

Lie Algebras, 2-Groups and Cotriangular Spaces

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Abstract

We describe the construction of a Lie algebra from a partial linear space with oriented lines of size 3, generalizing a construction by Kaplansky. We determine all suitable partial linear spaces and the resulting Lie algebras.

1 Lie oriented partial linear spaces

A *partial linear space* is an incidence structure (P, L) with *points* and *lines*, such that the point-line incidence graph does not contain quadrangles. Let (P, L) be a partial linear space with lines of size 3, and let \mathbb{F} be a field. Construct an algebra $\mathcal{L} := \mathcal{L}_{\mathbb{F}}(P, L)$ with bilinear multiplication on the \mathbb{F} -vector space \mathbb{F}^P with basis P by defining the multiplication on the basis:

$$xy = \begin{cases} z & \text{if } \ell = \{x, y, z\} \in L, \\ 0 & \text{otherwise } (x = y \text{ or } x, y \text{ noncollinear}). \end{cases}$$

Kaplansky [9] used this construction for symplectic spaces over fields of order 2, and found some new classes of Lie algebras in characteristic 2 (see also Rotman and Weichsel [11, 12]). The algebra \mathcal{L} is commutative. So, \mathcal{L} can only be a Lie algebra if the characteristic of \mathbb{F} is equal to 2. In [5] it has been proven that the construction produces a Lie algebra precisely in the cases studied by Kaplansky: (P, L) is a subspace of the partial linear space of hyperbolic lines of a symplectic space over \mathbb{F}_2 .

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If one associates a cyclic orientation to the lines of the partial linear space, one might obtain Lie algebras over fields of arbitrary characteristic different from 2. In this paper, we investigate the Lie algebras arising from this construction.

Let (P, L) be a partial linear space with lines of size 3, where each line $\ell = \{x, y, z\} \in L$ is provided with a cyclic orientation $\sigma(\ell)$, one of the two cyclic permutations (x, y, z) or (y, x, z) . For a field \mathbb{F} of characteristic different from 2 construct an algebra $\mathcal{L} := \mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ with bilinear multiplication on the \mathbb{F} -vector space \mathbb{F}^P with basis P by defining the multiplication on the basis:

$$[x, y] = \begin{cases} z & \text{if } \ell = \{x, y, z\} \in L \text{ and } \sigma(\ell) = (x, y, z), \\ -z & \text{if } \ell = \{x, y, z\} \in L \text{ and } \sigma(\ell) = (y, x, z), \\ 0 & \text{otherwise (} x = y \text{ or } x, y \text{ noncollinear)}. \end{cases}$$

We call \mathcal{L} the *Kaplansky algebra* of the *oriented* partial linear space (P, L, σ) over \mathbb{F} . This algebra has a bilinear and antisymmetric multiplication, but does not necessarily satisfy the Jacobi identity.

If $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ is a Lie algebra for some field \mathbb{F} of characteristic different from 2, then (P, L, σ) is called a *Lie oriented partial linear space* and σ is called a *Lie orientation* on (P, L) . If (P, L, σ) is a Lie oriented partial linear space for some σ , then we call (P, L) *Lie orientable*.

1.1 The examples

A partial linear space is called *connected* when its point-line incidence graph is connected. The Kaplasky algebra of a partial linear space is a direct sum of Kaplansky algebras associated to the connected components of the partial linear space.

As we shall see below, the connected Lie orientable partial linear spaces are of four types, that we describe here.

- (a) $\mathcal{T}(\Omega, \Omega')$, the partial linear space obtained from two disjoint sets Ω, Ω' by taking as points the subsets A of $\Omega \cup \Omega'$ with $|A \cap \Omega| = 2$, and as lines the triples $\{A, B, C\}$ of points, where $A + B + C = 0$ in the binary vector space $2^{\Omega \cup \Omega'}$.
- (b) $\mathcal{S}p(V, f)$, the partial linear space obtained from a vector space V over \mathbb{F}_2 provided with a symplectic form f by taking as points the vectors outside the radical of f , and as lines the hyperbolic lines.
- (c) $\mathcal{O}(V, Q)$, the partial linear space obtained from a vector space V over \mathbb{F}_2 provided with a quadratic form Q by taking as points the vectors where Q is nonzero and that lie outside the radical of f , the symplectic form associated to Q , and as lines the elliptic lines.

- (d) $\mathbb{P}V \setminus \mathbb{P}W$, the partial linear space obtained by taking, for a vector space V over \mathbb{F}_2 with subspace W of codimension 3, the points of $\mathbb{P}V$ (the projective space associated to V) that are not in $\mathbb{P}W$, and the lines of $\mathbb{P}V$ that are disjoint from $\mathbb{P}W$. Here $\mathbb{P}W$ can be empty.

1.2 Flipping

Let $p \in P$. Define σ_p by

$$\sigma_p(\ell) = \begin{cases} \sigma(\ell) & \text{if } p \notin \ell, \\ (\sigma(\ell))^{-1} & \text{if } p \in \ell. \end{cases}$$

Then $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ and $\mathcal{L}_{\mathbb{F}}(P, L, \sigma_p)$ are isomorphic by the algebra isomorphism that maps p to $-p$ and fixes all other basis elements. So, if we simultaneously reverse the orientation of all lines in L through one point p , we end up with an isomorphic algebra. This operation will be called *flipping at p* . Similarly, we can define flipping at a subset S of P as the operation that reverses the orientation of a line l exactly $|l \cap S|$ times. Note that these automorphisms of the algebra generate an elementary abelian 2-group.

Two orientations of (P, L) are called *flipping equivalent*, if one can be obtained from the other by a flipping.

1.3 Main result

Our main result is the following theorem.

Theorem 1.1 *Let (P, L, σ) be a connected Lie oriented partial linear space. Then (P, L) is isomorphic to one of the spaces (a) $\mathcal{T}(\Omega, \Omega')$, (b) $\mathcal{S}p(V, f)$, (c) $\mathcal{O}(V, Q)$, (d) $\mathbb{P}V \setminus \mathbb{P}W$ as described above. Conversely, each of these spaces is Lie orientable, and admits up to flipping a unique Lie orientation σ . For any field \mathbb{F} of characteristic different from 2 the resulting Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ is a Lie algebra, provided that in case (d) the field \mathbb{F} has characteristic 3.*

In the final section of this paper we identify the Lie algebras among the Kaplansky algebras with some classical Lie algebras.

The families (a)–(c) also occur in Kaplansky's original construction. Part of the results of this note were given earlier in [10].

2 Binary orthogonal geometries

A *subspace* of a partial linear space (P, L) is a subset S of the point set with the property that each line meeting S in at least 2 points is contained in S . We often identify a subspace S with the partial linear space with point set S whose lines are the lines of (P, L) contained in S .

If (P, L) is equipped with an orientation then S is naturally equipped with an orientation by restricting the orientation to the lines in S . One easily checks that the Kaplansky algebra of a subspace of (P, L) is isomorphic to the subalgebra of the Kaplansky algebra generated by the points of the subspace.

In this section we prove that the partial linear spaces of types (a), (b), and (c) are Lie orientable. As each of the spaces in (a) and (b) is isomorphic to a subspace of some partial linear space $\mathcal{O}(V, Q)$ of type (c), see for example [8], it suffices to handle case (c).

2.1 A Lie orientation on the orthogonal geometries

Let (P, L) be the partial linear space $\mathcal{O}(V, Q)$ and let $\phi : V \times V \rightarrow \mathbb{F}_2$ be a (not necessarily symmetric) bilinear form with the property that $\phi(v, v) = Q(v)$ for all $v \in V$. For $\ell = \{x, y, z\} \in L$ define $\sigma = \sigma_\phi$ by

$$\sigma(\ell) = \begin{cases} (x, y, z) & \text{if } \phi(x, y) + \phi(y, z) + \phi(z, x) = 0; \\ (x, z, y) & \text{if } \phi(x, y) + \phi(y, z) + \phi(z, x) = 1. \end{cases}$$

As a line $l \in L$ consists of a triple of nonsingular vectors, we have $\phi(x, y) + \phi(y, x) = \phi(x + y, x + y) + \phi(x, x) + \phi(y, y) = 1$ so that σ is well defined.

Since $x + y + z = 0$ we have $\phi(x, x) + \phi(x, y) + \phi(x, z) = 0$ so that $\phi(x, y) = \phi(y, z) = \phi(z, x)$.

The multiplication on the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ can now be expressed as follows. For any $u, v \in P$ we have

$$[u, v] = Q(u + v) \cdot (-1)^{\phi(u, v)} \cdot (u + v),$$

where $+$ is addition inside V .

Proposition 2.1 *For (P, L, σ) as above, the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ is a Lie algebra for every field \mathbb{F} . In particular, σ is a Lie orientation.*

Proof. Let \mathbb{F} be a field. To prove the lemma, it suffices to check that the Jacobi identity holds in the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$.

Let u, v, w be three points in P . Then

$$\begin{aligned} [[u, v], w] &= [Q(u + v) \cdot (-1)^{\phi(u, v)} \cdot (u + v), w] \\ &= Q(u + v + w) \cdot (-1)^{\phi(u + v, w)} \cdot Q(u + v) \cdot (-1)^{\phi(u, v)} \cdot (u + v + w) \\ &= Q(u + v + w) \cdot Q(u + v) \cdot (-1)^{\phi(u, v) + \phi(v, w) + \phi(w, u) + 1} \cdot (u + v + w). \end{aligned}$$

Inside the subspace of V generated by u, v , and w we easily check that either $Q(u + v + w) = 0$ or $Q(u + v) + Q(v + w) + Q(w + u) = 0$, and the Jacobi identity $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$ follows. \square

2.2 E_2 -groups

The above construction of σ is closely related to a well-known construction of extra-special groups and the following generalization from an orthogonal space, see [7].

Definition 2.2 A group E is called an E_2 -group, if it is a 2-group containing a normal subgroup Z of order 2 such that E/Z is elementary abelian. The subgroup Z of E is called the *scalar subgroup* of E .

Indeed, given a binary quadratic space (V, Q) and bilinear form $\phi : V \times V \rightarrow \mathbb{F}_2$ with $\phi(v, v) = Q(v)$ we can equip the set $E = V \times \mathbb{F}_2$ with the following multiplication for (v, ϵ) and (w, η) in E :

$$(v, \epsilon) \cdot (w, \eta) = (v + w, \epsilon + \eta + \phi(v, w)).$$

This multiplication turns E into an E_2 -group with scalar subgroup $Z = \{0\} \times \mathbb{F}_2$. Hall [7] shows that, up to isomorphism, every E_2 -group can be obtained in this way.

The group E carries enough information to recover the oriented orthogonal geometry as described above. Indeed, the cyclic subgroups of order 4 and quaternion subgroups of order 8 of E are in one-one correspondence (and hence can be identified) with the points and lines, respectively, of $\mathcal{O}(V, Q)$. This partial linear space is therefore called the *geometry of E* . The orientation σ defined above on the orthogonal geometry $\mathcal{O}(V, Q)$ can be recovered as follows. If p, q and r are three collinear points, with generators $\tilde{p} = (v, 0)$, $\tilde{q} = (u, 0)$ and $\tilde{r} = (w, 0)$, respectively, then the orientation of the line equals (p, q, r) if $\tilde{p}\tilde{q} = \tilde{r}$ and (p, r, q) otherwise.

The connection between the group E and the Kaplansky Lie algebra \mathcal{L} obtained from the orthogonal of E can be made explicit by the use of the group algebra $\mathbb{F}[E]$ of E .

Indeed, we can identify the Kaplansky Lie algebra \mathcal{L} with a subalgebra of the Lie algebra associated to the group algebra of E . To this end we use the following construction by Plesken as described in [2].

Let G be a group and $\mathbb{F}[G]$ its group algebra. Then the Lie bracket $[\cdot, \cdot] : \mathbb{F}[G] \rightarrow \mathbb{F}[G]$ defined by

$$[g, h] = gh - hg, \text{ for all } g, h \in G,$$

defines a Lie algebra on the vector space $\mathbb{F}[G]$ which we denote by $\mathbb{F}[G]_{\text{Lie}}$. For $g \in G$, denote the element $g - g^{-1}$ of $\mathbb{F}[G]$ by \hat{g} . Then

$$[\hat{g}, \hat{h}] = \widehat{gh} - \widehat{gh^{-1}} - \widehat{g^{-1}h} + \widehat{g^{-1}h^{-1}}.$$

So, the linear subspace spanned by \hat{g} for $g \in G$ of $\mathbb{F}[G]$ is a Lie subalgebra of $\mathbb{F}[G]_{\text{Lie}}$, which we denote by $\mathcal{L}_{\mathbb{F}}(G)$. This Lie (sub)algebra will be called the *Plesken Lie algebra* of G .

We turn our attention to the Plesken Lie algebra of the E_2 -group E . The nonzero generators of this Lie algebra are those \hat{g} where g has order 4. If e is an element of order 4, then $\hat{e} = (1 - z)e$, where $z = (0, 1)$ is the involution of the scalar subgroup Z . Note that $1 - z$ is a central element of the group algebra, and

$$(1 - z)^2 = 1 - 2z + z^2 = 2(1 - z).$$

Since $ze = e^{-1}$, we also have $z\hat{e} = e^{-1} - e = -\hat{e}$. Furthermore, we can obtain a basis of $\mathcal{L}_{\mathbb{F}}(E)$ by taking the algebra elements corresponding to a set of representatives for the points (i.e., subgroups of order 4) of E . Let R be the set of elements of order 4 of the form $(v, 0)$ with $v \in V$. As we have seen above, these elements satisfy the following rules in the Plesken Lie algebra: If e, f and g are three elements of R with $ef = z^\alpha g$ for some $\alpha \in \{0, 1\}$, then

$$[\hat{e}, \hat{f}] = (1 - z)^2(ef - fe) = (1 - z)^2(z^\alpha g - z^{1+\alpha} g) = z^\alpha(1 - z)^3 g = 4 \cdot (-1)^\alpha \hat{g}.$$

If we identify the subgroup $\langle r \rangle$ for $r \in R$ with the corresponding point of the geometry of E , then the above shows that the map $\hat{r} \mapsto 4\langle r \rangle$ for $r \in R$ induces an isomorphism between the Plesken Lie algebra of E and the Kaplansky Lie algebra of the geometry of E , provided the characteristic of \mathbb{F} is not two. Thus we have obtained the following result.

Proposition 2.3 *Let E be an E_2 -group and \mathbb{F} a field of characteristic different from 2. Then the Plesken Lie algebra of E over the field \mathbb{F} is isomorphic to the Kaplansky Lie algebra of the geometry of E over the field \mathbb{F} .*

If \mathbb{F} is of characteristic 2 and we take E as in the proposition, then the ordinary multiplication on the subalgebra $\langle eZ \mid e \in R \rangle_{\mathbb{F}}$ of $\mathbb{F}[E/Z]$ turns out to satisfy the Jacobi identity. This subalgebra is then isomorphic to the Kaplansky Lie algebra of the oriented partial linear space of E ; see [5].

3 Octonions and projective geometries

In this section we describe a class of Lie algebras arising as Kaplansky algebras from the geometry of points and lines of a binary projective space missing some fixed codimension 3 subspace; see 1.1(d).

We start with the Fano plane. It is well known that the 7 lines of the Fano plane can be oriented in such a way that the corresponding Kaplansky algebra over a field \mathbb{F} is isomorphic to an algebra of octonions over \mathbb{F} modulo its center $\mathbb{F} \cdot 1$, e.g. see [1, 3].

Indeed, if the 7 points of the Fano plane are called e_1, \dots, e_7 , then the octonions can be described as the 8-dimensional algebra over \mathbb{F} with basis $1, e_1, \dots, e_7$ and multiplication defined by the orientation as given in Figure 3.1.

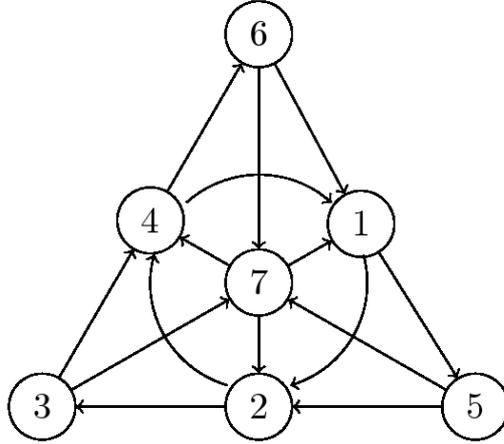


Figure 3.1: Oriented Fano plane

This orientation encodes the following multiplication table.

\times	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

For fields \mathbb{F} of characteristic 3, the induced multiplication on the quotient of the octonion algebra by its center $\mathbb{F} \cdot 1$ satisfies the Jacobi identity. In particular, the Fano plane is Lie orientable.

Next, let V be a binary vector space and let W be a subspace of V of codimension 3. Let $\mathbb{P}V$ and $\mathbb{P}W$ be the corresponding projective spaces and let $\mathbb{P}V \setminus \mathbb{P}W$ denote the geometry of points and lines of $\mathbb{P}V$ disjoint from $\mathbb{P}W$. Fix a subspace F_0 of the partial linear space $\mathbb{P}V \setminus \mathbb{P}W$ isomorphic to the Fano plane. Provide F_0 with a Lie orientation σ_0 . Then σ_0 can be extended to a Lie orientation σ on $\mathbb{P}V \setminus \mathbb{P}W$ in the following way.

Each point p of $\mathbb{P}V \setminus \mathbb{P}W$ is equal or noncollinear to a unique point in F_0 which we denote by \bar{p} . If $l = \{p, q, r\}$ is a line then $\bar{l} = \{\bar{p}, \bar{q}, \bar{r}\}$ is a line in F_0 . For each line $l = \{p, q, r\}$ of $\mathbb{P}V \setminus \mathbb{P}W$ define $\sigma(l)$ to be the cycle (p, q, r) if and only if $\sigma(\bar{l}) = (\bar{p}, \bar{q}, \bar{r})$.

It is now straightforward to check that σ is a Lie orientation. So, we have proved the following.

Proposition 3.1 *Suppose V is a binary vector space and W a subspace of codimension 3 in V . Then $\mathbb{P}V \setminus \mathbb{P}W$ admits a Lie orientation such that for fields of characteristic 3 the corresponding Kaplansky algebra is a Lie algebra.*

4 Partial linear spaces with a Lie orientation

In the previous sections we showed that the partial linear spaces of types (a)–(d) are Lie orientable. In this section we show that no other connected partial linear spaces are Lie orientable.

Let $\Pi = (P, L)$ be a connected partial linear space admitting a Lie orientation σ , such that for the field \mathbb{F} of characteristic $\neq 2$ the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(P, L, \sigma)$ is a Lie algebra.

As the intersection of subspaces of (P, L) is again a subspace, we can define the subspace *generated* by a subset X of P to be the intersection of all subspaces containing X . A subspace generated by (the union of) two intersecting lines is called a *plane*.

Proposition 4.1 *All planes of Π are isomorphic to a dual affine plane or a Fano plane.*

If the space Π contains a Fano space, then the characteristic of the field \mathbb{F} equals 3.

Proof. Let $\{a, b, c\}$ and $\{a, d, e\}$ be two lines on the point a . We may assume that their orientations are (a, b, c) and (a, d, e) . By the Jacobi identity we have

$$[[b, c], d] + [[c, d], b] + [[d, b], c] = e + [[c, d], b] + [[d, b], c] = 0.$$

Any product of standard basis elements is either 0 or plus or minus a standard basis element, and therefore, so are the second and third terms of this sum. Hence, either one of the terms is $-e$ and the other is 0, or both are e and the characteristic of \mathbb{F} equals 3.

Case 1: Suppose $[[c, d], b] = -e$ and $[[d, b], c] = 0$ (the case where these values are switched is symmetric). Then c and d must be collinear; call the third point on that line $f = [c, d]$ (the case where $f = -[c, d]$ leads to an isomorphic oriented partial linear space). Then b, f , and e must be on a line as well, with $[b, f] = e$. The plane containing only the points and lines mentioned so far is the dual affine plane of order 2, and with this choice for σ it satisfies the Jacobi identity.

If, in addition to the aforementioned points and lines, we have b collinear to d , then the third point on that line (say $g = [b, d]$) cannot be collinear to c . The Jacobi identity on g, d , and a then requires that $\{a, g, f\}$ is a line, oriented (a, g, f) . This violates the Jacobi identity on a, b , and d . Hence the

dual affine plane of order 2 is the only plane that can lead to a Lie algebra for fields with characteristic other than 3.

Case 2: Now suppose that the characteristic of \mathbb{F} equals 3 and $[[c, d], b] = [[d, b], c] = e$. We cannot have $[c, d] = \pm[d, b]$; for, if that would be the case, then by the axiom of partial linear spaces, $c = b$. So let $f = [c, d]$ and $g = [d, b]$ (again, the cases where $f = -[c, d]$ and/or $g = -[d, b]$ lead to isomorphic oriented partial linear spaces). The Jacobi identity on a, b and d then requires that $[a, g] = f$. This is the last line of the Fano plane. No more lines can be added, since each point is already collinear with every other point. This plane, depicted in Figure 3.1, also satisfies the Jacobi identity. \square

Proposition 4.2 *Let (P, L, σ) be an oriented partial linear space. Then σ is a Lie orientation if and only if for every plane of (P, L) , the restriction of σ to the plane induces a Lie orientation on this plane.*

Proof. Let (P', L') be a plane of (P, L) . Clearly $\mathcal{L}_{\mathbb{F}}(P', L', \sigma|_{L'})$ is a subalgebra of \mathcal{L} . Hence, it is certainly necessary for \mathcal{L} to be a Lie algebra that $\mathcal{L}_{\mathbb{F}}(P', L', \sigma|_{L'})$ be a Lie algebra. To show sufficiency, suppose that every $\mathcal{L}_{\mathbb{F}}(P', L', \sigma|_{L'})$ is a Lie algebra for planes (P', L') . Let $p, q, r \in P$. We will show that the Jacobi identity holds for p, q and r . If there are no two intersecting lines containing p, q and r , then the Jacobi identity certainly holds. Otherwise, they are contained in a plane; then the Jacobi identity holds because of the assumption. So the Jacobi identity holds on all triples of basis elements. Since the Jacobi identity is linear, we are done. \square

Proposition 4.1 gives us a powerful tool to analyze the partial linear spaces that can be Lie oriented. Indeed, the connected partial linear spaces in which any two lines generate a dual affine or Fano plane have been classified. They come in two families. The spaces containing only dual affine planes are called *cotriangular*. They have been classified by Hall [8] and are the spaces as described in cases (a), (b) and (c) of Theorem 1.1. The partial linear spaces containing at least one Fano plane are the spaces $\mathbb{P}V \setminus \mathbb{P}W$ obtained by removing the points and lines of a projective space $\mathbb{P}V$, for V a binary vector space, that meet a proper subspace W of V nontrivially. This follows from the results of [6] and [4].

The following lemma shows that if the space $\mathbb{P}V \setminus \mathbb{P}W$ is Lie orientable, then the subspace W has codimension at most 3.

Lemma 4.3 *Let $\dim V = 4$. Then $\mathbb{P}V$ has no Lie orientation.*

Proof. Suppose the partial linear space (P, L) of points and lines in the 4-dimensional vector space V over \mathbb{F}_2 has a Lie orientation σ . Let $\ell = \{a, b, c\}$ and $m = \{x, y, z\}$ be two disjoint lines in $\mathbb{P}V$ with $\sigma(\ell) = (a, b, c)$ and

$\sigma(m) = (x, y, z)$. Let ax denote the third point on the line on a and x , and similarly for the other pairs of points in $\ell \times m$. We have now named all 15 points of P . By potentially flipping with respect to ax , we may assume that $\sigma(\{a, x, ax\}) = (a, x, ax)$. Similarly for each other line k connecting m to ℓ we can assume that the orientation is such that $\sigma(k)$ maps $k \cap \ell$ to $k \cap m$.

We define projections $\pi_\ell: P \setminus m \rightarrow \ell$ and $\pi_m: P \setminus \ell \rightarrow m$, mapping the point pq , with $p \in \ell$ and $q \in m$, to p or to q respectively (and fixing ℓ and m , respectively). If a plane Π contains m or ℓ , then the restriction of σ to Π is fully determined by the choices that we have made. Indeed, take for Π the plane containing ℓ and x . The Jacobi identity on a, c and x tells us that

$$\begin{aligned} 0 &= [[a, c], x] + [[c, x], a] + [[x, a], c] \\ &= [-b, x] + [cx, a] + [-ax, c] \\ &= -bx + [cx, a] - [ax, c]. \end{aligned}$$

Since all three of these terms are $\pm bx$, we find $[cx, a] = -[ax, c] = -bx$. This determines the value of σ on the lines $\{cx, bx, a\}$ and $\{c, bx, ax\}$. Indeed $\sigma(\{cx, bx, a\}) = (cx, bx, a)$ and $\sigma(\{c, bx, ax\}) = (c, bx, ax)$. The Jacobi identity on a, b and x additionally gives the value on (cx, b, ax) on the line $\{cx, b, ax\}$. We see that π_ℓ is orientation-reversing on these lines.

The same is true if we take for Π a different plane containing ℓ , and if we take for Π a plane containing m , then π_m is orientation-reversing on the lines in there as well.

Now take a plane containing only one point of both ℓ and m , e.g. the plane containing a, x and by . The Jacobi identity on these three points cannot be satisfied anymore:

$$[[a, x], by] + [[x, by], a] + [[by, a], x] = [ax, by] - [bz, a] + [cy, x] = [ax, by] = \pm cz.$$

□

Using the results of [8] and [6] in the finite case and Theorem 1.1 of [4], we obtain the classification part of Theorem 1.1: any connected Lie oriented partial linear space is of one of the four types given there.

5 Uniqueness of the Lie orientation

Next we consider the various ways we can define a Lie orientation on the partial linear spaces in the conclusion of Theorem 1.1. In the previous sections we have seen that each of these spaces admits a Lie orientation; now we prove that this orientation is unique up to flipping.

Let $\Pi = (P, L)$ be one of the partial linear spaces of Theorem 1.1. An *even* collection of lines of Π is a set of lines that covers each point an even number of times. (Notice that the four lines in a dual affine plane of Π form an even set of lines.)

Proposition 5.1 *In Π any finite even collection of lines is the sum (in the binary vector space 2^L) of even collections of size four.*

Proof. From the classification result in Theorem 1.1 it follows that Π is locally finite. This means that any finite subset of points of P generates a finite subspace of Π . In particular, any finite even set of lines is contained in a finite subspace of Π . This implies that we can assume Π to be finite and use induction on the number of points of Π .

Let H be a (geometric) hyperplane of $\Pi = (P, L)$ different from P (arbitrary, or to be chosen later), i.e., a proper subspace of (P, L) meeting each line in at least one point. Given a finite even collection \mathcal{C} of lines in Π , we show that \mathcal{C} is the sum of an even collection in H and some even collections of size 4. For then, as each nontrivial space Π does contain hyperplanes, we have proved the proposition by induction.

Let \mathcal{G} be the subgraph of the collinearity graph of Π induced on the set $P \setminus H$. Each line in L not contained in H determines a unique edge of the graph \mathcal{G} . Hence, the collection \mathcal{C} determines a set of edges of a subgraph C of \mathcal{G} . As \mathcal{C} is an even collection, each vertex of C has even degree.

Any triangle x, y, z in \mathcal{G} is contained in a dual affine plane of Π meeting H in a line. So, the sum of the three lines on this triangle is equal to the sum of an even 4-set and a line in H . Hence, to prove that \mathcal{C} is the sum of an even collection in H and some even collections of size 4, it suffices to prove that the graph C can be triangulated.

We distinguish cases (a)–(d) as in the classification of Theorem 1.1.

In case (d), where $\Pi = \mathbb{P}V \setminus \mathbb{P}W$, choose H a hyperplane not containing W . The points outside $\mathbb{P}W$ fall into at least 7 cosets of W , and two points are joined by an edge when they lie in distinct cosets. Given a cycle $\cdots \sim w \sim x \sim y \sim z \sim \cdots$ in C , we can pick a point p outside H in a coset not containing w, x, y , or z and see that the cycle is the sum of the shorter cycle $\cdots \sim w \sim p \sim z \sim \cdots$ and the triangles formed by w, x, p , and x, y, p , and y, z, p . Repeating this process will eventually provide a triangulation of the cycle. This shows that in this case C can indeed be triangulated.

In case (a), we have Π isomorphic to $T(\Omega, \Omega')$. First suppose $|\Omega| > 3$. Let $\alpha \in \Omega$, and let H be the hyperplane of points not containing α . The complement of H consists of the points A with $\alpha \in A$, and two points A, A' with $A \cap \Omega = \{\alpha, \beta\}$ and $A' \cap \Omega = \{\alpha, \beta'\}$ are adjacent when $\beta \neq \beta'$. Given a minimal cycle $\cdots \sim w \sim x \sim y \sim z \sim \cdots$ we have $\beta, \gamma \in \Omega$ with $w \cap \Omega = y \cap \Omega = \{\alpha, \beta\}$ and $x \cap \Omega = z \cap \Omega = \{\alpha, \gamma\}$. Now the point $p = \{\alpha, \delta\}$ for some δ distinct from α, β, γ lies outside H and is adjacent to each of w, x, y, z . Now we can argue as in the previous case, and find that C can be triangulated.

Next suppose $|\Omega| = 3$. Now the collinearity graph of the partial linear space is complete tripartite. Let H be the hyperplane of points not contain-

ing $\alpha' \in \Omega'$. Given a minimal cycle $\cdots \sim w \sim x \sim y \sim z \sim \cdots$ the points w, x, y, z must lie in two of the three parts of the tripartition, and we can pick a point p adjacent to w, x, y, z in the third part, and continue as in the previous cases.

In case (b), $\Pi = \mathcal{S}p(V, f)$ for some binary symplectic space (V, f) , two points x, y are joined by an edge when $f(x, y) = 1$. Put $R := \text{Rad}(f)$. Then $\dim V/R$ is even. If $\dim V/R = 2$ or 4 , we are in case $T(\Omega, \Omega')$ with $|\Omega| = 3$ or 6 , which was treated already. So assume $\dim V/R \geq 6$. Pick a point a and put $H = a^\perp = \{b \in V \setminus \{0\} \mid f(a, b) = 0\}$. Since a is in an even number of lines of \mathcal{C} , we can add even 4-sets to \mathcal{C} or remove them, in order to remove all lines that pass through a from \mathcal{C} . Given a cycle $\cdots \sim w \sim x \sim y \sim z \sim \cdots$ in C , we can find a point p in the complement of H adjacent to each of w, x, y, z when a does not lie in the span of w, x, y, z and continue as above.

So suppose that a lies in the span of w, x, y, z . If $a = w + x + y + z$, then $f(x, a) = 0$, which yields a contradiction with $x \notin H = a^\perp$. Hence, we can assume that $a = w + z$ and $f(w, z) = 1$. Since we had removed all lines on a from \mathcal{C} , the edge wz is a diagonal in the cycle. This shows that if $x_0 \sim x_1 \sim \cdots \sim x_{g-1} \sim x_0$ is a shortest cycle in C , then we can make C smaller by adding triangles, unless $g = 6$ and $a = x_0 + x_3 = x_1 + x_4 = x_2 + x_5$. But now modulo even sets of size 4 the sum of the lines on the edges of this 6-cycle is zero.

In case (c), $\Pi = \mathcal{O}(V, Q)$, we have the subspace of $\mathcal{S}p(V, f)$ where f is the symmetric bilinear form associated to Q , induced by the points x with $Q(x) = 1$. In low dimensions this space is one of the examples seen already. Let \mathcal{P} be the set of projective points x with $Q(x) = 0$, and put $S = R \cap \mathcal{P}$ so that Q induces a nondegenerate quadratic form on V/S . We may assume that Q is nonempty. If $\dim V/S = 2$, then Q is hyperbolic and we have $PG(n, 2) \setminus PG(-1, 2)$. If $\dim V/S = 3$, then we have $T(4, \Omega')$. If $\dim V/S = 4$, then either Q is hyperbolic, and we have a disconnected space (two copies of $T(3, \Omega')$), or Q is elliptic, and we have $T(5, \Omega')$. If $\dim V/S = 5$, then we have $T(6, \Omega')$. If $\dim V/S = 6$ and Q is hyperbolic, then we have $T(8, \Omega')$.

So, assume $\dim V/S \geq 6$. Fix a point a with $Q(a) = 1$. Again by adding or removing 4-sets to or from \mathcal{C} we can assume that no line of \mathcal{C} contains a , and we take H to be a^\perp . Consider a cycle $\cdots \sim w \sim x \sim y \sim z \sim \cdots$ in C . If it has length 6 and opposite vertices sum to a , then as before, modulo even sets of size 4, the sum of the lines on the edges of this 6-cycle is zero. Otherwise, for a suitable choice of w , we have $a \neq w + z$, and then a is independent of w, x, y, z . If w, x, y, z are dependent, then they lie in a plane, and the three edges wx, xy, yz can be replaced by the single edge wz . So we may assume that a, w, x, y, z are independent. Then V contains a 6-dimensional nondegenerate elliptic subspace containing a, w, x, y and z . Within this subspace we can find, by explicit inspection, a (unique) point p

with $f(p, a) = f(p, w) = f(p, x) = f(p, y) = f(p, z) = 1$ and $Q(p) = 1$. But that implies that we have found a way to triangulate the cycle.

This finishes the proof of the proposition. \square

Corollary 5.2 *Up to flipping, the Lie orientation of the partial linear space Π is unique.*

Proof. Clearly we can restrict to the case where the partial linear space Π is connected.

First assume that Π is finite. If Π consists of a single plane, then the statement is true, as one verifies directly.

Let N be the point-line incidence matrix of Π . Consider two Lie orientations of Π . Let x be the binary row vector indexed by L with $x_\ell = 1$ when the two orientations differ on the line $\ell \in L$. We have to show that x is in the row span of N , given that the restriction of x to the set of lines in any plane is in the row span of N restricted to that same set of lines.

To this end, it suffices to show that if y is a binary row vector orthogonal to the row space of N , then y is the sum of such vectors that are zero outside the set of lines in a plane. In other words, if we have an even collection of lines, then that collection is the sum of even collections contained in a plane.

In the dual affine plane there is only one nonempty such collection, the set of all four lines. In the Fano plane any such collection is the set of four lines missing some point. So, in the finite connected case, the corollary follows from Proposition 5.1.

Now assume Π to be infinite. Using the classification result of Theorem 1.1, it is clear that any finite set of points of Π is contained in a finite subspace of Π . Again assume that we have two Lie orientations on L . We want to assign to each point $p \in P$ a value v_p equal to 0 or 1 in such a way, that $v_x + v_y + v_z = 1 \pmod{2}$ for a line $\ell = \{x, y, z\}$, if and only if the two orientations differ on ℓ . Indeed, switching at the points with value 1 will then map one orientation to the other.

Put a well-ordering on the points, and express for each line the value of the largest point in the values of the smaller points. A contradiction, namely $v_p = 0 = 1$ at some point p , must involve finitely many steps, and hence the points involved live in a finite subspace of Π , which, by the above, is not possible. So, we can find a set of points, the points with value 1, such that switching at this set maps one orientation to the other. Hence, also for infinite Π the corollary holds. \square

The above corollary finishes the proof of Theorem 1.1.

6 The structure of the Kaplansky algebras

In this final section we determine the structure and isomorphism types of the Kaplansky Lie algebras that we have constructed earlier.

Let $\Pi = (P, L, \sigma)$ be a Lie oriented partial linear space. For $p, q \in P$ we write $p \equiv q$ if and only if $p^\perp = q^\perp$. (For a point $p \in P$, we denote by p^\perp the set of all points equal to or not collinear to p .) The partial linear space Π is called *reduced* if and only if all \equiv -equivalence classes are trivial, i.e. consist of a single point.

The quotient geometry Π/\equiv is the geometry with as point set the \equiv -equivalence classes and as lines the triples of \equiv -equivalences obtained from lines of Π . Clearly, this quotient geometry is again Lie oriented.

Proposition 6.1 *Suppose Π is a connected Lie oriented partial linear space, \mathbb{F} a field and $\mathcal{L}_{\mathbb{F}}(\Pi)$ the Kaplansky Lie algebra of Π over \mathbb{F} . Then $\mathcal{L}_{\mathbb{F}}(\Pi)$ is simple if and only if Π is reduced.*

Proof. Suppose Π is connected and reduced. Let I be a nontrivial ideal of $\mathcal{L}_{\mathbb{F}}(\Pi)$. If I contains a point $p \in P$, then by connectedness of Π it contains all points of P and thus is equal to $\mathcal{L}_{\mathbb{F}}(\Pi)$. Thus assume that $P \cap I = \emptyset$ and pick $p_1, \dots, p_n \in P$ with $n > 1$ minimal such that $0 \neq x = \alpha_1 p_1 + \dots + \alpha_n p_n \in I$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{F}^*$. Since $p_1 \not\equiv p_n$, we can assume, up to permuting 1 and n , that there is a $p \in P$ collinear with p_n but not with p_1 . But then $[p, x] = \alpha_1 [p, p_1] + \alpha_2 [p, p_2] + \dots + \alpha_n [p, p_n] = \alpha_2 [p, p_2] + \dots + \alpha_n [p, p_n] \in I$. Since $[p, p_i]$ is either 0, $+q$ or $-q$ for some point $q \in P$, and $[p, p_k] \neq \pm [p, p_n]$ for $k < n$, we find $0 \neq [p, x] \in I$, which contradicts the minimality of n . Hence, if $I \neq 0$, then it equals $\mathcal{L}_{\mathbb{F}}(\Pi)$, proving simplicity of $\mathcal{L}_{\mathbb{F}}(\Pi)$.

Now suppose Π is connected but not reduced. Then Π is a geometry as in the conclusion of Theorem 1.1. In each of the cases (a)-(d) of the conclusion of Theorem 1.1 we can find a subspace $\Pi_0 = (P_0, L_0, \sigma_0)$ of Π meeting each \equiv -class in just one point.

The orientation σ_0 , the restriction of σ to L_0 , can be extended to an orientation $\hat{\sigma}$ on the whole of Π in the following way. If $l = \{x, y, z\}$ is a line in L , then there exists a unique line $l' = \{x', y', z'\}$ in Π_0 such that $x' \equiv x$, $y' \equiv y$ and $z' \equiv z$. Now let $\sigma'(l) = (x, y, z)$ if and only if $\sigma(l') = (x', y', z')$. By 4.2 we find that this orientation $\hat{\sigma}$ is a Lie orientation. As all Lie orientations are flipping equivalent, we can, in order to analyze the structure of the corresponding Kaplansky algebra, assume that $\hat{\sigma} = \sigma$. But then the two subspaces

$$I_- = \langle p - p' \mid p \equiv p' \in P \rangle$$

and

$$I_+ = \langle p + p' \mid p \equiv p' \in P \rangle$$

are clearly ideals of $\mathcal{L}_{\mathbb{F}}(\Pi)$. So $\mathcal{L}_{\mathbb{F}}(\Pi)$ is not simple. \square

We investigate the structure of $\mathcal{L}_{\mathbb{F}}(\Pi)$ somewhat further.

Suppose Π is connected but not reduced and I_- and I_+ are the two ideals as defined above.

Clearly, if \mathbb{F} is of characteristic 2, then $I_- = I_+$ is abelian. From now on we assume that the characteristic of \mathbb{F} is not 2. Let Π_1 be a subspace of Π meeting each \equiv -class in all but one point. Such subspaces exist. Indeed, we can obtain $\Pi_1 = (P_1, L_1)$ by intersecting P with an appropriate hyperplane of the projective space in which Π naturally embeds. As above, we can easily check that

$$J_- = \langle p - p' \mid p \equiv p', p \in P_1, p' \notin P_1 \rangle$$

and

$$J_+ = \langle p + p' \mid p \equiv p', p \in P_1, p' \notin P_1 \rangle$$

are ideals of $\mathcal{L}_{\mathbb{F}}(\Pi)$. Moreover, if we extend \mathbb{F} with $\sqrt{2}$, then over this extension J_- , J_+ and $\mathcal{L}_{\mathbb{F}}(\Pi_1)$ are isomorphic. Indeed, the map

$$\begin{aligned} P_1 &\rightarrow J_{\pm}; \\ p &\mapsto \frac{1}{2}\sqrt{2}(p \pm p'), \end{aligned}$$

induces an isomorphism from $\mathcal{L}_{\mathbb{F}}(\Pi_1)$ to J_{\pm} . Repeating this process we obtain the following result.

Proposition 6.2 *Suppose Π is a connected Lie oriented partial linear space, \mathbb{F} a field and $\mathcal{L}_{\mathbb{F}}(\Pi)$ the Kaplansky Lie algebra of Π over \mathbb{F} .*

If the characteristic of \mathbb{F} is not 2, then, possibly after extending \mathbb{F} by $\sqrt{2}$, the Kaplansky Lie algebra $\mathcal{L}_{\mathbb{F}}(\Pi)$ is a direct sum of pairwise commuting simple Kaplansky Lie algebras all isomorphic to $\mathcal{L}_{\mathbb{F}}(\Pi/\equiv)$.

For the remainder of this section we will consider the case where Π is connected and reduced and the Kaplansky Lie algebra $\mathcal{L}_{\mathbb{F}}(\Pi)$ is simple and identify these Lie algebras with some classical Lie algebras.

Theorem 6.3 *Let \mathbb{F} be a field of characteristic different from 2 and $\Pi = (P, L, \sigma)$ a finite reduced Lie oriented partial linear space with associated Kaplansky Lie algebra $\mathcal{L} = \mathcal{L}_{\mathbb{F}}(\Pi)$. Then \mathcal{L} is isomorphic to*

- (a) $\mathfrak{so}(n, \mathbb{F})$ if (P, L) is isomorphic to $\mathcal{T}(\{1, \dots, n\}, \emptyset)$ with $n > 4$.
- (b) $\mathfrak{sl}(2^n, \mathbb{F})$ if (P, L) is isomorphic to $\mathcal{S}p(V, f)$ for some nondegenerate binary symplectic space (V, f) of dimension $2n$.
- (c) $\mathfrak{so}(2^n, \mathbb{F})$ if (P, L) is isomorphic to $\mathcal{O}(V, Q)$ for some nondegenerate binary orthogonal space (V, Q) of dimension $2n$ and maximal Witt index.
- (d) $\mathfrak{sp}(2^{n-1}, \mathbb{F})$ if (P, L) is isomorphic to $\mathcal{O}(V, Q)$ for some nondegenerate binary orthogonal space (V, Q) of dimension $2n$ and Witt index $n - 1$.

Proof. Let $e_{i,j}$ be the $n \times n$ -matrix with value 1 at position i, j and 0 else. Then the matrices $e_{i,j} - e_{j,i}$ with $i < j$ form a basis for $\mathfrak{so}(n, \mathbb{F})$ and can be identified with the elements $\{i, j\}$ of the Kaplansky Lie algebra corresponding to $\mathcal{T}(\{1, \dots, n\}, \emptyset)$. This proves the first assertion.

Now consider the case that (P, L) is isomorphic to $\mathcal{O}(V, Q)$, where (V, Q) is a nondegenerate binary quadratic space of dimension $2n$ and maximal Witt index. The E_2 -group E obtained from a bilinear form $\phi : V \times V \rightarrow \mathbb{F}_2$ with $Q(v) = \phi(v, v)$ is then isomorphic to the subgroup $E(n) \leq \mathrm{GL}_{2^n}(\mathbb{F})$ defined as

$$E(n) = E(1)^{\otimes n} = E(1) \otimes \cdots \otimes E(1),$$

where

$$E(1) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

Denote by $\mathcal{L}_{\mathbb{F}}(E)$ the Plesken Lie algebra over \mathbb{F} of E . Let $\psi : E \rightarrow E(n)$ be an isomorphism and $\bar{\psi} : \mathcal{L}_{\mathbb{F}}(E) \rightarrow \mathfrak{gl}(2^n, \mathbb{F})$ the linear map defined by $\bar{\psi}(\hat{e}) = \psi(e) - \psi(e^{-1})$ for all $e \in E$. Then $\bar{\psi}$ is a Lie algebra homomorphism. So the Plesken algebra and hence also the Kaplansky Lie algebra defined by E are isomorphic to the image of $\bar{\psi}$. Notice that the elements of order 4 in $E(n)$ are represented by skew-symmetric matrices. So, by construction $\bar{\psi}$ maps all elements of E to elements of $\mathfrak{so}(2^n, \mathbb{F})$. Moreover, as $\mathcal{L}_{\mathbb{F}}(E)$ and $\mathfrak{so}(2^n, \mathbb{F})$ have the same dimension, we find these Lie algebras to be isomorphic. This proves statement (c).

Now consider the element

$$J_n := \begin{pmatrix} 0_{n-1} & I_{n-1} \\ -I_{n-1} & 0_{n-1} \end{pmatrix}$$

of $E(n)$ where 0_m and I_m represent the $2^m \times 2^m$ zero and identity matrix, respectively, and let e be its preimage in E . Then the centralizer of e in $\mathcal{L}_{\mathbb{F}}(E)$ contains the Plesken algebra corresponding to a subspace Δ of Π isomorphic to $\mathcal{S}p(W, f)$ for some binary $2(n-1)$ -dimensional nondegenerate symplectic space (W, f) .

The centralizer of J_n in $\mathfrak{so}(2^n, \mathbb{F})$ consists of all matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with $A = -A^\top$ and $B = B^\top$. The latter centralizer is isomorphic to $\mathfrak{gl}(2^{n-1}, \mathbb{F})$. Indeed, mapping $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ to $A + B$ yields an isomorphism. As the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(\Delta)$ is a simple subalgebra of this centralizer having the same dimension as $\mathfrak{sl}(2^{n-1}, \mathbb{F})$, it readily follows that $\mathcal{L}_{\mathbb{F}}(\Delta)$ and $\mathfrak{sl}(2^{n-1}, \mathbb{F})$ are isomorphic, which prove (b).

Next consider the element

$$\begin{pmatrix} 0_{n-1} & J_{n-1} \\ J_{n-1} & 0_{n-1} \end{pmatrix}$$

in $\mathfrak{so}(2^n, \mathbb{F})$ and its preimage f in E . The centralizer of e and f in the Plesken algebra contains the Plesken algebra corresponding to a subspace of Π isomorphic to $\mathcal{O}(W, Q')$ for some nondegenerate binary quadratic space (W, Q') of dimension $2n - 2$ and of Witt index $n - 2$. This space is mapped by $\bar{\psi}$ into the subalgebra of $\mathfrak{so}(2^n, \mathbb{F})$ consisting of all matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with $A = -A^\top$ and $B = B^\top$, and $(A + B)J_{n-1} = J_{n-1}(A + B)^\top$. As the latter subalgebra is isomorphic with $\mathfrak{sp}(2^{n-1}, \mathbb{F})$ and the dimensions fit, we have proved (d). \square

It remains to identify the Lie algebras arising from the Fano plane.

Theorem 6.4 *Suppose $\Pi = (P, L, \sigma)$ is the Fano plane equipped with a Lie orientation σ and \mathbb{F} a field of characteristic 3. Then, after possibly adding $\sqrt{-1}$ to \mathbb{F} , the Kaplansky algebra $\mathcal{L}_{\mathbb{F}}(\Pi)$ is isomorphic to the simple Lie algebra of type A_2 over \mathbb{F} .*

Proof. Take any point $p \in P$ as a generator for a Cartan subalgebra; if $\{p, q, r\} \in L$ and $p^{\sigma(\ell)} = q$, then $q + \sqrt{-1}r$ and $q - \sqrt{-1}r$ span eigenspaces of p . These eigenspaces form the images of a set of root spaces under modding out the centre. This finishes the proof. \square

We end this paper with the observation that in the latter case, where $\Pi = (P, L, \sigma)$ is the Fano plane equipped with a Lie orientation σ and \mathbb{F} is a field of characteristic 3, the automorphism group of the Lie algebra $\mathcal{L}_{\mathbb{F}}(\Pi)$ contains a group of Lie type G_2 ; see Section 3.

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