Pessimal gossiping

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Consider n ladies, each with a different gossip item. They communicate by telephone, and whenever two ladies talk, each tells the other all she knows. The question about the minimum number of calls needed to spread all gossip was answered around 1971, by several independent groups. For $n \geq 4$ the answer is $2n-4$.

In this note we look at the question about the longest possible series of calls, when in each call at least one participant learns something new. The answer turns out to be $n(n-1)/2$.

That $n(n-1)/2$ is a lower bound, is shown by the following scenario: Let the participants be $A_1, \ldots, A_n$. All calls involve A_1. For $i = 2, \ldots, n$ she calls $A_i, A_{i-1}, \ldots, A_2$, for a total of $1+2+\cdots+(n-1) = n(n-1)/2$ calls. Of course there are many other scenarios.

We show that $n(n-1)/2$ is an upper bound. Since each of the n participants must learn $n-1$ items, and at least one item is learned on each call, there are at most $n(n-1)$ calls. If during a call $1+e$ items of gossip are transmitted, we say that there were e extra items. We will show that in all calls together at least $n(n-1)/2$ extra items occur, by pointing at (at least) one extra item for each unordered pair A, B of participants, or, rather, at at least half an extra item for each ordered pair (A, B) of participants. Then the total number of calls is at most $n(n-1) - n(n-1)/2 = n(n-1)/2$.

We shall assign a call $i(A, B)$ to each ordered pair (A, B) of participants in such a way that a call with e extra items occurs as $i(C, D)$ for at most $2e$ ordered pairs (C, D). If A, B start out knowing a, b, respectively, the call $i(A, B)$ will be a call during which a and b are exchanged, or be the call where B first learns a. If a and b are exchanged during the call where B first learns a, this will be the call $i(A, B)$.

Number the calls in time order. There is a canonical call where a and b are exchanged: At the end B knows both a and b. Trace calls backward from B to uniquely find a strictly increasing series of calls numbered $i_1, ..., i_h$ and a maximal series of ladies X_0, X_1, ..., $X_h = B$, such that call i_j is between ladies X_{j-1} and X_j, where X_j knows both a and b at the end of the call, but did not know both at the start of the call. Now at call i_1 the ladies X_0 and X_1 exchanged a and b. The call $i(A, B)$ will be either i_1 or i_h. The series of calls $i_1, ..., i_h$ will be called the canonical path of a to B.

We shall compare paths $i_1, ..., i_h$ in reverse lexicographic order (revlex), that is lexicographic order on the reversed paths $i_h, ..., i_1$. For two paths where one is a tail of the other, the smaller path is the one that starts with the smallest element, i.e., the one that is longest.

Let R be the set of items that B learned during call number i. Let a_0 be some element of R (to be fixed later) that revlex minimizes the canonical path to B, and put $i(A,B) = i$ for all $a \in R \setminus \{a_0\}$.

Suppose $i_1, ..., i_h = i$ is the revlex minimal one among the canonical paths to B ending in i, where this is the canonical path of a to B, and let i_1 be a call between P and Q, where P is the participant learning b. If $p \in R$, then $i_1, ..., i_h$ is the canonical path of p to B. Now pick $a_0 = p$, and put $i(P,B) = i_1$ as promised. If $p \notin R$ then pick $a_0 \in R$ arbitrarily, and define $i(A_0, B) = i_1$. (In this case the canonical path for (P,B) is revlex smaller than $i_1, ..., i_h$, and does not use the call i_1. The call $i(P,B)$ will occur somewhere on that path.)

This definition of $i(A,B)$ has the properties (i) that one of the participants in call $i(A,B)$ learns gossip item a, and if this participant is not B, then the other learns b, and (ii) that for each B and each call i not involving B one may have $i(A,B) = i$ for at most one A.

Consider call i between participants P and Q, where Q, P learn the sets of items S, T of sizes s, t (respectively). Then $s + t$ items were transmitted, so $s + t - 1$ extra items, and we are allowed to have $i(A,B) = i$ for at most $2(s + t - 1)$ ordered pairs (A,B). (Call them 'such pairs'.) Such pairs will be among (A,Q) for $a \in S$, (B,P) for $b \in T$, (A,B) for $a \in S$, $b \in T \setminus \{q\}$, and (B,A) for $b \in T$, $a \in S \setminus \{p\}$.

The number of such pairs (A,Q) for $a \in S$, and (A,B) for $a \in S$, $b \in T \setminus \{q\}$ is at most $s + t - 1$. Indeed, if $q \in T$ then these sets have sizes at most s and $t - 1$, respectively, and we are happy. If $q \notin T$ then for precisely one $a \in S$ we have $i(A,Q) \neq i$, and we are happy again. Similarly, the number of such pairs (B,P) for $b \in T$, and (B,A) for $b \in T$, $a \in S \setminus \{p\}$ is at most $s + t - 1$. Altogether at most $2(s + t - 1)$, as desired. □

References

[1] A. E. Brouwer, http://www.win.tue.nl/~aeb/math/gossip.html.

[2] Bart Frenk, *Tropical varieties, maps and gossip*, Ph.D. thesis, Eindhoven Univ. of Techn., 2013-03-13.