

# On $q$ -analogues and stability theorems

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**Abstract.** In this survey recent results about  $q$ -analogues of some classical theorems in extremal set theory are collected. They are related to determining the chromatic number of the  $q$ -analogues of Kneser graphs. For the proof one needs results on the number of  $0$ -secant subspaces of point sets, so in the second part of the paper recent results on the structure of point sets having few  $0$ -secant subspaces are discussed. Our attention is focussed on the planar case, where various stability results are given.

**Mathematics Subject Classification (2010).** 05D05, 05B25.

**Keywords.**  $q$ -analogues, Erdős-Ko-Rado theorem,  $q$ -Kneser graph, chromatic number, blocking set, stability results.

## 1. Introduction

In many branches of mathematics certain structures that have a numerical parameter (for example the size, or order of the structure) are studied. The first question is to bound the value of this parameter, the second is to find examples meeting the bound. Quite often the extremal examples are unique or one can classify them. A stability theorem says that when a structure is “close” to being extremal, then it can be obtained from an extremal one by changing it a little bit. In some cases we can get a clearer picture of the spectrum of the possible values of the numerical parameter by finding (or classifying) the second, third, ... extremal values of the numerical parameter.

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The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP 4.2.1./B-09/KMR-2010-0003. The third author gratefully acknowledges this support. The fourth author was partly supported by OTKA Grant K 81310. The first and fourth authors were partly supported by the ERC project No. 227701 DISCRETECONT.

### 1.1. Stability results in extremal graph theory

Let us first discuss some stability results in extremal graph theory, since this is the guiding model for such problems. This is not a complete survey at all, for a more complete one, see Simonovits [56].

The first results in extremal graph theory were about graphs containing no complete graphs on  $p+1$  vertices as a subgraph. Complete graphs on  $p+1$  vertices will be denoted by  $K_{p+1}$ . The maximum number of edges in a  $K_{p+1}$ -free graph was determined by Turán in the forties [68]. He also proved that his upper bound is attained if and only if the graph is the so-called Turán graph  $T(n, p)$ . This graph has  $n$  vertices, the vertices are partitioned in  $p$  almost equal classes and there is an edge between any two points in different classes. Here almost equal means that the sizes of the classes are either  $\lfloor n/p \rfloor$  or  $\lceil n/p \rceil$ . The Turán graph has roughly  $\binom{n}{2} - p\binom{n/p}{2} \sim (1 - \frac{1}{p})\binom{n}{2}$  edges. The exact number of edges of the Turán graph will be denoted by  $t(n, p)$ . The particular case  $p = 2$  of Turán's theorem was already discovered in 1907 by Mantel. It says that a triangle-free graph on  $n$  vertices has at most  $\lfloor n^2/4 \rfloor$  edges and in case of equality the graph is bipartite with almost equal classes. Turán's results were founding the field of extremal graph theory. Zykov re-discovered Turán's theorem in the late forties. The following theorem shows the stability of Turán graphs.

**Theorem 1.1.** (Erdős–Simonovits, see [56], first stability theorem) *For every  $\varepsilon > 0$  there is a  $\delta > 0$  and a threshold  $n(\varepsilon)$  such that if  $n > n(\varepsilon)$ , and the graph  $G_n$  on  $n$  vertices does not contain a  $K_{p+1}$  and the number of edges of  $G_n$  is greater than  $t(n, p) - \delta n^2$ , then  $G_n$  can be obtained from  $T(n, p)$  by changing (deleting and/or adding) at most  $\varepsilon n^2$  edges.*

Instead of the complete graph there is a similar theorem for any excluded subgraph  $H$  whose chromatic number is  $p+1$  ( $p \geq 2$ ). We will concentrate on the simplest case  $p = 2$ , that is the case of Mantel's theorem, and try to see what type of questions were posed (and answered) in extremal graph theory. One natural question is about the number of triangles. Rademacher proved that when a graph  $G_n$  on  $n$  vertices contains  $t(n, 2) + 1 = \lfloor \frac{n^2}{4} \rfloor + 1$  edges, then it contains at least  $\lfloor n/2 \rfloor$  triangles. Then Erdős [21, 22] proved that there is a positive constant  $c_1$  so that there is an edge which is contained in  $c_1 n$  triangles. He also proved the existence of another constant  $c_2 > 0$ , such that a graph  $G_n$  with  $t(n, 2) + k$  edges,  $k < c_2 n$ , has at least  $k \lfloor n/2 \rfloor$  triangles. The exact value of  $c_2$ , which is  $1/2$ , was found by Lovász and Simonovits [47, 48]. Finding a bound for the number of triangles is also meaningful if the graph  $G_n$  has fewer edges than the bipartite Turán graph  $T(n, 2)$ . Erdős [23] proved that when  $G_n$  has  $\lfloor n^2/4 \rfloor - \ell$  edges and contains at least one triangle then it contains at least  $\lfloor n/2 \rfloor - \ell - 1$  triangles. Another natural question is to determine a bound on  $\ell$  so that a graph  $G_n$  having  $\lfloor n^2/4 \rfloor - \ell$  edges and containing no triangles should be bipartite. It is not difficult to show that for  $\ell < \lfloor n/2 \rfloor - 1$  this is the case. This is a particular case of some results of Hanson and Toft [38]. More generally, it can be asked how many edges should

be deleted from a triangle-free graph to obtain a bipartite graph. Answers can be found in Erdős, Győri, and Simonovits [26]. Under additional local conditions on the graph  $G_n$  on  $n$  vertices one can obtain sharp results for the global structure of the graph. For example, Gallai proved that when  $G_n$  is triangle-free and the minimum degree is greater than  $2n/5$ , then  $G_n$  must be bipartite.

## 1.2. Stability results in finite geometry

In light of the previous results some classical theorems in finite geometry can also be considered as stability theorems. Usually they are called Segre type theorems by finite geometers. Also here this is not intended to be a complete survey. This means that we do not give the full history of the problems, just the starting point and the best results.

The Segre type stability theorems in finite geometry are one-sided in the sense that one assumes that the numerical parameter (typically the size of the set) is either smaller or larger than in case of the extremal examples and only deletion or addition of points is allowed (but not both) when we modify a nearly extremal point set to obtain an extremal one. In some cases we try to generalize the stability theorems given here in the spirit of the Erdős–Simonovits stability theorem.

Let us begin with some definitions and notation. The finite field with  $q$  elements ( $q = p^h$ ,  $p$  prime) will be denoted by  $\text{GF}(q)$ . We denote the projective (resp. affine) plane coordinatized over  $\text{GF}(q)$  by  $\text{PG}(2, q)$  (resp.  $\text{AG}(2, q)$ ). We say that the line  $\ell$  is an  $i$ -secant of a point set  $B$ , if  $\ell$  contains exactly  $i$  points of  $B$ . A 0-secant is also called a *skew line* or an *external line* (to  $B$ ) and a 1-secant is called a *tangent line*.

Let us continue with the classical example in finite geometry: Segre's theory of arcs, see [54, 55, 41]. An *arc* is a set of points no three of which are collinear. Using combinatorial arguments Bose proved that an arc in  $\text{PG}(2, q)$  can have at most  $q + 1$  or  $q + 2$  points according as  $q$  is odd or even. These values can be attained, a famous theorem of Segre shows that  $(q + 1)$ -arcs in  $\text{PG}(2, q)$ ,  $q$  odd are curves of degree two. For  $q$  even, one can add a point, the nucleus, to any  $(q + 1)$ -arc, thus obtaining a  $(q + 2)$ -arc. Very often  $(q + 1)$ -arcs are called *ovals*,  $(q + 2)$ -arcs are called *hyperovals*. For  $q$  even, there are nonisomorphic hyperovals if  $q > 8$ .

Then Segre went on to study nearly extremal arcs and proved the following stability theorem.

**Theorem 1.2.** (Segre [55]) *If  $A$  is an arc in  $\text{PG}(2, q)$  with  $|A| \geq q - \sqrt{q} + 1$  when  $q$  is even and  $|A| \geq q - \sqrt{q}/4 + 7/4$  when  $q$  is odd, then  $A$  is contained in an arc of maximum size (that is, in an oval or hyperoval).*

There are several improvements on Segre's bound, see Thas [66], Voloch [69, 70]. The result is sharp when  $q$  is an even square, see [18, 27, 10]. A complete bibliography can be found for example in Hirschfeld [41] or [42]. The paper of Hirschfeld and Korchmáros [42] also contains a beautiful stability type improvement of Segre's theorem for  $q$  even.

**Theorem 1.3.** (Hirschfeld and Korchmáros [42]) *For  $q$  even, let  $A$  be an arc in  $\text{PG}(2, q)$  with  $q - 2\sqrt{q} + 5 < |A| < q - \sqrt{q} + 1$ . Then  $A$  is contained in an arc of size  $q + 2$  or  $q - \sqrt{q} + 1$ .*

Besides arcs, other well-studied objects in finite geometry are blocking sets. A *blocking set* is a point set intersecting each line. It is called *minimal* when no proper subset of it is a blocking set. It is easy to see that the smallest blocking sets of projective planes are lines.

Using combinatorial arguments Bruen proved that the second smallest examples of minimal blocking sets of  $\text{PG}(2, q)$  (and more generally, planes of order  $q$ ) have at least  $q + \sqrt{q} + 1$  points. When  $q$  is a square, minimal blocking sets of this size exist; they are the points of a Baer subplane, that is a subplane of order  $\sqrt{q}$ . In general, minimal blocking sets of size  $\frac{3}{2}(q + 1)$  of  $\text{PG}(2, q)$  always exist if  $q$  is odd, see [5]. Similarly, in  $\text{PG}(2, q)$  there are minimal blocking sets of size  $\frac{3}{2}q + 1$ , when  $q$  is even, see [5].

There are lots of interesting results on blocking sets, for a survey see [5] and [58]. Many of them concentrate on *small* blocking sets of  $\text{PG}(2, q)$ , these are blocking sets whose cardinality is less than  $\frac{3}{2}(q + 1)$ . In some cases small minimal blocking sets are characterized.

**Theorem 1.4.** (1) (Blokhuis, [4]) *If  $q = p$  prime, then the small minimal blocking sets in  $\text{PG}(2, p)$  are lines;*  
 (2) (Szőnyi, [59]) *If  $q = p^2$ ,  $p$  prime, then the small minimal blocking sets in  $\text{PG}(2, p^2)$  are lines and Baer subplanes;*  
 (3) (Polverino, [53]) *If  $q = p^3$ ,  $p$  prime, then small minimal blocking sets in  $\text{PG}(2, p^3)$  have size  $q + 1 = p^3 + 1$ ,  $p^3 + p^2 + 1$  or  $p^3 + p^2 + p + 1$  and they are unique.*

In general, it is known that the sizes of small minimal blocking sets can take only certain values.

**Theorem 1.5.** (Szőnyi, [59]) *The size of a small minimal blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime, is 1 modulo  $p$ .*

There are important improvements on the above result, see Sziklai [57].

## 2. $q$ -analogues

In extremal combinatorics, the  $q$ -analogues of questions about sets and subsets are questions about vector spaces and subspaces. For the set case we denote the set of  $k$ -element subsets of an  $n$ -element sets  $X$  by  $\binom{X}{k}$ ; it has size  $\binom{n}{k}$ . For a prime power  $q$ , and an  $n$ -dimensional vector space  $V$  over  $\text{GF}(q)$ , let  $\left[ \begin{smallmatrix} V \\ k \end{smallmatrix} \right]$  denote the family of  $k$ -subspaces of  $V$ . This collection has size  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q$ , where

$$\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}$$

is the Gaussian (or  $q$ -binomial) coefficient. We shall omit the subscript  $q$  since it causes no ambiguity.

### 2.1. The set case

In 1961, Erdős, Ko and Rado [24] proved that if  $\mathcal{F}$  is a  $k$ -uniform intersecting family of subsets of an  $n$ -element set  $X$ , then  $|\mathcal{F}| \leq \binom{n-1}{k-1}$  when  $2k \leq n$ . Furthermore they proved that if  $2k + 1 \leq n$ , then equality holds if and only if  $\mathcal{F}$  is the family of all subsets containing a fixed element  $x \in X$ . A set system  $\mathcal{F}$  is called  $r$ -wise  $t$ -intersecting if for all  $F_1, \dots, F_r \in \mathcal{F}$  we have  $|F_1 \cap \dots \cap F_r| \geq t$ . For  $r = 2$  and  $t = 1$  the system is simply called *intersecting*.

Note that the non-uniform version of the EKR-theorem is an exercise: if  $\mathcal{F}$  is an intersecting set system on an  $n$ -element set, then  $|\mathcal{F}| \leq 2^{n-1}$ . Indeed, pair each subset  $A$  with its complement. Since  $\mathcal{F}$  is intersecting,  $\mathcal{F}$  can contain at most one element from each pair, hence it can contain at most half of the subsets. Of course, taking all subsets containing a fixed element gives an intersecting set system of size  $2^{n-1}$ . The same argument works in the  $k$ -uniform case for  $n = 2k$  and shows  $|\mathcal{F}| \leq \frac{1}{2} \binom{2k}{k} = \binom{2k-1}{k-1}$ .

For any family  $\mathcal{F}$  of sets the *covering number*  $\tau(\mathcal{F})$  is the minimum size of a set that meets all  $F \in \mathcal{F}$ . The result of Erdős, Ko and Rado states that to obtain an intersecting family of maximum size, one has to consider a family with  $\tau(\mathcal{F}) = 1$  when  $2k + 1 \leq n$ .

There are several versions of the Erdős, Ko, Rado theorem: it was extended to  $t$ -intersecting uniform set systems (see Deza [16]),  $r$ -wise intersecting set systems (see Frankl [28]). For a survey of this type of results, see Frankl [31] or Tokushige [67].

Hilton and Milner determined the maximum size of an intersecting family with  $\tau(\mathcal{F}) \geq 2$ .

**Theorem 2.1.** (Hilton and Milner [40]) *Let  $\mathcal{F} \subset \binom{X}{k}$  be a  $k$ -uniform intersecting family with  $k \geq 2$ ,  $n \geq 2k + 1$  and  $\tau(\mathcal{F}) \geq 2$ . Then  $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ . The families achieving that size are*

(i) *for any  $k$ -subset  $F$  and  $x \in X \setminus F$  the family*

$$\{F\} \cup \{G \in \binom{X}{k} : x \in G, F \cap G \neq \emptyset\},$$

(ii) *if  $k = 3$ , then for any 3-subset  $S$  the family*

$$\{F \in \binom{X}{3} : |F \cap S| \geq 2\}.$$

In the set case the *Kneser graph*  $K_{n:k}$  has vertex set  $\binom{X}{k}$ , where  $X$  is an  $n$ -element set. Two vertices are adjacent iff the corresponding  $k$ -sets are disjoint. Of course, for  $n < 2k$  the Kneser-graphs are just cocliques. Note that  $K_{5:2}$  is the Petersen graph.

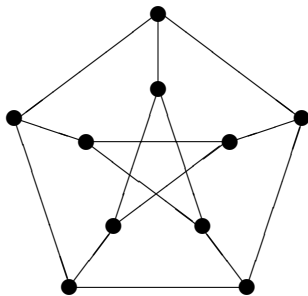


Figure 1

It is surprisingly difficult to determine the chromatic number of Kneser-graphs for  $n \geq 2k$ .

**Theorem 2.2.** (Lovász [46]) *For  $n = 2k + r$ ,  $r \geq 0$ , the chromatic number of  $K_{n:k}$  is  $r + 2$ .*

The proof uses topological methods. Simpler proofs are available now, see Bárány [1], Greene [37]. Recently an elementary (that is: not topological) proof was found by Matoušek [49].

## 2.2. Vector spaces

The vertex set of the  $q$ -Kneser graph  $qK_{n:k}$  is  $\binom{V}{k}$ , where  $V$  is an  $n$ -dimensional vector space over  $GF(q)$ . Two vertices of  $qK_{n:k}$  are adjacent if and only if the corresponding  $k$ -subspaces are disjoint (i.e., meet in 0).

In 1975, Hsieh [43] proved the  $q$ -analogue of the theorem of Erdős, Ko and Rado for  $2k + 1 \leq n$ . Greene and Kleitman [36] found an elegant proof for the case where  $k \mid n$ , settling the missing  $n = 2k$  case. Combining their results gives that the size of a maximum size coclique in  $qK_{n:k}$  is at most  $\binom{n-1}{k-1}$  for  $n \geq 2k$ . Hsieh's proof also showed that for  $n \geq 2k + 1$  (or  $n \geq 2k + 2$  when  $q = 2$ ), equality is achieved only for point-pencils, i.e., all  $k$ -subspaces through a given 1-dimensional subspace. A short proof of both the bound and the characterization that works also for  $q = 2$  and  $n = 2k + 1$  can be found in Godsil, Newman [35] and in [52].

The non-uniform version of the  $q$ -analogue of the EKR theorem is relatively easy if one uses the description of maximum size cocliques in  $qK_{2k:k}$ , given by Newman [52]. Note that in the set case maximum size cocliques have no structure for  $n = 2k$ : one can arbitrarily choose one  $k$ -set from each complementary pair of  $k$ -subsets.

**Theorem 2.3.** (Newman [52]) *The maximum size cocliques in  $qK_{2k:k}$  are either point-pencils or their duals.*

The dual of a point pencil is the set of  $k$ -subspaces contained in a hyperplane.

**Theorem 2.4.** ([8]) *Let  $\mathcal{F}$  be an intersecting family of subspaces of a vector space  $V$  of dimension  $n$ . Then*

(i) *if  $n$  is odd, then*

$$|\mathcal{F}| \leq \sum_{i > n/2} \binom{n}{i},$$

(ii) *if  $n$  is even, then*

$$|\mathcal{F}| \leq \binom{n-1}{n/2-1} + \sum_{i > n/2} \binom{n}{i}.$$

*For odd  $n$  equality holds only if  $\mathcal{F} = \left[ \begin{smallmatrix} V \\ >n/2 \end{smallmatrix} \right]$ . For even  $n$  equality holds only if  $\mathcal{F} = \left[ \begin{smallmatrix} V \\ >n/2 \end{smallmatrix} \right] \cup \{F \in \left[ \begin{smallmatrix} V \\ n/2 \end{smallmatrix} \right] : E \leq F\}$  for some  $E \in \left[ \begin{smallmatrix} V \\ 1 \end{smallmatrix} \right]$ , or if  $\mathcal{F} = \left[ \begin{smallmatrix} V \\ >n/2 \end{smallmatrix} \right] \cup \left[ \begin{smallmatrix} U \\ n/2 \end{smallmatrix} \right]$  for some  $U \in \left[ \begin{smallmatrix} V \\ n-1 \end{smallmatrix} \right]$ .*

Note that Theorem 2.4 also follows from the profile polytope of intersecting families which was determined implicitly by Bey [2] and explicitly by Gerbner and Patkós [34].

A family  $\mathcal{F}$  of  $k$ -subspaces of  $V$  is called  $t$ -intersecting if  $\dim(F_1 \cap F_2) \geq t$  for any  $F_1, F_2 \in \mathcal{F}$ . In 1986, Frankl and Wilson proved the following result giving the maximum size of a  $t$ -intersecting family of  $k$ -spaces for  $2k - t \leq n$ .

**Theorem 2.5.** (Frankl and Wilson [32]) *Let  $V$  be a vector space over  $GF(q)$  of dimension  $n$ . For any  $t$ -intersecting family  $\mathcal{F} \subseteq \left[ \begin{smallmatrix} V \\ k \end{smallmatrix} \right]$  we have*

$$|\mathcal{F}| \leq \binom{n-t}{k-t} \text{ if } 2k \leq n,$$

and

$$|\mathcal{F}| \leq \binom{2k-t}{k} \text{ if } 2k - t \leq n \leq 2k.$$

*These bounds are best possible.*

A more general approach, settling the case of equality in the Frankl-Wilson theorem, was found by Tanaka [63]. His approach also gives the Erdős-Ko-Rado theorem for several graphs, see [64, 65].

For  $r$ -wise intersecting systems of subspaces (that is when  $F_1 \cap \dots \cap F_r \neq 0$  for any  $F_1, \dots, F_r \in \mathcal{F}$ ) Chowdhury and Patkós proved the following result, which is the  $q$ -analogue of a theorem of Frankl [28].

**Theorem 2.6.** (Chowdhury and Patkós [15]) *Suppose  $\mathcal{F} \subset \left[ \begin{smallmatrix} V \\ k \end{smallmatrix} \right]$  is  $r$ -wise intersecting and  $(r-1)n \geq rk$ . Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

*Moreover, equality holds if and only if  $\mathcal{F} = \{F \in \left[ \begin{smallmatrix} V \\ k \end{smallmatrix} \right] : E \subset F\}$  for some fixed one-dimensional subspace  $E$ , unless  $r = 2$  and  $n = 2k$ .*

The introduction of [15] also contains some historical comments on these problems.

We shall discuss in more detail the  $q$ -analogue of Theorem 2.1 (Hilton–Milner theorem) and Theorem 2.2 (Lovász’ theorem).

Let the *covering number*  $\tau(\mathcal{F})$  of a family  $\mathcal{F}$  of subspaces of  $V$  be defined as the minimal dimension of a subspace of  $V$  meeting all elements of  $\mathcal{F}$  nontrivially.

Let us first remark that for a fixed 1-subspace  $E \leq V$  and a  $k$ -subspace  $U$  with  $E \not\leq U$  the family  $\mathcal{F}_{E,U} = \{U\} \cup \{W \in \binom{V}{k} : E \leq W, \dim(W \cap U) \geq 1\}$  is not maximal as we can add all subspaces in  $\binom{E+U}{k}$ . We will say that  $\mathcal{F}$  is an *HM-type family* if

$$\mathcal{F} = \left\{ W \in \binom{V}{k} : E \leq W, \dim(W \cap U) \geq 1 \right\} \cup \binom{E+U}{k}$$

for some fixed  $E \in \binom{V}{1}$  and  $U \in \binom{V}{k}$  with  $E \not\leq U$ . Note that the size of an HM-type family is

$$|\mathcal{F}| = f(n, k, q) := \binom{n-1}{k-1} - q^{k(k-1)} \binom{n-k-1}{k-1} + q^k. \quad (2.1)$$

The  $q$ -analogue of the theorem of Hilton–Milner is the following. (For brevity it will be called  $q$ -HM theorem.)

**Theorem 2.7.** ([8]) *Let  $V$  be an  $n$ -dimensional vector space over  $GF(q)$ , and let  $k \geq 3$ . If  $q \geq 3$  and  $n \geq 2k + 1$  or  $q = 2$  and  $n \geq 2k + 2$ , then for any intersecting family  $\mathcal{F} \subseteq \binom{V}{k}$  with  $\tau(\mathcal{F}) \geq 2$  we have  $|\mathcal{F}| \leq f(n, k, q)$  (with  $f(n, k, q)$  as in (2.1)). When equality holds, either  $\mathcal{F}$  is an HM-type family, or  $k = 3$  and*

$$\mathcal{F} = \mathcal{F}_3 = \{F \in \binom{V}{3} : \dim(S \cap F) \geq 2\}$$

for some  $S \in \binom{V}{3}$ .

Furthermore, if  $k \geq 4$ , then there exists an  $\epsilon > 0$  (independent of  $n, k, q$ ) such that if  $|\mathcal{F}| \geq (1-\epsilon)f(n, k, q)$ , then  $\mathcal{F}$  is a subfamily of an HM-type family.

If  $k = 2$ , then a maximal intersecting family  $\mathcal{F}$  of  $k$ -spaces with  $\tau(\mathcal{F}) > 1$  is the family of all lines in a plane, and the conclusion of the theorem holds.

The last assertion in the theorem shows the stability of the Hilton–Milner families. We mention that R. Ellis has similar (unpublished) results for  $t$ -intersecting systems of subspaces, see [20].

The  $q$ -HM theorem can be applied to determine the chromatic number of the  $q$ -analogue of the Kneser-graphs, that is the  $q$ -analogue of Lovász’ theorem.

**Theorem 2.8.** ([8]) *If  $k \geq 3$  and  $q \geq 3$ ,  $n \geq 2k + 1$  or  $q = 2$ ,  $n \geq 2k + 2$ , then for the chromatic number of the  $q$ -Kneser graph we have  $\chi(qK_{n:k}) = \binom{n-k+1}{1}$ . Moreover, each colour class of a minimum colouring is a point-pencil and the points determining a colour are the points of an  $(n - k + 1)$ -dimensional subspace.*

For  $k = 2$ , the chromatic number was determined earlier by Chowdhury, Godsil, and Royle [14]. For that case they proved  $\chi(qK_{n:k}) = \binom{n-1}{1}$  and characterized the minimal colourings, without any restriction on  $q$ .



In Lovász' theorem there is no difference between the  $n = 2k$  and  $n > 2k$  cases but in the vector space case the situation is different for  $n = 2k$ . In this case maximum size cocliques of the  $q$ -Kneser graph are described by Newman [52], see Theorem 2.3: *they are either point-pencils or their duals*. Point-pencils are denoted as  $P^*$ , their duals as  $H^*$ . Regarding the chromatic number of  $qK_{2k:k}$  we conjecture that it is always  $q^k + q^{k-1}$ . One can indeed colour  $qK_{2k:k}$  with this number of colours. For example, fix a  $(k+1)$ -subspace  $T$  and a cover of  $T$  with points and  $k$ -subspaces. A proper colouring of  $qK_{2k:k}$  is obtained by taking all families  $P^*$  where  $P$  is one of the points in this cover, and all families  $H^*$  where  $H$  is a hyperplane that contains some  $k$ -subspace in this cover. If we fix a  $(k-1)$ -subspace  $S$  in  $T$  and take  $s$   $k$ -subspaces on  $S$ , where  $1 \leq s \leq q$ , and cover the rest with points, then we have  $(q+1-s)q^{k-1}$  colours of type  $P^*$  and  $sq^{k-1}$  colours of type  $H^*$  where  $H$  does not contain  $T$ , and these suffice. So the chromatic number  $\chi$  satisfies  $\chi \leq (q+1-s)q^{k-1} + sq^{k-1} = q^k + q^{k-1}$ .

Very recently we obtained the following result. The special case  $k = 2$  is already contained in a paper by Eisfeld, Storme, and Sziklai [19].

**Theorem 2.9.** ([9]) *If  $k < q \log q - q$  then the chromatic number of  $qK_{2k:k}$  equals  $q^k + q^{k-1}$ .*

In the same paper we also show that a minimal colouring of  $qK_{2k:k}$  that only uses colour classes of type  $P^*$  and  $H^*$ , must be one of the examples given at the beginning of the previous section. Again, for  $k = 2$  this was shown already in [19].

**Proposition 2.10.** ([9]) *Let  $\mathcal{P}$  be a set of points and  $\mathcal{H}$  a set of hyperplanes such that  $\{P^* \mid P \in \mathcal{P}\} \cup \{H^* \mid H \in \mathcal{H}\}$  is a colouring of  $qK_{2k:k}$  where  $k \geq 2$ . Then  $|\mathcal{P}| + |\mathcal{H}| \geq q^k + q^{k-1}$ . If equality holds, then  $\mathcal{P}$  and  $\mathcal{H}$  are nonempty, no  $H \in \mathcal{H}$  contains a  $P \in \mathcal{P}$ , and there are a  $(k-1)$ -space  $S$  and a  $(k+1)$ -space  $T$  containing  $S$ , such that  $\mathcal{P} \subset T \setminus S$  and  $\bigcap \mathcal{H} \supseteq S$ .*

For the proof, one needs a Hilton-Milner type theorem also for  $n = 2k$ . Actually, this is the point where for  $n = 2k$  the condition  $k < q \log q - q$  enters.

**Theorem 2.11.** *Let  $\mathcal{F}$  be a maximal coclique in  $qK_{2k:k}$  of size*

$$|\mathcal{F}| > \left(1 + \frac{1}{q}\right) \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-1} \begin{bmatrix} k-1 \\ 1 \end{bmatrix}.$$

*Then  $\mathcal{F}$  is an EKR family  $P^*$  or a dual EKR family  $H^*$ .*

For  $k \gg q \log q$  the statement in the theorem is empty, since then the right hand side is larger than the size of an EKR-family.

For  $k = 3$ , we managed to prove a much more precise result for maximal cocliques than the analogue of Hilton-Milner theorem.

**Theorem 2.12.** ([9]) *Let  $V = V(6, q)$  be a 6-dimensional vector space over  $GF(q)$ . Let  $\mathcal{F}$  be a maximal intersecting family of planes in  $V$ . Then we have one of the following four cases:*

(i)  $|\mathcal{F}| = \binom{5}{2} = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ , and  $\mathcal{F}$  is either the collection  $P^*$  of all planes on a fixed point  $P$ , or the collection  $H^*$  of all planes in a fixed hyperplane  $H$  of  $V$ .

(ii)  $|\mathcal{F}| = 1 + q(q^2 + q + 1)^2 = q^5 + 2q^4 + 3q^3 + 2q^2 + q + 1$ , and  $\mathcal{F}$  is either the collection  $\pi^*$  of all planes that meet a fixed plane  $\pi$  in at least a line, or the collection  $(P, S)^*$  of all planes that are either contained in the solid  $S$ , or contain the point  $P$  and meet  $S$  in at least a line (where  $P \subset S$ ), or  $\mathcal{F}$  is the collection  $(L, H)^*$  of all planes that either contain the line  $L$ , or are contained in the hyperplane  $H$  and meet  $L$  (where  $L \subset H$ ).

(iii)  $|\mathcal{F}| = 3q^4 + 3q^3 + 2q^2 + q + 1$  and  $\mathcal{F}$  is the collection  $(P, \pi, H)^*$  of all planes on  $P$  that meet  $\pi$  in a line, and all planes in  $H$  that meet  $\pi$  in a line, and all planes on  $P$  in  $H$  (where  $P \subset \pi \subset H$ ).

(iv)  $\mathcal{F}$  is smaller.

This theorem suggests that a (close to) complete classification of all maximal cocliques in  $qK_{6;3}$  might be feasible. For general  $k$  the situation is of course more complicated, but there is the following nice correspondence with the thin case, conjectured by Brouwer, and proved by Jan Draisma [11].

Let  $V = V(n, q)$  and fix a basis  $v_i$ ,  $i = 1, \dots, n$  of  $V$ . For each subset  $I$  of  $\{1, \dots, n\}$  define the subspace  $V_I = \langle v_i : i \in I \rangle$ . Given a maximal coclique  $C$  in the ordinary Kneser graph  $K_{n;k}$  ( $n \geq 2k$ ), let  $C(q)$  be the collection of all  $k$ -dimensional subspaces of  $V$  that intersect all  $V_I$ ,  $I \in C$ .

**Theorem 2.13.** (J. Draisma) *The set  $C(q)$  is a maximal coclique in  $qK_{n;k}$ .*

*Proof.* Let  $\bigwedge V$  be the exterior algebra of  $V$ . Map  $k$ -subspaces of  $V$  to projective points in  $P(\bigwedge V)$  via

$$\psi : U = \langle u_1, \dots, u_k \rangle \mapsto \langle u_1 \wedge \dots \wedge u_k \rangle.$$

Now  $U \cap U' \neq 0$  if and only if  $\psi U \wedge \psi U' = 0$ . For  $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$  with  $i_1 < \dots < i_k$ , let  $v_K = v_{i_1} \wedge \dots \wedge v_{i_k}$ . The  $v_K$  form a basis for (the degree  $k$  part of)  $\bigwedge V$ . The  $k$ -space  $U$  with  $\psi U = \langle u \rangle$ , where  $u = \sum \alpha_K v_K$ , intersects  $V_I$  for all  $I \in C$  if and only if  $\alpha_K = 0$  whenever  $K$  is disjoint from some  $I \in C$ . (Indeed, if  $I \cap K = \emptyset$ , then the coefficient of  $v_{I \cup K}$  in  $u \wedge v_I$  is  $\pm \alpha_K$ .) Since  $C$  is maximal, this condition is equivalent to  $\psi U \in \langle v_I \mid I \in C \rangle$ . If  $U, U' \in C(q)$ , then  $\psi U \in \langle v_I \mid I \in C \rangle$  implies  $\psi U \wedge \psi U' = 0$  so that  $U \cap U' \neq 0$ .  $\square$

For example, if  $C$  is the Hilton-Milner example (case (i) of Theorem 2.1), then  $C(q)$  is an HM-type family.

Theorem 2.9 above shows that the chromatic number of  $qK_{6;3}$  equals  $q^3 + q^2$  for  $q \geq 5$ . In fact, the restriction on  $q$  is superfluous.

**Theorem 2.14.** ([9]) *The chromatic number of the graph  $qK_{6;3}$  equals  $q^3 + q^2$ .*

Let us now return to the proof of Theorem 2.8. If only point-pencils are used in the colouring as colour classes then the result follows immediately from the following result of Bose and Burton.

**Theorem 2.15.** (Bose & Burton [3]) *If  $V$  is an  $n$ -dimensional vector space over  $GF(q)$  and  $\mathcal{E}$  is a family of 1-subspaces of  $V$  such that any  $k$ -subspace of  $V$  contains at least one element of  $\mathcal{E}$ , then  $|\mathcal{E}| \geq \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$ . Furthermore, equality holds if and only if  $\mathcal{E} = \begin{bmatrix} H \\ 1 \end{bmatrix}$  for some  $(n-k+1)$ -subspace  $H$  of  $V$ .*

If we have fewer than  $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$  point pencils, then the uncoloured  $k$ -spaces are coloured by smaller colour classes (recall that  $n > 2k$ ). So we need a lower bound on the number of uncoloured  $k$ -subspaces and hope that it contradicts the bound in the  $q$ -analogue of Hilton-Milner theorem. Let us first give two natural extensions of the Bose-Burton result, each of which can be used in the proof.

**Proposition 2.16.** *If  $V$  is an  $n$ -dimensional vector space over  $GF(q)$  and  $\mathcal{E}$  is a family of  $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} - \varepsilon$  1-subspaces of  $V$ , then the number of  $k$ -subspaces of  $V$  that are disjoint from all  $E \in \mathcal{E}$  is at least  $\varepsilon q^{(k-1)(n-k+1)} / \begin{bmatrix} k \\ 1 \end{bmatrix}$ .*

*Proof.* Induction on  $k$ . For  $k = 1$  there is nothing to prove. Next, let  $k > 1$  and count incident pairs (1-space,  $k$ -space), where the  $k$ -space is disjoint from all  $E \in \mathcal{E}$ :

$$N \begin{bmatrix} k \\ 1 \end{bmatrix} \geq \left( \begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} + \varepsilon \right) \varepsilon q^{(k-2)(n-k+1)} / \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \geq \varepsilon q^{(k-1)(n-k+1)}.$$

□

Another version of the previous result is the following.

**Proposition 2.17.** *If  $V$  is an  $n$ -dimensional vector space over  $GF(q)$  and  $\mathcal{E}$  is a family of  $\begin{bmatrix} m \\ 1 \end{bmatrix}$  1-spaces, then the number of  $l$ -spaces disjoint from all  $E \in \mathcal{E}$  is at least  $N(m, l, n) = q^{lm} \begin{bmatrix} n-m \\ l \end{bmatrix}$ , the number of  $l$ -spaces disjoint from an  $m$ -space, with equality for  $l = 1$  if and only if the elements of  $\mathcal{E}$  are all different, and for  $l > 1$  if and only if  $\mathcal{E}$  is the set of 1-subspaces in an  $m$ -space.*

*Proof.* Induction on  $l$ . For  $l = 1$  there is nothing to prove. For  $l > 1$  take a 1-space  $P \notin \mathcal{E}$ . By induction, the number of  $l$ -spaces on  $P$  disjoint from all  $E \in \mathcal{E}$  is at least  $N(m, l-1, n-1)$ , and varying  $P$  we find at least  $N(m, l-1, n-1) \left( \begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} m \\ 1 \end{bmatrix} \right) / \begin{bmatrix} l \\ 1 \end{bmatrix} = N(m, l, n)$   $l$ -spaces. If we have equality, then the elements of  $\mathcal{E}$  are all different in the local space at  $P$ , for every  $P \notin \mathcal{E}$ , and we have a subspace (of dimension  $m$ ). □

Actually, a slightly better, but precise bound was found by Metsch, using some results of Szőnyi and Weiner [71] that will be discussed in the next section.

**Theorem 2.18.** (Metsch [50]) *If  $V$  is an  $n$ -dimensional vector space over  $GF(q)$  and  $\mathcal{E}$  is a family of  $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} - \varepsilon$  1-subspaces of  $V$ , then the number of  $k$ -subspaces of  $V$  that are disjoint from all  $E \in \mathcal{E}$  is at least  $\varepsilon q^{(k-1)(n-k)}$ .*

### 3. Stability of blocking sets

#### 3.1. Planar results

A *blocking set*  $B$  of  $\text{PG}(2, q)$  is a set of points intersecting each line in at least one point. Lines intersecting  $B$  in exactly one point are called *tangents*. A point is *essential* to  $B$ , if it is on at least one tangent. The blocking set is *minimal* if all of its points are essential. Geometrically this means that there is a tangent line at each point. The minimal blocking set  $B$  is *small*, if  $|B| < 3(q+1)/2$ .

If we delete few (say,  $\varepsilon$ ) points from a blocking set, then we get a point set intersecting almost all except of few (at most  $\varepsilon q$ ) lines. A point set which is “close” to be a blocking set is a point set that intersects almost all lines. However, it may happen that we delete a lot of points from a minimal blocking set and yet the number of external lines is small. The reason for this is that there are blocking sets having very few tangents at each point. The number of tangents at each point can be 1, which happens for blocking semiovals. For example, there are semiovals which are unions of parabolas for  $q \equiv 1 \pmod{4}$  (see Section 3.3 in [33]). If we delete  $\varepsilon$  parabolas completely, then the resulting set has  $\varepsilon q$  0-secants and to block these lines one needs at least  $q/2$  points. This shows that we have to be careful by choosing sensible bounds in a stability theorem for sets having few external lines. For blocking sets of size at most  $2q$  the problem indicated above does not occur, so it is natural to try to formulate stability results for such blocking sets. The simplest case is to consider the stability of lines, that is to look for conditions that guarantee the existence of a large collinear subset in a set intersecting all except of few (at most  $\varepsilon q$ ) lines.

**Theorem 3.1.** (Erdős–Lovász [25]) *A point set of size  $q$  in a projective plane of order  $q$ , with fewer than  $\sqrt{q+1}(q+1-\sqrt{q+1})$  0-secants always contains at least  $q+1-\sqrt{q+1}$  points from a line.*

Note that this result is essentially sharp, since if we delete  $\sqrt{q}+1$  points from a Baer subplane, then the resulting set will have exactly  $q\sqrt{q}-q$  skew lines and it has at most  $\sqrt{q}+1$  collinear points. The proof is combinatorial, so the result is valid for any projective plane of order  $q$ . The result can easily be extended to sets of size  $q+k$ , where  $k \leq \sqrt{q}+1$ . For the sake of completeness, we prove this more general result here.

**Theorem 3.2.** (Erdős–Lovász [25]) *If  $S$  is a set of  $q+k$  points a projective plane of order  $q$  and the number of 0-secants is less than  $(\lfloor\sqrt{q}\rfloor+1-k)(q-\lfloor\sqrt{q}\rfloor)$ , where  $k \leq \sqrt{q}+1$ , then the set contains at least  $q+k-\lfloor\sqrt{q}\rfloor+1$  collinear points.*

The result is sharp for  $q$  square: deleting  $\sqrt{q}+1-k$  points from a Baer subplane gives this number of 0-secants.

*Proof.* Let  $P_1, \dots, P_{q+k}$  be the points of  $S$ , let  $\delta$  be the number of 0-secants. List the lines meeting  $S$ :  $L_1, \dots, L_{q^2+q+1-\delta}$ , and let  $e_i = |S \cap L_i|$ ,  $i > 0$ .

Order the lines in such a way that  $e_1 \geq e_2 \geq \dots$ . The standard counting argument gives

$$\sum_{i>0} e_i = (q+k)(q+1),$$

$$\sum_{i>0} e_i(e_i - 1) = (q+k)(q+k-1).$$

Using these inequalities we see that

$$\sum_{i>0} e_i(e_i - 1) \leq e_1 \sum_{i>0} (e_i - 1) = e_1((q+k)(q+1) - (q^2 + q + 1 - \delta)).$$

Looking at the set  $S$  from a point of  $L_1 \setminus S$  we see that  $\delta \geq (q + 1 - e_1)(e_1 - k)$  which gives that for  $\lfloor \sqrt{q} \rfloor + 1 \leq e_1 \leq q + k - \lfloor \sqrt{q} \rfloor$  we have  $\delta \geq (q - \lfloor \sqrt{q} \rfloor)(\lfloor \sqrt{q} \rfloor + 1 - k)$ .

When  $e_1 \leq \sqrt{q}$ , the previous inequality can be used and it implies that

$$\delta \geq (\lfloor \sqrt{q} \rfloor + 1 - k)(q - \lfloor \sqrt{q} \rfloor) + k(\lfloor \sqrt{q} \rfloor - 1) + (k - 1) \frac{k}{\sqrt{q}}.$$

The two bounds for  $\delta$  give the assertion in the theorem. □

This bound is purely combinatorial, similarly to the bound on the size of blocking sets, due to Bruen [13]. From now on, we shall work in Galois planes  $PG(2, q)$  (or  $AG(2, q)$ ). Let us begin with the Jamison, Brouwer-Schrijver theorem which can also be considered as a stability theorem for sets having exactly one 0-secant.

**Theorem 3.3.** (Jamison [44], Brouwer and Schrijver [12]) *A blocking set in  $AG(2, q)$  contains at least  $2q - 1$  points.*

The next result helps us estimate the number of 0-secants we get by deleting an essential point from a small blocking set. It is a consequence of Theorem 3.3.

**Proposition 3.4.** (Blokhuis and Brouwer [6]) *Let  $B$  be a blocking set in  $PG(2, q)$ ,  $|B| = 2q - s$  and let  $P$  be an essential point of  $B$ . Then there are at least  $s + 1$  tangents through  $P$ .*

Hence if we delete  $\varepsilon$  essential points from a small blocking set then we get at least  $\varepsilon(q/2)$  skew lines.

Let us first see an extension of the Erdős-Lovász theorem for Galois planes of prime order.

**Theorem 3.5.** ([62]) *Let  $B$  be a set of points of  $PG(2, q)$ ,  $q = p$  prime, with at most  $\frac{3}{2}(q+1) - \beta$  points, where  $\beta > 0$ . Suppose that the number  $\delta$  of 0-secants is less than  $(\frac{2}{3}(\beta + 1))^2/2$ . Then there is a line that contains at least  $q - \frac{2\delta}{q+1}$  points.*

The proof uses the lacunary polynomial method by Blokhuis [4] together with ideas from Szőnyi [60].

**Remark 3.6.** 1) Note that deleting  $\varepsilon$  points from a blocking set of size  $3(q+1)/2$  (for example, from a projective triangle) would give rise to a point set with few (roughly  $\varepsilon q/2$ ) 0-secants not containing a large collinear subset.

2) In the case corresponding to the Erdős, Lovász theorem, that is when  $|B| = q$ , we can allow roughly  $q^2/18$  0-secants to guarantee a collinear subset of size at least  $8q/9$  in  $B$ . The bound  $q^2/18$  can most likely be improved but we do need an upper bound of the form  $cq^2$ , ( $c \leq 1/2$ ) on the number of 0-secants as the constructions below show.

3) The result gives a non-trivial bound even for sets of size less than (but close to)  $q$ . For more details, see [62].

In the paper [62] the reader can find several examples for sets that have few 0-secants. All the examples can be obtained from a blocking set contained in the union of three lines by deleting quite a few points. In some cases they have less 0-secants than a set of the same size contained in the projective triangle. The examples also show that we cannot expect  $\frac{\delta}{q+1}$  missing points from a line in Theorem 3.5. Here we only give explicitly the simplest constructions, for more details and more general constructions the reader is referred to [62].

**Construction 3.7.** ([62]) *Assume that  $3|(q-1)$  and let  $H$  be a subgroup of  $GF(q)^*$ ,  $|H| = \frac{q-1}{3}$ . Furthermore, let  $B$  be the set of size  $q+2$ , where*

$$B = \{(0, h) | h \in H\} \cup \{(h, 0) | h \in H\} \cup \{(h) | h \in H\} \cup \{(0, 0)\} \cup (0) \cup (\infty).$$

*Then the number of 0-secants to  $B$  is  $\frac{2}{9}(q-1)^2$ . Add  $k < \frac{q+17}{6}$  ideal points not in  $B$  to obtain  $B'$ . Then the total number of 0-secants to  $B'$  is  $(\frac{2}{3}(q-1) - k)\frac{1}{3}(q-1)$ .*

In general, one could choose a multiplicative subgroup  $H$  (of size  $\frac{q-1}{t}$ ) from the line  $Y = 0$ ,  $s$  cosets of  $H$  from the line  $X = 0$ , and the same  $s$  cosets from the ideal line. For  $t > 2$  one can achieve that the set is not contained in a projective triangle and has as many 0-secants as one could get by deleting  $(\frac{q-1}{2} - \frac{q-1}{2s})$  appropriate points from a projective triangle.

**Construction 3.8.** ([62]) *Let  $A$  and  $B$  be less than  $p$  and let  $B^*$  be the following set.*

$$B^* = \{(1, a) | 0 \leq a \leq A\} \cup \{(0, -b) | 0 \leq b \leq B\} \cup \{(\infty)\} \cup \{(c) | 0 \leq c \leq A+B\}.$$

*Then  $B^*$  has  $2(A+B) + 4$  points and the total number of 0-secants to  $B^*$  is  $(q-1-A-B)(q-A-B-2)$ .*

For  $A = B = \frac{p}{4}$ , the number of 0-secants of  $B^*$  is roughly  $\frac{p^2}{4}$ , which is what one would get for a set contained in a projective triangle.

There are more general stability theorems for relatively small blocking sets. For example, the famous theorem by Jamison, Brouwer-Schrijver on affine blocking sets of  $AG(2, q)$  says that when the number of 0-secants is precisely one, then the set has to have at least  $2q-1$  points. The following result uses the affine blocking set theorem. The lower bound  $\frac{7}{6}q$  on  $|B|$  comes

from the fact that in this case the number of 0-secants is at most  $2q$ , hence they can form a  $(\delta, 3)$ -arc.

**Proposition 3.9.** (1) *Assume that for the size of  $B$ ,  $\frac{3}{2}q - 2 \leq |B| \leq 2q - 2$  holds and  $\delta < 2(2q - 1 - |B|)$ . Then  $B$  can be obtained from a blocking set by deleting at most one point.*

(2) *If  $\frac{7}{6}q \leq |B| < \frac{3}{2}q - 3$ , and  $\delta \leq 3(2q - 1 - |B|) - (q + a + 1)/2$ , (where  $a = 1$  for  $q$  even,  $a = 0$  for  $q$  odd), then  $B$  can be obtained from a blocking set by deleting at most two points.*

*Proof.* (1) can be found in [7]. For the sake of completeness we recall the proof based on Theorem 3.3. If not all 0-secants pass through a point, then one can block the  $\delta$  0-secants, with precisely one exception  $\ell$ , by at most  $\delta/2$  points. Then we get a blocking set of size  $|B| + \delta/2$  in the affine plane obtained by taking  $\ell$  as the line at infinity. It has at least  $2q - 1$  points, so we get  $\delta \geq 2(2q - 1 - |B|)$ , contradicting our assumption.

(2) Our aim is to add few points to  $B$  so that we obtain an affine blocking set. To do this, let us block the 0-secants greedily. There are two possibilities: either for some  $k \geq 3$  we end up with  $k$  concurrent lines, so we find an affine blocking set of size at most  $|B| + \lfloor \frac{\delta - k}{k} \rfloor + (k - 1)$  or we block at least three lines at each step until a dual  $h$ -arc is obtained,  $h \leq q + 1 + a$ . In the former case the size of the affine blocking set is clearly less than  $2q - 1$ . In the latter case the remaining  $h$  lines can be blocked by  $h/2$  points with precisely one exception. In total we need at most  $(\delta - h)/3 + h/2$  points, which is less than  $2q - 1 - |B|$ .  $\square$

Part (1) is best possible since  $q - 1$  points on a fixed line together with  $m$  points on a different line on one of the earlier points leave  $h = 2(q - m)$  0-secants, and equality holds.

When the size of the set is close to  $q$ , we have better bounds.

**Theorem 3.10.** ([61]) *Let  $B$  be a point set in  $\text{PG}(2, q)$ ,  $q \geq 81$ , of size less than  $\frac{3}{2}(q + 1)$ . Denote the number of 0-secants of  $B$  by  $\delta$ , and assume that*

$$\delta < \min \left( (q - 1) \frac{2q + 1 - |B|}{2(|B| - q)}, \frac{1}{3}q\sqrt{q} \right). \tag{3.2}$$

*Then  $B$  can be obtained from a blocking set by deleting at most  $\frac{2\delta}{q}$  points.*

When  $|B|$  is close to  $\frac{7}{6}q$ , then the bound in Proposition 3.9 is weaker than the one in Theorem 3.10. The point when the two bounds are the same is roughly  $|B| = (1 + c)q$ , where  $c$  is the smaller root of  $6c^2 - 6c + 1 = 0$ , that is when  $c$  is  $(3 - \sqrt{3})/6 = 0,211\dots$ , so we could have put this as a lower bound on  $|B|$  in Proposition 3.9. Because of the min in the bound for  $\delta$ , this result is weaker than the Erdős-Lovász theorem, when  $|B| < q + \frac{2}{3}\sqrt{q}$ .

Let us finish this part with a stability result for Baer subplanes.

**Theorem 3.11.** ([61]) *Let  $B$  be a point set in  $\text{PG}(2, q)$ ,  $81 \leq q$  a square, with cardinality  $q + k$ ,  $0 \leq k \leq \sqrt{q}$ . Assume that the number,  $\delta$  of skew lines of  $B$  is less than  $(q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$ , where  $c$  is an integer,*

$1 \leq c \leq (\sqrt{4 - 2\sqrt{2}} - 1)\sqrt{q} - 4$ . Then  $B$  contains at least  $q + 1 - (\sqrt{q} - k + c)$  points from a line or at least  $q + k - c + 1$  points from a Baer subplane.

The theorem looks complicated but it says something relatively simple. If we delete  $\sqrt{q} + 1 - k + c$  points from a Baer subplane and add  $c$  points outside then we get a set of size  $q + k$  which has roughly  $(q - \sqrt{q}) \cdot c$  more 0-secants than a set obtained from a Baer subplanes by deleting just  $\sqrt{q} + 1 - k$  points. More precisely, the resulting set has at least  $(q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$  0-secants. The theorem says that when we have less than this number of 0-secants then the set either contains a large collinear subset or it is indeed obtained by deleting less than  $\sqrt{q} + 1 - k + c$  points from, and adding less than  $c$  point outside to, the subplane. In the theorem  $c$  can be a small constant times  $\sqrt{q}$ .

### 3.2. Higher dimensions

Almost nothing is known in higher dimensions. Of course, Propositions 2.16 and 2.17 can be regarded as relatively simple one sided stability theorems but they are not even sharp as Proposition 2.18 shows. Besides these we can only mention the following result of Dodunekov, Storme, and Van de Voorde [17].

**Theorem 3.12.** (Dodunekov, Storme, and Van de Voorde [17]) *If  $S$  is a set of  $q + k$  points in  $\text{PG}(n, q)$ , ( $k < (q - 2)/3$ ), and there are at most  $q^{n-1}$  skew hyperplanes to  $S$ , then there are at least  $q^{n-1} - kq^{n-2}$  skew hyperplanes and they pass through a point. If  $q$  is a prime then  $S$  contains  $q$  collinear points.*

The proof uses the following useful remark.

**Theorem 3.13.** (Dodunekov, Storme, and Van de Voorde [17]) *The number of tangent hyperplanes through an essential point of a blocking set  $B$  of size  $q + k + 1$ ,  $|B| \leq 2q$ , in  $\text{PG}(n, q)$  is at least  $q^{n-1} - kq^{n-2}$ .*

Recently, Harrach and Storme started to investigate systematically higher dimensional extensions of the planar stability results. As an illustration we mention the following result (in the dual version).

**Theorem 3.14.** (Harrach and Storme [39]) *Let  $S$  be a set of points in  $\text{PG}(n, q)$ ,  $q \geq 143$ ,  $|S| \leq q + \sqrt{q}$ . If the number of skew hyperplanes to  $S$  is at most  $\lfloor \frac{1}{3}\sqrt{q} - 1 \rfloor q^{n-1}$ , then  $S$  can be completed to a blocking set of  $\text{PG}(n, q)$  by adding at most  $\frac{1}{3}\sqrt{q}$  points.*

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