j) \leq \mathrm{rk}\,(E_i \otimes E_j) = \mathrm{rk}\, E_i \times \mathrm{rk}\, E_j = \mu_i \mu_j$. And if $i = j$, then $\mathrm{rk}\,(E_i \circ E_i) \leq \frac{1}{2}\mu_i(\mu_i + 1)$. Indeed, if the rows of E_i are linear combinations of μ_i rows, then the rows of $E_i \circ E_i$ are linear combinations of the $\mu_i + \frac{1}{2}\mu_i(\mu_i - 1)$ rows that are the elementwise products of any two of these μ_i rows. ■

For strongly regular graphs with $q_{11}^1 = 0$ we obtain Seidel's bound: $v \leq \frac{1}{2} f(f+3)$. But in case $q_{11}^1 > 0$, Neumaier's result states that the bound can be improved to $v \leq \frac{1}{2} f(f+1)$.

Subsets of association schemes.

The last subject of this section is a result of DELSARTE (1973) (Theorem 3.3, p. 26) on subsets in association schemes. For $Y \subseteq X$, $Y \neq \emptyset$, we define

$$a_i = \frac{1}{|Y|} \chi^{\top} A_i \chi,$$

where χ is the characteristic vector of Y. In other words, a_i is the average degree of the subgraph of (X,R_i) induced by Y. Clearly $a_0 = 1$, and $\sum_{i=0}^{d} a_i = |Y|$. The vector $a = (a_0, \cdots, a_d)$ is called the *inner distribution* of Y.

3.7. Theorem. *The inner distribution a of an nonempty subset of an association scheme satisfies* $aQ \geq 0$.

Proof. $|Y| \sum_{i=0}^{d} a_i Q_{ij} = \sum_{i=0}^{d} \chi^{\top} Q_{ij} A_i \chi = n \chi^{\top} E_j \chi \geq 0$, since E_j is positive semidefinite. ■

This inequality leads to Delsarte's linear programming bound, as we shall see in the next section.

As an application we have the following result (DELSARTE (1973), Theorem 3.9, p. 32).

3.8. Theorem. *Let $\{\{0\}, I_1, I_2\}$ be a partition of $\{0, \cdots, d\}$, and assume that Y and Z are nonempty subsets of X such that the inner distribution b of Y satisfies $b_i = 0$ for $i \in I_1$, and the inner distribution c of Z satisfies $c_i = 0$ for $i \in I_2$. Then $|Y| \cdot |Z| \le |X|$, and equality holds if and only if for all $i \ne 0$ we have $(bQ)_i = 0$ or $(cQ)_i = 0$.*

Proof. Define $\beta_i = \mu_i^{-1} |Z|^{-1} \sum_j c_j Q_{ji}$. Then $\beta_0 = 1$, $\beta_i \ge 0$ for all i, and $\sum_i \beta_i Q_{ki} = n_k^{-1} |Z|^{-1} n c_k$. Now we have $|Y| = \sum_k b_k = (bQ)_0 \le \sum_{i,k} b_k Q_{ki} \beta_i = \sum_i Q_{0i} \beta_i = \dfrac{n}{|Z|}$. ■

Let us investigate a special case of this situation somewhat closer. Let, for $I \subseteq \{0, \cdots, d\}$, the *I-sphere* around the point $x \in X$ be the set $\{ y \in X | (x,y) \in R_i \text{ for some } i \in I \}$. A nonempty subset Y of X is called *perfect* (more precisely, *I-perfect*) when the *I*-spheres around its points form a partition of *X*.

3.9. Theorem. (‘Lloyd’s theorem’, cf. LLOYD (1957), DELSARTE (1973), p. 63). *Let Y be I-perfect, with inner distribution a. Then $\sum_{i \in I} P_{ji} = 0$ for all $j \ne 0$ such that $(aQ)_j \ne 0$.*

Proof. Apply the previous theorem, with for *Z* an *I*-sphere. If *c* is the inner distribution of *Z*, then

$$|Z| \mu_j^{-1} (cQ)_j = \mu_j^{-1} \sum_i \sum_{g,h \in I} n_g p_{ih}^g Q_{ij} = \sum_i \sum_{g,h \in I} p_{gh}^i P_{ji} = \sum_{g,h \in I} P_{jg} P_{jh} = (\sum_{i \in I} P_{ji})^2. \ ■$$

4. Applications.

In this section we discuss some special types of association scheme and their significance to other fields of combinatorics.

Distance regular graphs.

Consider a connected simple graph with vertex set X of diameter d. Define $R_i \subset X^2$ by $(x,y) \in R_i$ whenever x and y have graph distance i. If this defines an association scheme, then the graph (X, R_1) is called *distance-regular*. The corresponding association scheme is called *metric*. By the triangle inequality, $p_{ij}^k = 0$ if $i + j < k$ or $|i - j| > k$. Moreover, $p_{i\,j}^{i+j} > 0$. Conversely, if the intersection numbers of an association scheme satisfy these conditions, then (X, R_1) is easily seen to be distance-regular.

Many of the association schemes that play a rôle in combinatorics are metric. In fact, all the examples treated in this chapter are metric. Strongly regular graphs are obviously metric. The line graph of the Petersen graph and the Hoffman-Singleton graph are easy examples of distance-regular graphs that are not strongly regular.

Any k-regular graph of diameter d has at most

$$1 + k + k(k-1) + \cdots + k(k-1)^{d-1}$$

vertices, as is easily seen. Graphs for which equality holds are called *Moore graphs*. Moore graphs are distance-regular, and those of diameter 2 were dealt with in Theorem 2.3. Using the rationality conditions DAMERELL (1973) and BANNAI & ITO (1973) showed:

4.1. Theorem. *A Moore graph with diameter $d \ge 3$ is a $(2d+1)$-gon.*

A strong non-existence result of the same nature is the theorem of FEIT & G. HIGMAN (1964) about finite generalized polygons. A *generalized m-gon* is a point-line geometry such that the incidence graph is a connected, bipartite graph of diameter m and girth $2m$. It is called *regular* of order (s,t) for certain (finite or infinite) cardinal numbers s, t if each line is incident with $s+1$ points and each point is incident with $t+1$ lines. (It is not difficult to prove that if each point is on at least three lines, and each line has at least three points (and $m < \infty$), then the geometry is necessarily regular, and in fact $s = t$ in case m is odd.) From such a regular generalized m-gon of order (s,t), where s and t are finite and $m \ge 3$, we can construct a distance-regular graph with valency $s(t+1)$ and diameter $d = [\frac{1}{2} m]$ by taking the collinearity graph on the points.

4.2. Theorem. *A finite generalized m-gon of order* (s,t) *with* $s > 1$ *and* $t > 1$ *satisfies* $m \in \{2,3,4,6,8\}$.

Proofs of this theorem can be found in FEIT & HIGMAN (1964), KILMOYER & SOLOMON (1973) and ROOS (1980); again the rationality conditions do the job. The Krein conditions yield some additional information:

4.3. Theorem. *A finite regular generalized m-gon with* $s > 1$ *and* $t > 1$ *satisfies* $s \le t^2$ *and* $t \le s^2$ *if* $m = 4$ *or* 8*; it satisfies* $s \le t^3$ *and* $t \le s^3$ *if* $m = 6$.

This result is due to HIGMAN (1971 1974) and HAEMERS & ROOS (1981). For each $m \in \{2,3,4,6,8\}$ infinitely many generalized m-gons exist. (For $m = 2$ we have trivial structures - the incidence graph is complete bipartite; for $m = 3$ we have (generalized) projective planes; an example of a generalized 4-gon of order (2,2) with collinearity graph $T(6)$ can be described as follows: the points are the pairs from a 6-set, and the lines are the partitions of the 6-set into three pairs, with obvious incidence.)

Many association schemes have the important property that the eigenvalues P_{ij} can be expressed in terms of orthogonal polynomials. An association scheme is called *P-polynomial* if there exist polynomials f_k of degree k with real coefficients, and real numbers z_i such that $P_{ik} = f_k(z_i)$. Clearly we may always take $z_i = P_{i1}$.

By the orthogonality relation (3.2.iii) we have

$$\sum_i \mu_i f_j(z_i) f_k(z_i) = \sum_i \mu_i P_{ij} P_{ik} = n n_j \delta_{jk},$$

which shows that the f_k are orthogonal polynomials.

4.4. Theorem. *An association scheme is metric if and only if it is P-polynomial.*

Proof. Let the scheme be metric. Theorem 1.1 gives

$$A_1 A_i = p_{1i}^{i-1} A_{i-1} + p_{1i}^{i} A_i + p_{1i}^{i+1} A_{i+1}.$$

Since $p_{1i}^{i+1} \ne 0$, A_{i+1} can be expressed in terms of A_1, A_{i-1} and A_i. Hence for each j there exists a polynomial f_j of degree j such that

$$A_j = f_j(A_1).$$

Using this we have $P_{ij} E_i = A_j E_i = f_j(A_1) E_i = f_j(A_1 E_i) E_i = f_j(P_{i1}) E_i$, hence $P_{ij} = f_j(P_{i1})$.
Now suppose that the scheme is P-polynomial. Then the f_j are orthogonal polynomials, and therefore they satisfy a 3-term recurrence relation (see SZEGÖ (1959) p. 42)

$$\alpha_{j+1} f_{j+1}(z) = (\beta_j - z) f_j(z) + \gamma_{j-1} f_{j-1}(z).$$

Hence

$$P_{i1} P_{ij} = -\alpha_{j+1} P_{i\,j+1} + \beta_j P_{ij} + \gamma_{j-1} P_{i\,j-1} \quad \text{for } i = 0, \cdots, d.$$

Since $P_{i1} P_{ij} = \sum_l p_{1j}^{l} P_{il}$ and P is nonsingular, it follows that $p_{1j}^{l} = 0$ for $|l - j| > 1$. Now the full metric property easily follows by induction. ■

This result is due to DELSARTE (1973) (Theorem 5.6, p. 61). There is also a result dual to this theorem, involving so-called Q-polynomial and cometric schemes. However, just as the intersection numbers p_{ij}^{k} have a combinatorial interpretation while the Krein parameters q_{ij}^{k} do not, the metric schemes have the combinatorial description of distance-regular graphs, while there is no combinatorial interpretation for the cometric property. For more information on P- and Q-polynomial association schemes, see DELSARTE (1973) and BANNAI & ITO (1984); for distance-regular graphs, see the forthcoming book by Brouwer, Cohen & Neumaier.

The Hamming scheme and the linear programming bound for error-correcting codes.

Let $X = \mathbf{Q}^d$, the set of all vectors of length d with entries in $\mathbf{Q}$, where $\mathbf{Q}$ is some set of size q. Define $R_i \subset X^2$ by $(x,y) \in R_i$ if the Hamming distance between x and y (i.e., the number of coordinates in which x and y differ) equals i. This defines an association scheme, the *Hamming scheme* $H(d,q)$. The Hamming scheme is easily seen to be metric, and hence by (4.4) P-polynomial. The orthogonal polynomials involved are the Kravcuk polynomials $K_j(x)$.

4.5. Theorem. *For the Hamming scheme $H(d,q)$ we have*

$$P_{ij} = Q_{ij} = K_j(i) = \sum_{k=0}^{j} (-1)^k (q-1)^{j-k} \binom{i}{k} \binom{n-i}{j-k}.$$

See DELSARTE (1973) (p. 38) or MACWILLIAMS & SLOANE (1977) for proofs. From (4.5) we see that $P = Q$, so the Hamming scheme is self-dual.

A subset $Y \subset X$ of $H(d,q)$, such that $Y^2 \cap R_i = \emptyset$ for $i = 1, \cdots, \delta-1$ and $Y^2 \cap R_\delta \neq \emptyset$ (i.e., a subset Y such that the minimum Hamming distance between two vectors of Y equals δ), is nothing but an error-correcting code with parameters $(d, |Y|, \delta)$ over the alphabet $\mathbf{Q}$ (cf. the chapter of coding theory). Let $a = (a_0, \cdots, a_d)$ be the inner distribution of Y. Then $\sum_i a_i = |Y|$, $a_0 = 1$, $a_1 = \cdots = a_{\delta-1} = 0$, $a \geq 0$ (by definition), and $aQ \geq 0$ by (3.7). Consider $a_\delta, \cdots, a_d$ as variables and define

$$a^* = 1 + \max \sum_{i=\delta}^{d} a_i \ \textit{ subject to } \ K_j(1) + \sum_{i=\delta}^{d} a_i K_j(i) \geq 0,\ j = 0, \cdots, d, \textit{ and } \ a_i \geq 0,\ i = \delta, \cdots, d.$$

Then clearly $|Y| \leq a^*$. So a^* is an upper bound for the number of codewords with a given length and minimum distance. This bound, due to DELSARTE (1973), is called the linear programming bound, since the value of a^* can be computed by linear programming. Of course, the above-mentioned inequalities are not the only ones satisfied by the a_i, and by adding extra inequalities to the system, one may obtain sharper bounds on $|Y|$. For details and more applications to coding theory, see DELSARTE (1973), MACWILLIAMS & SLOANE (1977), BEST, BROUWER, MACWILLIAMS, ODLYZKO & SLOANE (1978), BEST & BROUWER (1977).

The Johnson scheme and *t*-designs.

Let the set X consist of all subsets of size d of a set M, where $|M| = m \geq 2d$. Define relation $R_i \subset X^2$ by $(x,y) \in R_i$ if the Johnson distance between x and y (i.e., the cardinality of $x \setminus y$) equals i. This defines an association scheme, the *Johnson scheme* $J(m,d)$. Since the Johnson distance between x and y equals twice the Hamming distance between (the characteristic vectors of) x and y, it follows that also the Johnson distance satisfies the triangle inequality, so that the Johnson scheme is metric. Note that the graph (X, R_1) is complete for $d = 1$, and is the triangular graph $T(m)$ for $d = 2$. The following result (due to OGASAWARA (1965) and YAMAMOTO et al. (1965)) gives some parameters:

4.6. Theorem. *For the Johnson scheme $J(m,d)$ the following hold:*

$$P_{ij} = \frac{n_j}{\mu_i} Q_{ji} = E_j(i) = \sum_{k=0}^{j} (-1)^{j-k} \binom{d-k}{j-k} \binom{d-i}{k} \binom{m-d+k-i}{k} = \sum_{k=0}^{j} (-1)^k \binom{i}{k} \binom{d-i}{j-k} \binom{m-d-i}{j-k},$$

$$\mu_i = \binom{m}{i} - \binom{m}{i-1}, \quad n_j = \binom{d}{j} \binom{m-d}{j}.$$

Here $E_j(x)$ is a so-called Eberlein polynomial. It has degree $2j$ in the indeterminate x, and degree j in the indeterminate $x(m+1-x)$. Since $P_{i1} = d(m-1) - i(m+1-i)$, $E_j(i)$ indeed has degree j in P_{i1} as required by the definition of P-polynomial scheme.

The graph (X, R_d) of a Johnson scheme is called a Kneser graph. A subset Y of X such that any two elements of Y have non-empty intersection is a coclique (independent set) of the Kneser graph. By use of Theorem 3.7 and the above formulas, it can be deduced that

$$|Y| \leq \binom{m-1}{d-1},$$

the famous result of ERDŐS, KO & RADO (1961).

A t-(m,d,λ) design is a subset $Y \subset X$ of the Johnson scheme $J(m,d)$ such that each t-element subset of M is contained in precisely λ elements of Y. DELSARTE (1973) (Theorem 4.7, p. 51) proved the following:

4.7. Theorem. *A non-empty subset Y of the Johnson scheme $J(m,d)$ with inner distribution $a = (a_0, \cdots, a_d)$ is a t-(m,d,λ) design if and only if*

$$\sum_{i=0}^{d} a_i Q_{ij} = 0 \quad \text{for } j = 1, \cdots, t. \tag{*}$$

Just as we did in case of the Hamming scheme, we can define

$$a_* = 1 + \min \sum_{i=1}^{d} a_i \ \textit{ subject to}$$

$$a_i \geq 0 \textit{ for } i = 1, \cdots, d, \textit{ and } \sum_{i=0}^{d} a_i Q_{ij} = 0 \textit{ for } j = 1, \cdots, t, \textit{ and } \sum_{i=0}^{d} a_i Q_{ij} \geq 0 \textit{ for } j = t+1, \cdots, d,$$

where the Q_{ij} are found in (4.6). Now we have the linear programming (lower) bound for the number of blocks in a t-design: $|Y| \geq a_*$. Using the simplex algorithm, HAEMERS & WEUG (1974) showed that this inequality implies the non-existence of the designs with parameters 4-(17,8,5), 4-(23,11,λ), $\lambda = 6,12$, 4-(24,12,15), 6-(19,9,λ), $\lambda \leq 10$, 6-(20,10,λ), $\lambda = 7,14$. In certain other cases, such as 5-(19,9,7), KÖHLER (1985) ruled out the existence of t-designs by showing that no solution of the above system of inequalities corresponds to an actual design.

If we replace the restrictions $(aQ)_i = 0$ by restrictions $a_i = 0$ $(1 \leq i < \frac{1}{2}\delta)$ in this system of inequalities, and maximize $\sum a_i$, we obtain upper bounds for the cardinality of codes with minimum distance δ and constant weight d.

We might also require both $a_i = 0$ and $(aQ)_j = 0$ for suitable i, j, and obtain results for t-designs with restricted block intersections, such as quasisymmetric block designs. By this method it follows for instance that a quasisymmetric 2-(29,7,12) design with block intersections 1 and 3 cannot exist (HAEMERS (1975)). For details and more results we refer to DELSARTE (1973), MACWILLIAMS & SLOANE (1977), BEST, BROUWER, MACWILLIAMS, ODLYZKO & SLOANE (1978), CAMERON & VAN LINT (1980), CALDERBANK (1988). We also point out that several results of §3 in Godsil's chapter can also be obtained by use of the framework of Johnson schemes.

DELSARTE (1973) generalized the notion of t-designs to subsets of arbitrary association schemes satisfying (*). (Equivalently, a t-design is a subset Y of X such that its characteristic vector χ satisfies $E_j \chi = 0$ for $j = 1, \cdots, t$.) He shows that a t-design in the Hamming scheme is what is known as an orthogonal array of strength t (see the chapter on Block Designs). Thus, we also have a linear programming bound for orthogonal arrays. More generally, one may give an interpretation of the classical concept of t-design in terms of ranked posets in the obvious way, and then prove for each of the eight known infinite families of P- and Q-polynomial association schemes that a subset is a classical t-design if and only if it is a Delsarte $f(t)$-design (where usually $f(t) = t$), see DELSARTE (1976) and STANTON (1986).

Imprimitive schemes.

In section 2 we saw how to associate to a quasi-symmetric 2-design an association scheme whose points are the blocks of the design. In the previous subsection we saw how t-designs give rise to interesting subsets of the Johnson scheme. Let us give one more example of a relation between designs and association schemes. Let N be the incidence matrix of a square 2-design. Then, defining $A_0 = I$,

$$A_1 = \begin{pmatrix} 0 & N \\ N^{\mathrm{T}} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

and $A_3 = J - I - A_1 - A_2$, we obtain a 3-class association scheme. It is *imprimitive*, that is, the union of some of the R_i form a non-trivial equivalence relation (here R_2 is an equivalence relation). Another imprimitive association scheme we have seen is the line graph of the Petersen graph (there having maximal distance is an equivalence relation).

Given an imprimitive association scheme one may produce new association schemes; on the one hand, there is a natural way to give the set of equivalence classes the structure of an association scheme (the 'quotient scheme'), and on the other hand, each equivalence class together with the restrictions of the original relations becomes an association scheme (a 'subscheme' of the original scheme). See (1989), §2.4

for details.

The group case.

We very briefly discuss some relations between association schemes and finite permutation groups.

Let **G** be a permutation group acting on a set X. Then **G** has a natural action on X^2, the orbits of which are called orbitals. Suppose **G** acts generously transitive on X, that is, for any $x,\ y \in X$ there exists an element of **G** interchanging x and y. Then the orbitals form an association scheme.

(Without any requirements on **G**, the orbitals form a coherent configuration (see § 3). The coherent configuration is homogeneous if **G** is transitive. We get an association scheme in the sense of Delsarte when the permutation character is multiplicity free.)

For any $x \in X$, the number of orbitals equals the number of orbits on X of $\mathbf{G}_x$ (the subgroup of **G** of permutations fixing x). This number is called the *rank* of **G**. Thus, the number of classes in the association scheme is one less than the rank of **G**. We can also transfer other permutation group theoretic terminology and results to the theory of association schemes. For instance, the Bose-Mesner algebra is in the group case known as the *centralizer algebra*, and all standard results on this centralizer algebra (cf., e.g., WIELANDT (1964)) have their direct analogue for the Bose-Mesner algebra.

The Hamming and Johnson schemes are derived from generously transitive permutation representations as discussed above; for instance, the Johnson scheme is derived from the representation of the symmetric group $\mathrm{Sym}(m)$ on the d-element subsets of an m-set.

If a metric association scheme belongs to the group case, then the corresponding distance-regular graph is called *distance-transitive*. In other words, a graph is distance-transitive when its group of automorphisms is transitive on pairs of vertices with a given distance. Distance-transitive strongly regular graphs are known as rank 3 graphs. A rank 3 permutation group is generously transitive if and only if it has even order; consequently every rank 3 permutation group of even order provides a strongly regular graph. All such strongly regular graphs have recently been classified, see KANTOR & LIEBLER (1982), LIEBECK (1985 1987), and LIEBECK & SAXL (1986).

Euclidean representations of graphs and association schemes

It is possible to represent a graph in a Euclidean space and obtain combinatorial information from facts in Euclidean geometry. For example, let G be a graph with adjacency matrix A with smallest eigenvalue s. If $m \geq -s$, then the matrix $A + mI$ is positive semidefinite, and hence we can write $A + mI = Z^{\top} Z$, for some real $f \times n$ matrix Z (where f is the rank of $A + mI$). This yields a representation $\bar{\ }: G \to \mathbb{R}^f$ sending the vertex x of G to column x of Z. For this representation we have for the inner products

$$(\bar{x}, \bar{y}) = (A + mI)_{xy} = \begin{cases} m \text{ if } y = x \\ 1 \text{ if } y \in \Gamma(x) \\ 0 \text{ otherwise.} \end{cases}$$

The above shows that a lower bound on the smallest eigenvalue of G yields a Euclidean representation of G with known inner products. But conversely, suppose we have a Euclidean representation of G with inner products m, 1, 0 for equal, adjacent and nonadjacent vertices, respectively. Then the Gram matrix of this representation is $mI + A$, and since Gram matrices are positive semidefinite, it follows that A has smallest eigenvalue $\geq -m$.

As a very simple application, consider a graph G with smallest eigenvalue -1. We find a representation by vectors of norm 1 and it follows that having distance at most one is an equivalence relation on G, that is, G is a union of cliques.

As a second application, consider graphs G with smallest eigenvalue ≥ -2. Our representation (for $m = 2$) satisfies

$$(\bar{x}, \bar{y}) = \begin{cases} 2 \text{ if } y = x \\ 1 \text{ if } y \in \Gamma(x) \\ 0 \text{ otherwise} \end{cases}$$

and hence the $\mathbb{Z}$-span of the image $\overline{G}$ is a root lattice. But all root lattices are known (they are direct sums of the lattices A_n, D_n, E_6, E_7, E_8) and this makes it possible to classify all such graphs G. See CAMERON, GOETHALS, SEIDEL & SHULT (1976).

For an association scheme we have a natural supply of positive semidefinite matrices, namely the idempotents, and we can play the same game. Let $(X, \mathbf{R})$ be an association scheme, E an idempotent, and consider the map $\overline{}: X \to \mathbb{R}^n$ sending $x \in X$ to $E_{\cdot x}$, column x of E. (Then the dimension of the span of the image of X under this map is the rank of E.) For this representation we have for the inner products:

$$(\overline{x}, \overline{y}) = E_x . E_{\cdot y} = E_{xy}.$$

In particular, if $E = E_j$, then $E = \frac{1}{n}\sum_i Q_{ij} A_i$ and we find that

$$(\overline{x}, \overline{y}) = Q_{ij} \quad \text{whenever } (x,y) \in R_i.$$

Suppose a graph G has a representation $\overline{}$ in $\mathbb{R}^f$ satisfying

$$(\overline{x}, \overline{y}) = \begin{cases} p \text{ if } y = x \\ q \text{ if } d(x,y) = 1 \\ r \text{ if } d(x,y) = 2, \end{cases}$$

where $d(x,y)$ denotes the graph distance between the vertices x and y. Fix a vertex x of G, and define a representation $\hat{}$ of $\Gamma(x)$ by $\hat{y} = \overline{y} - \frac{q}{p}\overline{x}$. Then $(\hat{y}, \overline{x}) = 0$ for all $y \in \Gamma(x)$, and we have a representation $\hat{}$ of $\Gamma(x)$ in $\mathbb{R}^{f-1}$ satisfying

$$(\hat{x}, \hat{y}) = \begin{cases} p - q^2/p \text{ if } y = x \\ q - q^2/p \text{ if } d(x,y) = 1 \\ r - q^2/p \text{ otherwise.} \end{cases}$$

In a number of cases the above techniques suffice to determine all graphs with given parameters. For example, let G be a distance-regular graph with the same parameters as the Johnson graph $J(m,d)$ (with as vertices the d-subsets of a fixed m-sets, two d-subsets being adjacent when they meet in a $(d-1)$-set). From $Q_{ij} = P_{ji}\mu_j / k_i$ and the formulas given in Theorem 4.6 above, it follows that $Q_{i1} = m - 1 - im(m-1)/d(m-d)$, which is linear in i. It follows that every local graph $\Gamma(x)$ has smallest eigenvalue ≥ -2, so that the theory developed in (1976) applies. This enabled NEUMAIER (1985) and TERWILLIGER (1986) (independently) to completely classify such graphs G. (The result being that necessarily $G \cong J(m,d)$, except in the case $(m,d) = (8,2)$, where there are three other graphs, known as the Chang graphs, with the same parameters as the triangular graph $T(8)$.)

As a second example, due to IVANOV & SHPECTOROV (1989), consider a distance-regular graph G with the same parameters as the dual polar graph for $U(2d,r)$ (that is, the graph with as vertices the maximal totally isotropic subspaces in $PG(2d-1,q)$ provided with a nondegenerate Hermitean form, where $q = r^2$; two vertices are adjacent when they meet in a subspace of codimension one in both). The graph G is regular of valency $k = rm$, its local graphs $\Gamma(x)$ are regular of valency $\lambda = r-1$, and G has smallest eigenvalue $\theta_d = -m$ with multiplicity $f = k+1-m$, where $m = (q^d - 1)/(q-1)$. Using the idempotent E_d we find a representation of G in $\mathbb{R}^f$, and then of $\Gamma(x)$ in $\mathbb{R}^{f-1}$, with inner products $(\overline{x}, \overline{y}) = Q_{id} = c((-r)^d - 1)/(-r-1)$ (for some positive constant c) whenever $d(x,y) = i$. Now the Gram matrix M of the representation of $\Gamma(x)$ has eigenvalue 0 with multiplicity at least m, so that $\Gamma(x)$ has eigenvalue λ with multiplicity at least m, so that $\Gamma(x)$ has at least m connected components. It follows that $\Gamma(x)$ is a union of cliques, and that our graph G is the collinearity graph of a near polygon with classical parameters. If $d \geq 3$ then every such G is known, and we find that G must be the $U(2d,r)$ dual polar graph.

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