Variations on a theme by Weetman

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Abstract

We show for many strongly regular graphs Δ, and for all Taylor graphs Δ except the hexagon, that locally Δ graphs have bounded diameter.

1 Locally Δ graphs

Let Δ be a graph. A graph Γ is called *locally* Δ if for each vertex x of Γ, the subgraph of Γ induced on the set $\Gamma(x)$ of neighbours of x (in Γ) is isomorphic to Δ. Locally Δ-graphs have been studied for many particular graphs Δ. (An extensive study for small graphs Δ can be found in Hall [H].) If Δ is finite, then the first question is whether locally Δ graphs must be of bounded size, or can be infinite. (By König's Lemma, if there are arbitrarily large locally Δ graphs, then there also are infinite locally Δ graphs.) Weetman [W0] showed that for regular graphs Δ of girth at least 6, there exist infinite locally Δ graphs Γ. On the other hand, in [W] he showed for many graphs Δ of diameter 2 and girth at most 5 that locally Δ graphs have bounded diameter. Here I want to extend Weetman's results a little bit.

2 Weetman's argument

Suppose that Γ is a graph that is locally Δ. We would like to derive a bound on the diameter $\operatorname{diam}\Gamma$ of Γ.

Let us call the graph Δ κ-*Weetman* if whenever x is a vertex of Δ, and C is a nonempty subgraph of Δ such that $d(x, C) \geq 2$ and C has minimal valency at least κ, then the number of vertices y with $y \sim x$ and $d(y, C) = 1$ is larger than κ. (Here $\sim$ denotes adjacency.) We call Δ *Weetman* if it is κ-Weetman for all $\kappa > 0$.

Note that Δ is 0-Weetman if and only if it has diameter at most 2.

Suppose $x_0 \sim x_1 \sim \ldots \sim x_i \sim \ldots$ is a geodesic in Γ. Define $C_i := \Gamma(x_i) \cap \Gamma_{i-1}(x_0)$, $A_i := \Gamma(x_i) \cap \Gamma_i(x_0)$ and $B_i := \Gamma(x_i) \cap \Gamma_{i+1}(x_0)$, where $\Gamma_j(x)$ denotes the set of vertices at distance j from x. If Δ is κ-Weetman, and C_i has minimal valency at least κ whenever $d(x_0, x_i) = i$, then locally at x_i we find that $|\Gamma(x_{i+1}) \cap A_i| \geq \kappa + 1$, and consequently C_{i+1} has minimal valency at least $\kappa + 1$.

It follows, for example, that if Δ is κ-Weetman for all $\kappa \geq 0$, then $\operatorname{diam}\Gamma \leq |\Delta|$. (For more precise statements, see Weetman [W].)

3 Strongly regular graphs

Suppose any two nonadjacent vertices of Δ have $\mu > 0$ common neighbours. Then C_2 is regular of valency μ, and we obtain a diameter bound when Δ is κ-Weetman for all $\kappa \geq \mu$.

Theorem 3.1 (Weetman [W]) *Let Δ be a strongly regular graph with parameters (v, k, λ, μ). Then Δ is Weetman whenever one of the following holds:*
(i) $v \leq 2k + 1$,
(ii) $\mu > \lambda$,
(iii) Δ is the collinearity graph of a partial geometry with parameters (s, t, α).

Proof: Let x be a vertex and C be a subgraph contained in $\Delta_2(x)$. If C has minimal valency κ, then it has at least $\kappa + 1$ vertices. For (i), observe that C and x have at least $|C|.\mu/(k - \lambda - 1) = |C|.k/(v - k - 1) \geq |C|$ common neighbours. For (ii), observe that if C' is a vertex z of C together with its (at least) κ neighbours, then C' and x have at least $|C'|.\mu/\max(\mu, \lambda + 1) = |C'|$ common neighbours. For (iii), observe that if z is a vertex of C and L is a line on z with at least $\kappa/(t+1)$ points in C distinct from z, then $L \cap C$ and x have at least $(1 + \kappa/(t+1)).(t+1) = t + 1 + \kappa$ common neighbours, if $\alpha > 1$. (Indeed, if M is a line on x then there are at least $|L \cap C|.\alpha/\alpha$ of these common neighbours on M in case $L \cap M = \emptyset$, and at least $|L \cap C|.(\alpha - 1)/(\alpha - 1)$ in case $L \cap M \neq \emptyset$ and $\alpha \neq 1$.) If $\alpha = 1$, then it does not suffice to look at L alone, but we can order the $t+1$ lines L_i on z and the $t+1$ lines M_i on x in such a way that $L_i \cap M_i = \emptyset$. We find $t + 1 + \kappa$ common neighbours of C and x: $t + 1$ common neighbours of z and x, and for each $y \in C \cap L_i$ a common neighbour of y and x on M_i. □

At least one of these criteria applies to each strongly regular graph on fewer than 16 vertices, and in fact Weetman [W] conjectured that all strongly regular graphs are Weetman. The first few cases not settled by the above proposition are $(16, 6, 2, 2)$ (the Shrikhande graph), $(28, 12, 6, 4)$ (the three Chang graphs), $(36, 15, 6, 6)$, $(45, 12, 3, 3)$, $(49, 18, 7, 6)$, $(57, 24, 11, 9)$.

Let us first improve condition (ii) above so as to include the case $\lambda = \mu$.

Theorem 3.2 *Let Δ be a strongly regular graph with parameters (v, k, λ, μ). If $\lambda = \mu$, then Δ is Weetman.*

Proof: Let x be a vertex of Δ, and let C be a subgraph of $\Delta_2(x)$ with minimal valency κ. Let A be the set of common neighbours of x and C, and assume that $|A| \leq \kappa$. For $z \in C$, let C' be the set of neighbours of z in C. Then x and C'

have at least $|C'|.\mu/\max(\lambda,\mu) = |C'|$ common neighbours. It follows that C is regular of valency κ. If $a \in A$ and $z \in C$ then the μ common neighbours of a and z lie in C. It follows that A is a coclique. Count paths $a \sim y \sim b$ for distinct $a, b \in A$, and $y \neq x$. There are $\kappa(\kappa-1)(\mu-1)$ such paths, and $|C|\mu(\mu-1)$ with $y \in C$. Since $\lambda = \mu = 1$ is impossible, it follows that $|C| \leq \kappa(\kappa-1)/\mu$. On the other hand, counting vertices at distance 0, 1 and 2 of a given vertex in C we find $|C| \geq 1 + \kappa + \kappa(\kappa - 1 - \lambda)/\mu$. Contradiction. □

Let Δ be a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue s. Let x be a vertex of Δ, and let C a subgraph of $\Delta_2(x)$ with minimal valency κ. (We may assume that C is connected.) Let A be the set of common neighbours of x and C. Assume that $|A| \leq \kappa$. Our aim is to derive a contradiction. (And we shall succeed in this at least for $v \leq 195$, so that every strongly regular graph on at most 195 vertices is Weetman.) Distinguish two cases, depending on whether the subgraph C is complete or not.

A. C is complete

Proposition 3.3 *Δ is μ-Weetman if and only if Δ does not contain a vertex x and a $(2\mu+1)$-clique D not containing x, such that $|D \cap \Delta(x)| = \mu$.*

Proof: If $\kappa = \mu$, then every vertex of C is adjacent to every vertex of A, so that A and C must be cliques. Thus, $D := A \cup C$ is the required $(2\mu+1)$-clique. □

Thus, if there is a strongly regular graph Δ with parameters $(400, 21, 2, 1)$, or, more generally, a strongly regular graph Δ with $\mu = 1$, $\lambda > 0$, then it is not 1-Weetman. The smallest v for which the existence of such D, x cannot be ruled out by the standard methods is $v = 196$, where the parameter set $(196, 39, 14, 6)$ for pseudo Latin square graphs $L_3(14)$ survived our attacks.

What are the standard methods?

Lemma 3.4 *Let D be a clique of size d in the strongly regular graph Δ. Then*

(i) $d \leq 1 + k/(-s)$, with equality if and only if each vertex x outside D has precisely $\mu/(-s)$ neighbours in D.

(ii) $(v-d) - d(k-d+1) + \binom{d}{2}(\lambda - d + 2) \geq 0$ with equality if and only if each vertex x outside D has either 1 or 2 neighbours in D.

(iii) $d \leq \lambda + 2$, and equality implies that each vertex x outside D has either 0 or 1 neighbours in D (so that in particular $v \geq d(k-d+2)$).

Proof: Part (i) is the Delsarte-Hoffman bound. Part (ii) follows by quadratic counting: Let x_i be the number of vertices outside D with i neighbours inside. Then $\sum x_i = v - d$ and $\sum i x_i = d(k-d+1)$ and $\sum \binom{i}{2} x_i = \binom{d}{2}(\lambda - d + 2)$. Now apply $\sum (i-1)(i-2)x_i \geq 0$. (In general, the strongest inequality is obtained by computing $\sum (i-p)(i-p-1)x_i \geq 0$, where $p = \lfloor \sum i x_i / \sum x_i \rfloor$. But in our applications only $p = 1$ occurred.) Part (iii) is clear. □

Proposition 3.5 *Let Δ, x and C be as above, and assume that C is a $(\kappa+1)$-clique. If x and C have at most κ common neighbours, then C is contained in a clique D of size $d = \kappa+1+a \geq \kappa+1+(\kappa+\mu)/(\kappa-\mu+2) \geq \mu+2\sqrt{2\mu-2}$, where x has a neighbours in D. Moreover, if $\mu > 1$, then $\lambda \geq d-1$.*

Proof: Let A be the set of common neighbours of x and C, and assume that $|A| \leq \kappa$. Let A_0 be the set of points in A adjacent to each point in C. If $A_0 = \emptyset$ then each point of A has at most μ neighbours in C, but $|C| > |A|$, contradiction. So $a := |A_0| > 0$. Any point of $A \setminus A_0$ is adjacent to at most $\mu - 1$ points of C. (Indeed, let $u \in A \setminus A_0$, $v \in A_0$, $w \in C$ where $w \not\sim u$. If u is adjacent to at least μ vertices of C, then, since u and v also have x as common neighbour, we find $u \sim v$. But then u and w have more than μ common neighbours, contradiction.) Hence we find $a(\kappa+1)+(\kappa-a)(\mu-1) \geq (\kappa+1)\mu$, i.e., $a(\kappa+2-\mu) \geq \kappa+\mu$, and we find a clique of size $\kappa+a+1$. If $a \geq 2$, then we also find $\lambda \geq \kappa+a$ by looking at an edge in A_0. $\square$

For example, in case $(126, 25, 8, 4)$ we have $s = -3$, and hence find that a clique has size at most 9. But if some clique has size 9 then each point outside has 1 or 2 neighbours inside. This rules out the case where C is complete.

B. C **is not complete**

We try to rule out the case where C is not complete by deriving contradictory upper and lower bounds on $c := |C|$. First of all we have an upper bound (for arbitrary C):

Lemma 3.6 *With notation as above, $|C| \leq (v-k-1).|A|/k$, with strict inequality when $\mu > 1$.*

Proof: Clearly, we have $|A| \geq \mu.|C|/(k-\lambda-1) = k.|C|/(v-k-1)$. If equality holds, then the common neighbours of two points $a, b \in \Delta(x)$, with $a \in A$ and $b \notin A$, must all lie in $\Delta(x) \cup \{x\}$. In particular, if $a \sim b$, then we find that each neighbour of a (resp. b) in $\Delta(x)$ is adjacent or equal to b (resp. a) since $\Delta(x)$ is regular of valency λ, and it follows that the edge ab lies in a $(\lambda+1)$-clique in $\Delta(x)$. But since $\Delta(x)$ is not complete, we can also take $a \not\sim b$. If $\mu > 1$, then for some $z \in \Delta(x)$ we have $a \sim z \sim b$, and by the above either az or bz lies in a $(\lambda+1)$-clique in $\Delta(x)$, contradiction, since such a clique is a connected component of $\Delta(x)$. $\square$

Next, a simple lower bound.

Lemma 3.7 *Let Δ, x and C be as above, and suppose that x and C do not have more than κ common neighbours. Assume that C is not complete (so that $\kappa > \mu$). Then $|C| \geq \max(2\kappa+2-\mu,\ 1+\kappa+\kappa(\kappa-\lambda-1)/\mu)$. If $\mu+1 \leq \kappa \leq 2\mu$, then we have the sharper bound $|C| \geq \max(\kappa+\mu+2,\ 1+\kappa+\kappa(2\mu-\lambda-1)/(\kappa-\mu))$.*

Proof: Let A be the set of common neighbours of x and C, so that $|A| \leq \kappa$. Note that any two adjacent (nonadjacent) vertices $y, z \in C$ have at least $\max(0,\, 2\mu - \kappa)$ common neighbours in A, and hence, if $\kappa \leq 2\mu$, at most $\lambda + \kappa - 2\mu$ (resp. $\kappa - \mu$) common neighbours in C. In both cases the first inequality follows by looking at two nonadjacent vertices of C, and the second by counting vertices at distance 0, 1 and 2 of a given vertex in C. $\square$

Sometimes the above lower bound can be sharpened by 1 or 2 by invoking induction on κ.

Lemma 3.8 *Let Δ, x and C be as above, and suppose that C is not complete and that x and C do not have more than κ common neighbours. If $|C| = 2\kappa + 2 - \mu$, or, more generally, if $|C|$ contains a vertex z such that $0 < |C \setminus (\{z\} \cup \Delta(z))| \leq \kappa + 1 - \mu$, then Δ is not $(\kappa - \mu)$-Weetman.*

Proof: Each nonneighbour of z in C has precisely μ common neighbours with z in C. Let D be this set of nonneighbours of z in C. Then D has minimal valency $\kappa - \mu$ and has at most $\kappa - \mu$ common neighbours with x (since the common neighbours of z and x are not adjacent to D). $\square$

Combining the above upper and lower bounds, we can improve the factor 2 in part (i) of Weetman's theorem above to $\frac{5}{2}$ (provided C is not complete).

Proposition 3.9 *Let Δ, x and C be as above, and assume that $v \leq \min(\frac{5}{2}k + \frac{k}{\mu} + 1,\, 3k + 1)$. If C is not complete, then x and C have more than κ common neighbours.*

Proof: Let A be the set of common neighbours of x and C, and assume that $|A| \leq \kappa$. Then $|C| \leq \frac{v-k-1}{k}\kappa$, with strict inequality if $\mu > 1$. On the other hand, if $\kappa \leq 2\mu$ then $|C| \geq \kappa + \mu + 2$. Combining this with the previous inequality, we find $(v - 2k - 1)\kappa \geq k(\mu + 2)$. Since we may assume that $v > 2k + 1$, this inequality holds in particular for $\kappa = 2\mu$, contradiction.

But if $\kappa \geq 2\mu$, then $|C| \geq 2 + 2\kappa - \mu$ and we find $(3k + 1 - v)\kappa \leq k(\mu - 2)$. Since we are assuming that $v \leq 3k + 1$, this inequality holds in particular for $\kappa = 2\mu$, contradiction again. $\square$

For example, in case (28,12,6,4) we have $s = -2$ so that cliques have size at most 7. But $\mu + 2\sqrt{2\mu - 2} > 8$, so C cannot be a clique. And $v < 5k/2$, so graphs with these parameters are Weetman. In the same way the cases (49,18,7,6) and (57,24,11,9) are ruled out.

Conclusion: all strongly regular graphs on fewer than 60 vertices are Weetman. The first few cases not settled by the above are $(64, 21, 8, 6)$, $(69, 20, 7, 5)$, $(81, 24, 9, 6)$, $(85, 14, 3, 2)$, $(85, 30, 11, 10)$ and $(100, 27, 10, 6)$, $(100, 33, 14, 9)$, $(100, 36, 14, 12)$.

The following proposition (combined with an upper bound for $|C|$) shows that we may assume that κ is small.

Proposition 3.10 *Let Δ, x and C be as above. Put $c := |C|$ and $l := v-k-1$. Then*

$$c\,(1+\frac{s+1}{l}) \geq 1+\kappa+\frac{l}{s+1}(1+\frac{\kappa s}{k}).$$

If equality holds, C is regular of valency κ.

Proof: Apply Delsarte's linear programming bound to $C \cup \{x\}$, where x gets (negative) weight $w := (s+1)c/l$ and the points of C get weight 1. We find $1 + \frac{\bar{\kappa}s}{k} + \frac{(c-\bar{\kappa}-1)(-s-1)}{l} \geq \frac{w^2}{c}$, i.e., $w + c \geq 1 + \bar{\kappa} + (1+\frac{\bar{\kappa}s}{k})\frac{l}{s+1}$, where $\bar{\kappa}$ is the average valency of C. Now apply $\bar{\kappa} \geq \kappa$. □

For example, in the case $(64, 21, 8, 6)$ we have $s = -3$, and by Proposition 3.5 C is not a clique. By this proposition we have $\kappa < 10$, and then Lemma 3.7 yields a contradiction. Thus, any such graph is Weetman.

On the other hand, we have an upper bound for $|C|$, since the pairs of vertices in A cannot have too many common neighbours:

Proposition 3.11 *Let Δ, x, C and A be as above, and let the subgraph induced on A have e edges. If $|A| = \kappa_0$, then*

$$|C|.\binom{\mu}{2}+\kappa_0\binom{2e/\kappa_0}{2}+(k-\kappa_0)\binom{(\kappa_0\lambda-2e)/(k-\kappa_0)}{2} \leq (\lambda-\mu)e+(\mu-1)\binom{\kappa_0}{2}.$$

Proof: Count 2-claws $y \sim a, b$, with $y \neq x$ and $a, b \in A$. The right hand side gives the total number, and the left hand side gives lower bounds for the number of such 2-claws with $y \in C$, $y \in A$ and $y \in \Delta \setminus A$, respectively. □

For example, in the case $(69, 20, 7, 5)$ we have $s = -3$, and Proposition 3.10 shows that any such graph is κ-Weetman for $\kappa \geq 10$. Again by Proposition 3.10 we find $|C| \geq 20$ if $\kappa = 9$. The above upper bound for $|C|$ then yields a contradiction for all e. Smaller κ are disposed of in a similar way, so we find that all such graphs are Weetman. In fact this proposition, together with the foregoing, shows that all strongly regular graphs on at most 100 vertices are Weetman, except possibly graphs with parameters $(85, 14, 3, 2)$. Tightening the bound just a little bit more is enough to settle also this last case.

Indeed, in the case $(85, 14, 3, 2)$ we find $\kappa = 5$ by the foregoing, and then $|C| \geq 12$ by Lemma 3.8, and then $|C| = 12$, $e = 3$ by Proposition 3.11. The following proposition rules out this possibility.

Proposition 3.12 *Let Δ be a strongly regular graph with parameters as above, and let D be a subgraph on p vertices with at least m edges. Then we have*

$$1+\frac{2m}{p}.\frac{s}{k}+(p-1-\frac{2m}{p}).\frac{-s-1}{v-k-1} \geq 0.$$

In particular, if Δ, x, C and A are as above, and $|A| \leq \kappa$, $|C| = c$, then this holds for $D = C \cup A$, with $p = c + \kappa$ and $m = c\kappa/2 + c\mu + e$ (if A has at least e induced edges).

Proof: Apply Delsarte's Linear Programming bound. □

This suffices to show that all strongly regular graphs on at most 195 vertices are Weetman.

4 Taylor graphs

A Taylor graph is a distance-regular antipodal 2-cover of a complete graph, that is, a distance-regular graph with intersection array $\{k, \mu, 1;\ 1, \mu, k\}$. Any Taylor graph Δ is locally strongly regular, and the only case in which it is locally a union of cliques is the case where Δ is a hexagon.

Lemma 4.1 *Let Δ be a Taylor graph. If Δ is not isomorphic to the hexagon, then Δ is Weetman.*

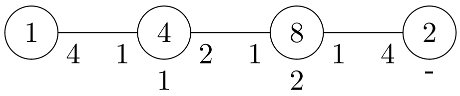
Proof: Let x be a vertex of Δ and C a subgraph of Δ such that $d(x, C) \geq 2$. Each vertex of $\Delta(x)$ has μ neighbours in $\Delta_2(x)$, and each vertex of $\Delta_2(x)$ has μ neighbours in $\Delta(x)$. Consequently, if C has minimal valency κ, and the set A of common neighbours of C and x has size at most κ, then C is a $(\kappa + 1)$-clique containing the antipode x' of x, and $|A| = \kappa$. If u, v are two neighbours of x', with $u \sim v$, $u \in C$, $v \notin C$, then u, v have no common neighbours in $\Delta(x)$, so have $\lambda - 1$ common neighbours in $\Delta(x')$. But the strongly regular subgraph $\Delta(x')$ has valency λ, and we see that it must be a union of cliques, contradiction. Thus, no such vertices u, v exist, and $C \setminus \{x'\}$ is regular of valency λ, so that $\lambda = \kappa - 1$, and C is a $(\lambda + 2)$-clique. If $w \in A$, then $\Delta(w) \cap \Delta(x')$ cannot contain an edge, so $\mu = 1$, and Δ is a hexagon, contrary to assumption. □

Theorem 4.2 *Let Δ be a Taylor graph, not the hexagon. Then the locally Δ graphs have bounded diameter.*

Proof: Suppose not. Then we can find for any natural number N a locally Δ graph Γ, and a geodesic $x_0 \sim x_1 \sim \ldots \sim x_N$ of length N in Γ. If x_0 and x_2 are not antipodes in $\Gamma(x_1)$, then the above lemma yields a contradiction for sufficiently large N. But if they are, then choose vertices y_i $(1 \leq i \leq N)$ in Γ such that $y_1 \sim x_0, x_1$ and $y_j \sim y_{j-1}, x_{j-1}, x_j$ $(2 \leq j \leq N)$. (This can be done since $\mu > 1$: y_j must be one of the μ common neighbours of x_j and y_{j-1} in $\Gamma(x_{j-1})$, but must be distinct from the antipode of x_{j+1} in $\Gamma(x_j)$.) The path $x_0 \sim y_1 \sim y_2 \sim \ldots \sim y_N \sim x_N$ has length $N + 1$, and either its first half or its last half is a geodesic. Thus, we can find arbitrarily long geodesics $z_0 \sim z_1 \sim z_2 \ldots$ such that z_1 is not isolated in the subgraph of common neighbours of z_0 and z_2.
□

5 Examples

Let Δ be the line graph of the Petersen graph, so that Δ is distance-regular with distance distribution diagram

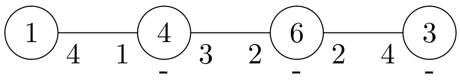


v=15.

Then there exist infinite graphs Γ that are locally Δ. Indeed, we can take for Γ the Cayley graph (with respect to the set S of generators) of the group G generated by the 15 vertices of Δ, with relations $e^2 = efg = 1$ for $e, f, g \in S$, forming a triangle in Δ. It is not difficult to check that G is infinite and that this Cayley graph is indeed locally Δ. [This example was shown to me by A.A. Ivanov, who referred to S.V. Shpectorov and D.V. Pasechnik.] Here Δ has diameter 3, and is an antipodal 3-cover of a complete graph. It has $\mu = 1$.

More generally, this argument shows that the line graph of a cubic graph such that any two disjoint edges are in disjoint circuits is the local graph of some infinite graph. (E.g., this applies to the line graph of the cube.)

Thus, it is still conceivable that all locally Δ graphs are finite whenever Δ is a (finite) distance-regular graph of diameter 3 with $\mu > 1$. In fact, in [BFS] it is shown that if Δ is the point-block incidence graph of the biplane on 7 points (with distance distribution diagram



v=14

), then there are precisely three locally Δ graphs (on 36, 48 and 108 vertices, respectively), while if Δ' is obtained from Δ by removing a single edge, then there are infinitely many finite (and hence also at least one infinite) locally Δ' graphs.

F. Buekenhout asked whether it is true that if all locally Δ graphs are finite, and Δ' is obtained from Δ by adding an edge, then all locally Δ' graphs are also finite. However, this is not the case. In [H] it is shown in (4.14.4) that if Δ is the graph obtained from the 2×3 grid by removing an edge not contained in a triangle, then the unique locally Δ graph is the graph $L(K_6 - 3K_2)$ on 12 vertices. And in (4.11) that if Δ' is the graph obtained from Δ by adding a diagonal to its unique quadrangle, then the locally Δ' graphs are precisely the distance 1-or-2-or-3 graphs of a finite or infinite cycle of length at least 10.

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