A lower bound for the Laplacian eigenvalues of a graph—proof of a conjecture by Guo

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Abstract

We show that if μ_j is the j-th largest Laplacian eigenvalue, and d_j is the j-th largest degree ($1 \le j \le n$) of a connected graph Γ on n vertices, then $\mu_j \ge d_j - j + 2$ ($1 \le j \le n-1$). This settles a conjecture due to Guo.

1 Introduction

Let Γ be a finite simple (undirected, without loops) graph on n vertices. Let $X = V\Gamma$ be the vertex set of Γ. Write $x \sim y$ to denote that the vertices x and y are adjacent. Let d_x be the degree (number of neighbors) of x.

The *adjacency matrix* A of Γ is the 0-1 matrix indexed by X with $A_{xy} = 1$ when $x \sim y$ and $A_{xy} = 0$ otherwise. The *Laplacian matrix* of Γ is $L = D - A$, where D is the diagonal matrix given by $D_{xx} = d_x$, so that L has zero row and column sums.

The eigenvalues of A are called *eigenvalues* of Γ. The eigenvalues of L are called *Laplacian eigenvalues* of Γ. Since A and L are symmetric, these eigenvalues are real. Since L is positive semidefinite (indeed, for any vector u indexed by X one has $u^\top Lu = \sum(u_x - u_y)^2$ where the sum is over all edges xy), it follows that the Laplacian eigenvalues are nonnegative. Since L has zero row sums, 0 is a Laplacian eigenvalue. In fact the multiplicity of 0 as eigenvalue of L equals the number of connected components of Γ.

Let $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_n = 0$ be the Laplacian eigenvalues. Let $d_1 \ge d_2 \ge \ldots \ge d_n$ be the degrees, ordered nonincreasingly. We will prove that $\mu_i \ge d_i - i + 2$ with basically one exception.

2 Exception

Suppose $\mu_m = 0 < d_m - m + 2$. Then $d_m \ge m - 1$, and we find a connected component with at least m vertices, hence with at least $m - 1$ nonzero Laplacian eigenvalues. It follows that this component has size precisely m, and hence $d_1 = \ldots = d_m = m - 1$, and the component is K_m. Now $\Gamma = K_m + (n - m)K_1$ is the disjoint union of a complete graph on m vertices and $n - m$ isolated points. We'll see that this is the only exception.

3 Interlacing

Suppose M and N are real symmetric matrices of order m and n with eigenvalues $\lambda_1(M) \geq \ldots \geq \lambda_m(M)$ and $\lambda_1(N) \geq \ldots \geq \lambda_n(N)$, respectively. If M is a principal submatrix of N, then it is well known that the eigenvalues of M interlace those of N, that is,

$$\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N) \quad \text{for} \quad i = 1, \ldots, m\,.$$

Less well-known, (see for example [3]) is that these inequalities also hold if M is the quotient matrix of N with respect to some partition $X_1, \ldots, X_m$ of $\{1, \ldots, n\}$. This means that $(M_{i,j})$ equals the average row sum of the block of N with rows indexed by X_i and columns indexed by X_j.

Let K be the point-line incidence matrix of a graph Γ. Then the Laplacian of Γ is $L = KK^\top$. But $KK^\top$ has the same nonzero eigenvalues as $K^\top K$, and interlacing for that latter matrix implies that the eigenvalues of L do not increase when an edge of Γ is deleted.

4 The lower bound

Theorem 1 *Let Γ be a finite simple graph on n vertices, with vertex degrees $d_1 \geq d_2 \geq \ldots \geq d_n$, and Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. If Γ is not $K_m + (n-m)K_1$, then $\mu_m \geq d_m - m + 2$.*

For the union of K_m and some isolated points we have $\mu_m = 0$ and $d_m = m - 1$.

The case $m = 1$ of this theorem ($\mu_1 \geq d_1 + 1$ if there is an edge) is due to Grone & Merris [1]. The case $m = 2$ ($\mu_2 \geq d_2$ if the number of edges is not 1) is due to Li & Pan [4]. The case $m = 3$ is due to Guo [2], and he also conjectured the general result.

Let us separate out part of the proof as a lemma.

Lemma 2 *Let S be a set of vertices in the graph Γ such that each vertex in S has at least e neighbors outside S. Let $m = |S|$, $m > 0$. Then $\mu_m \geq e$. If S contains a vertex adjacent to all other vertices of S, and $e > 0$, then $\mu_m \geq e + 1$.*

Proof Consider the principal submatrix L_S of L with rows and columns indexed by S. Let $L(S)$ be the Laplacian of the subgraph induced on S. Then $L_S = L(S) + D$ where D is the diagonal matrix such that D_{ss} is the number of neighbors of s outside S. Since $L(S)$ is positive semidefinite and $D \geq eI$, all eigenvalues of L_S are not smaller than e, and by interlacing $\mu_m \geq e$.

Now suppose that $S = \{s_0\} \cup T$, where s_0 is adjacent to all vertices of T. Throw away all edges entirely outside S, and possibly also some edges leaving S, so that each vertex of S has precisely e neighbours outside S. Also throw away all vertices outside S that now are isolated. Since these operations do not increase μ_m, it suffices to prove the claim for the resulting situation.

Consider the quotient matrix Q of L for the partition of the vertex set X into the $m+1$ parts $\{s\}$ for $s \in S$ and $X \setminus S$. We find, with $r = |X \setminus S|$,

$$Q = \begin{pmatrix} e+m-1 & -1 \ldots -1 & -e \\ -1 & & -e \\ \vdots & L_T & \vdots \\ -1 & & -e \\ -e/r & -e/r \ldots -e/r & em/r \end{pmatrix}.$$

Consider the quotient matrix R of L for the partition of the vertex set X into the 3 parts $\{s_0\}$, T, $X \setminus S$. Then

$$R = \begin{pmatrix} e+m-1 & 1-m & -e \\ -1 & e+1 & -e \\ -e/r & -e(m-1)/r & em/r \end{pmatrix}.$$

The eigenvalues of R are 0, $e+m$, and $e+me/r$, and these three numbers are also the eigenvalues of Q for (right) eigenvectors that are constant on the three sets $\{s_0\}$, T, $X \setminus S$. The remaining eigenvalues θ of Q belong to (left) eigenvectors perpendicular to these, so of the form $(0, u^\top, 0)$ with $\sum u = 0$. Now $L_T u = \theta u$, but $L_T = L(T) + (e+1)I$ and $L(T)$ is positive semidefinite, so $\theta \geq e+1$.

Since $me/r \geq 1$ (each vertex in S has e neighbors outside S and $|S| = m$, so at most me vertices in $X \setminus S$ have a neighbor in S), it follows that all eigenvalues of Q except for the smallest are not less than $e+1$. By interlacing, $\mu_m \geq e+1$. □

Proof (of the theorem). Since $\mu_m \geq 0$ we are done if $d_m \leq m-2$. So, suppose that $d_m \geq m-1$.

Let Γ have vertex set X, and let x_i have degree d_i ($1 \leq i \leq n$). Put $S = \{x_1, \ldots, x_m\}$. Put $e = d_m - m + 1$, then we have to show $\mu_m \geq e+1$.

Each point of S has at least e neighbours outside. If each point of S has at least $e+1$ neighbours outside, then we are done by the lemma. And if not, then a point in S with only e neighbours outside is adjacent to all other vertices in S, and we are done by the lemma, unless $e = 0$.

Suppose first that Γ is K_m with a pending edge attached, possibly together with some isolated vertices. Then Γ has Laplacian spectrum $m+1$, m^{m-2}, 1, 0^{n-m}, with exponents denoting multiplicities, and equality holds. And if Γ is $K_m + K_2 + (n-m-2)K_1$, it has spectrum m^{m-1}, 2, 0^{n-m}, and the inequality holds.

Let T be the set of vertices of S with at most $m-2$ neighbours in S. The case $T = \emptyset$ has been treated above. For each vertex $s \in T$ delete all edges except one between s and $X \setminus S$. Now the row of L_S indexed by s gets row sum 1. Since $d_m = m-1$ we can always do so. Also delete all edges inside $X \setminus S$, and possible isolated vertices. By interlacing, μ_i has not been increased, so it suffices to show that for the remaining graph $\mu_m \geq 1$.

Again consider the partition of X into $m+1$ parts consisting of $\{s\}$ for each $s \in S$, and $X \setminus S$, and let Q be the corresponding quotient matrix of L. By interlacing it suffices to show that the second smallest eigenvalue of Q is at least 1. Put $r = |X \setminus S|$ and $t = |T|$, then $0 < r \leq t$, and

$$Q = \begin{pmatrix} mI - J & -J & \mathbf{0} \\ -J & L_T & -\mathbf{1} \\ \mathbf{0}^\top & -\mathbf{1}^\top/r & t/r \end{pmatrix}$$

(J is the all-ones matrix, and $\mathbf{0}$ and $\mathbf{1}$ denote the all-zeros and the all-ones vector, respectively). Now Q has a 3×3 quotient matrix

$$R = \begin{pmatrix} t & -t & 0 \\ t-m & m-t+1 & -1 \\ 0 & -t/r & t/r \end{pmatrix}$$

The three eigenvalues of R are $0 \le x \le y$ (say). We easily have that

$$(1-x)(1-y) = \det(I-R) = t - 1 + (m-1)(t/r - 1) \ge 0\ ,$$

which implies that $x \ge 1$ (since $x \le y \le 1$ contradicts $x + y = \operatorname{trace} R > m+1$). These three values are also eigenvalues of Q with (right) eigenvectors constant over the partition. The remaining eigenvalues have (left) eigenvectors that are orthogonal to the characteristic vectors of the partition, and these eigenvalues remain unchanged if a multiple of J is added to a block of the partition of Q. So they are also eigenvalues of

$$Q' = \begin{pmatrix} mI & O & \mathbf{0} \\ O & L_T & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & 1 \end{pmatrix},$$

which are at least 1 since $L_T = L(T) + (m-t+1)I$ and $L(T)$ is positive semidefinite. So we can conclude that $\mu_m \ge 1$. □

5 Equality

There are many cases of equality (that is, $\mu_m = d_m - m + 2$), and we do not have a complete description.

For $m = 1$ we have equality, i.e., $\mu_1 = d_1 + 1$, if and only if Γ has a vertex adjacent to all other vertices.

For $m = n$ we have equality, i.e., $0 = \mu_n = d_m - m + 2$, if and only if the complement of Γ has maximum degree 1.

The path $P_3 = K_{1,2}$ has Laplace eigenvalues 3, 1, 0 and degrees 2, 1, 1 with equality for $m = 0, 1, 2$, and is the only graph with equality for all m.

The complete graph K_m with a pending edges attached at the same vertex has spectrum $a+m$, m^{m-2}, 1^a, 0, with exponents denoting multiplicities. Here $d_m = m-1$, with equality for m (and also for $m = 1$).

The complete graph K_m with a pending edges attached at each vertex has spectrum $\frac{1}{2}(m+a+1 \pm \sqrt{(m+a+1)^2 - 4m})^{m-1}$, $a+1$, $1^{m(a-1)}$, 0, with $\mu_m = a+1 = d_m - m + 2$.

The complete bipartite graph $K_{a,b}$ has spectrum $a+b$, a^{b-1}, b^{a-1}, 0. For ($a = 1$ or $a \ge b$) and $b \ge 2$ we have $d_2 = a = \mu_2$. This means that all graphs $K_{1,b}$, and all graphs between $K_{2,a}$ and $K_{a,a}$ have equality for $m = 2$.

The following describes the edge-minimal cases of equality.

Proposition 3 *Let Γ be a graph satisfying $\mu_m = d_m - m + 2$ for some m, and such that for each edge e the graph $\Gamma \setminus e$ has a different m-th largest degree or a different m-th largest eigenvalue. Then one of the following holds.*

(i) Γ *is a complete graph* K_m *with a single pending edge.*
(ii) $m = 2$ *and* Γ *is a complete bipartite graph* $K_{2,d}$*.*
(iii) Γ *is a complete graph* K_m *with a pending edges attached at each vertex. Here* $d_m = m + a - 1$*.*

Proof This is a direct consequence of the proof of the main result. □

Many further examples arise in the following way. Any eigenvector u of $L = L(\Gamma)$ remains eigenvector with the same eigenvalue if one adds an edge between two vertices x and y for which $u_x = u_y$. If Γ had equality, and adding the edge does not change d_m or the index of the eigenvalue μ_m, then the graph Γ' obtained by adding the edge has equality again.

The eigenvector for the eigenvalue $a+1$ for K_m with a pending edges attached at each vertex, is given by: 1 on the vertices of degree 1, and $-a$ on the vertices in the K_m. So, equality will persist when arbitrary edges between the outside vertices are added to this graph, as long as the eigenvalue keeps its index and d_m does not change.

The eigenspace of $K_{a,b}$ for the eigenvalue a is given by: values summing to 0 on the b-side, and 0 on the a-side. Again we can add edges.

For example, the graphs $K_{2,d}$ with $d \geq 2$ have $d_2 = d = \mu_2$ with equality for $m = 2$. Adding an edge on the 3-side of $K_{2,3}$ gives a graph with spectrum 5, 4, 3, 2, 0, and the eigenvalue 3 is no longer 2nd largest. Adding an edge on the 4-side of $K_{2,4}$ gives a graph with spectrum 6, 4, 4, 2, 2, 0, and adding two disjoint edges gives 6, 4, 4, 4, 2, 0, and in both cases we still have equality for $m = 2$.

References

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