The number of dominating sets of a finite graph is odd

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Let Γ be a finite graph with vertex set $V = V\Gamma$. A subset D of V is called *dominating* when each vertex in $V \setminus D$ has a neighbour in D. The following theorem answers a question by S. Akbari.

Theorem *The number of dominating sets of a finite graph is odd.*

Today, there are three proofs, by Andries Brouwer, Péter Csorba and Lex Schrijver, respectively. Let us give all three.

First proof: Let us write S^+ for the set of vertices in S or with a neighbour in S. By induction on $|V|$, and for fixed $|V|$ on $|S|$, we prove the following two claims for $S \subseteq V$:

(i) $\#\{D \mid S \subseteq D \subseteq V,\ D^+ = V\} \equiv \#\{E \mid E \subseteq V,\ E^+ = V \setminus S\} \pmod 2$,
(ii) $\#\{D \mid D \subseteq V \setminus S,\ D^+ = V\} \equiv \#\{E \mid E \subseteq V,\ V \setminus S \subseteq E^+\} \pmod 2$.

Indeed, if $S = \emptyset$ both (i) and (ii) are trivial. Assume $S \neq \emptyset$.

Let $U = S^+ \setminus S$ and $W = V \setminus S$. Then (i) is equivalent to

(i') $\#\{D \mid D \subseteq W,\ W \setminus U \subseteq D^+\} \equiv \#\{E \mid E \subseteq W \setminus U,\ E^+ = W\} \pmod 2$.

for $U \subseteq W$. But this is precisely (ii), with W instead of V, and since $|W| < |V|$ this holds by induction. This proves (i).

If we sum the equality (ii) over all $S \subseteq T$, where $T \subseteq V$, the left hand side counts pairs (D, S) with $D^+ = V$ and $S \subseteq T \setminus D$, so that each D is seen $2^{|T \setminus D|}$ times, which is 0 (mod 2) except when $T \subseteq D$. The right hand side counts pairs (E, S) with $V \setminus T \subseteq V \setminus S \subseteq E^+$, so that each E is seen $2^{|E^+ \setminus (V \setminus T)|}$ times, which is 0 (mod 2) except when $E^+ = V \setminus T$. The result is

$$\#\{D \mid T \subseteq D \subseteq V,\ D^+ = V\} \equiv \#\{E \mid E \subseteq V,\ E^+ = V \setminus T\} \pmod 2$$

which is precisely (i), but using the variable T instead of S. Since (i) holds, and by induction (ii) holds for all proper subsets S of T, it follows that (ii) also holds for $S = T$. This completes the proof of (i) and (ii).

Now we can prove the theorem. If $V = \emptyset$ then there is precisely one dominating set. Otherwise, let $x \in V$ and put $W = V \setminus x$ and $S = N(x)$, the set of neighbours of x. The dominating sets in V are the dominating sets D in W that intersect S, and the sets $E \cup \{x\}$ where $E \subseteq W$ with $W \setminus S \subseteq E^+$. By induction, the number of dominating sets (of the graph $\Gamma \setminus x$) in W is odd. Adding equation (ii) (with W instead of V) yields the desired conclusion. $\square$

Second proof: Let $n > 0$, and look at the simplicial complex P of all nonempty non-dominating sets. The Euler characteristic $\chi(P)$ is an alternating sum, and mod 2 one has $|P| = \chi(P)$. The Euler characteristic of a simplicial complex equals that of its barycentric subdivision. In this case that means that we go to the simplicial complex of all chains in the poset P.

Let $f(A)$ be the set of all vertices of Γ not equal or adjacent to anything in A. If A is non-dominating, then also $f(A)$ is non-dominating, and f defines a Galois correspondence so that f^2 is a closure operator.

Consider an increasing chain $C = (A_1, ..., A_m)$ in P. If all A_j in C are closed, then pair C with $(f(A_1), ..., f(A_m))$. Otherwise, if A_j is the last non-closed element in the chain, and $f^2(A_j) = A_{j+1}$ then pair C with $C \setminus A_{j+1}$, otherwise pair C with $C \cup f^2(A_j)$.

This pairing shows that the complex of all chains in the poset P has an even number of vertices, and hence $|P|$ is even. Including the empty set we see that the total number of non-dominating sets is odd, and therefore the number of dominating sets is odd. □

Third proof: Let

$$A := \{(S,T) \mid S,T \subseteq V,\ S \cap T = \emptyset,\ s \not\sim t \text{ for all } s \in S, t \in T\}.$$

A subset S of V is dominating precisely when $\#\{T \mid (S,T) \in A\}$ is odd, and hence the number of dominating sets equals $|A|$ (mod 2). But $(S,T) \in A$ iff $(T,S) \in A$, and $(S,T) = (T,S)$ only if $S = T = \emptyset$, so $|A|$ is odd. □