The number of dominating sets of a finite graph is odd

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1 Dominating sets

Let Γ be a finite graph with vertex set $V = V\Gamma$. A subset D of V is called *dominating* when each vertex in $V \setminus D$ has a neighbour in D. The following theorem answers a question by S. Akbari.

Theorem 1.1 *The number of dominating sets of a finite graph is odd.*

Proof: Let

$$A := \{(S,T) \mid S,T \subseteq V,\ S \cap T = \emptyset,\ s \not\sim t \text{ for all } s \in S, t \in T\}.$$

A subset S of V is dominating precisely when $\#\{T \mid (S,T) \in A\}$ is odd, and hence the number of dominating sets equals $|A|$ (mod 2). But $(S,T) \in A$ iff $(T,S) \in A$, and $(S,T) = (T,S)$ only if $S = T = \emptyset$, so $|A|$ is odd. □

Along the same lines one can give the actual number of dominating sets. Given a subset R of V, let $c(R)$ be the number of connected components of the subgraph of Γ induced by Γ on R.

Proposition 1.1 *The number of dominating sets in Γ is equal to $\sum_R 2^{c(R)}$, where the sum is over all subsets R such that the subgraph induced by Γ on R has no connected components of odd size.*

Proof: Let A be as above. For fixed S, the sum $\sum_{(S,T)\in A}(-1)^{|T|}$ is 1 when S is dominating, and 0 otherwise. Hence the number of dominating sets equals $\sum_{(S,T)\in A}(-1)^{|T|} = \sum_R \sum_{(S,T)\in A,\, R=S\cup T}(-1)^{|T|}$. The inner sum counts subsets T of R that are unions of connected components, where unions of even size count for 1 and unions of odd size for -1. So, given R, the sum is nonzero only when all connected components of R are even, and in that case the sum equals $2^{c(R)}$. □

This reproves the theorem since $\sum_R 2^{c(R)}$ has precisely one odd term.

Proposition 1.2 *Let $0 < m < 2^n$, m odd. Then there exists a graph Γ on n vertices with precisely m dominating subsets.*

Proof: Apply induction on n. If Γ has m dominating subsets, then consider the graphs Γ' and Γ'' on $n+1$ vertices obtained by adding a new vertex that is isolated (for Γ') or joined to all old vertices (for Γ''). Then Γ' and Γ'' have m and $2^n + m$ dominating subsets, respectively. □

2 Simplicial complexes

The arguments used fit naturally in the setting of simplicial complexes. A *simplicial complex* P here is a finite nonempty collection of finite nonempty sets such that if $\emptyset \neq X \subset Y \in P$ then $X \in P$. The *Euler characteristic* $\chi(P)$ is defined by $\chi(P) = \sum_{X \in P} (-1)^{|X|}$.

The *barycentric subdivision* of a simplicial complex P is the simplicial complex of which the elements are the nonempty chains (subsets totally ordered by inclusion) in P.

Second proof of the theorem. Let Γ be a graph on n vertices, $n > 0$, and look at the simplicial complex P of all nonempty non-dominating sets. The Euler characteristic $\chi(P)$ is an alternating sum, and mod 2 one has $|P| = \chi(P)$. The Euler characteristic of a simplicial complex equals that of its barycentric subdivision.

For any nonempty set S of vertices of Γ, let $f(S)$ be the set of all vertices of Γ not equal or adjacent to anything in S. If S is non-dominating, then also $f(S)$ is non-dominating, and f defines a Galois correspondence so that f^2 is a closure operator. (That is, for all S we have $S \subseteq f^2(S)$ and $f(S) = f^3(S)$. The set S is called *closed* if $S = f^2(S)$.)

Consider an increasing chain $C = (S_1, ..., S_m)$ in P. If all S_j in C are closed, then pair C with $(f(S_1), ..., f(S_m))$. Otherwise, if S_j is the last non-closed element in the chain, and $f^2(S_j) = S_{j+1}$ then pair C with $C \setminus S_{j+1}$, otherwise pair C with $C \cup f^2(S_j)$.

This pairing shows that the complex of all chains in the poset P has an even number of vertices, and hence $|P|$ is even. Including the empty set we see that the total number of non-dominating sets is odd, and therefore the number of dominating sets is odd. □

The simplicial complex P used in this proof is related to the neighbourhood complex $\mathcal{N}(\Delta)$ of a graph Δ as introduced by Lovász [1]. Indeed, the simplices of $\mathcal{N}(\Delta)$ are the nonempty subsets with a common neighbour in Δ, so that our P is $\mathcal{N}(\overline{\Gamma})$, the neighbourhood complex of the complementary graph.

References

[1] L. Lovász, *Kneser's Conjecture, Chromatic Number, and Homotopy*, J. Comb. Th. (A) (1978) **25** 319–324.