Heden's bound on maximal partial spreads

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24 Nov 1987

Abstract

We prove Heden's result that the deficiency δ of a maximal partial spread in $\mathrm{PG}(3,q)$ is greater than $1+\frac{1}{2}(1+\sqrt{5})q$ unless $\delta-1$ is a multiple of p, where $q=p^n$. When q is odd and not a square, we are able to improve this lower bound to roughly $\sqrt{3q}$.

0 Introduction

In this note we translate Heden [5] into geometry and find that the same theory now only takes one-fifth of the space. Having thus decoded [5], we proceed to apply the methods of Blokhuis and Brouwer [1] to improve Heden's result a little. A *spread* in $\mathrm{PG}(3,q)$ (the projective geometry of dimension 3 over the field $\mathbb{F}_q$) is a partition of the set of points into lines. An easy counting argument shows that a spread contains q^2+1 lines. A *partial spread* is a collection of pairwise disjoint lines that is not a spread. A *maximal partial spread* $\mathcal{S}$ is a partial spread such that no projective line is disjoint from each of its lines. Its deficiency δ is the number of lines 'missing' from $\mathcal{S}$, i.e. $q^2+1-|\mathcal{S}|$. Let $q=p^n$, where p is prime. Heden's result is:

Theorem 1 *For any maximal partial spread in* $\mathrm{PG}(3,q)$*, the deficiency* δ *satisfies* $\delta \geq 1+\sqrt{q}$*. If* p *does not divide* $\delta-1$ *and if* $\delta < \frac{1}{2}(q+1)$*, then* $\delta \geq 1+\frac{1}{2}(1+\sqrt{5})\sqrt{q}$*.*

We show:

Theorem 2 *If* q *is not a square, then for any maximal spread with deficiency* δ *in* $\mathrm{PG}(3,q)$*, we have*

$$\delta \geq \min(1+\sqrt{3q}, \sqrt{pq}-p+2)$$

1 Trivialities

Let $\mathcal{S}$ be a maximal partial spread with deficiency δ. Then $\delta(q+1)$ points of $\mathrm{PG}(3,q)$ are not covered by a line from $\mathcal{S}$. We call these points 'holes'. Dually (note that 'spread', 'partial spread' and 'deficiency' are self-dual concepts), all planes except for $\delta(q+1)$ contain a line from $\mathcal{S}$. Let us call the planes on a line of $\mathcal{S}$ 'rich' and the other planes 'poor'. The $q^2+1-\delta$ lines of $\mathcal{S}$ cover $q^2+1-\delta$ points in any poor plane, so that a poor plane has $(q^2+q+1)-(q^2+1-\delta)=q+\delta$ holes. Similarly, a rich plane has δ holes. Let L be any line not in $\mathcal{S}$, and suppose that it has h holes. Then L is hit by $q+1-h$ lines of $\mathcal{S}$, and hence L lies in $q+1-h$ rich planes, and in h poor planes. In particular, each line in a poor plane contains a hole, so that the set of holes in a poor plane forms a blocking set in that plane.

[Now standard results on blocking sets show $\delta \geq 1+\sqrt{q}$ (Bruen [3, 4] or even $\delta \geq \sqrt{2q}$ if q is not a square (Blokhuis and Brouwer [1]). Note that it follows that $q \neq 2$ since every blocking set in PG(2, 2) contains a line, i.e. each partial spread in PG(3, 2) can be extended to a spread.]

Also, the intersection of two poor planes is a line containing at least two holes. Finally, remark that a line contains at most δ holes (otherwise it cannot be on a rich plane and hence contains $q+1$ holes, and $\mathcal{S}$ would not be maximal).

2 The number of holes on a line

For a set A of points, let $H(A)$ be the set of holes in A, and let $h(A)$ be the cardinality of $H(A)$.

Lemma 1 (Heden's Lemma 11.1). *Let L, M be two skew lines. Then either $H(M)$ meets all rich planes on L, or it meets at most $\delta - h(L)$ of them.*

Proof. Suppose Π is a rich plane on L disjoint from $H(M)$. Then $h(H)=\delta$. If all planes on M meet $H(\Pi)$, then $\delta \geq q+1$ and the statement is trivial. Otherwise, some plane Π' on M meets Π in a line without holes, so that Π' is rich and $h(W)=\delta$. All the $h(L)$ poor planes on L meet $H(\Pi')$, so at most $\delta-h(L)$ rich planes on L can do so, and a forteriori at most $\delta-h(L)$ rich planes on L meet $H(M)$. □

Lemma 2 (Heden's Lemma 11.2). *Let L be a line such that $h(L)<\delta$ and $h(L)<q$. Then there is a line M skew to L such that $H(M)$ meets at least $(q+1-h(L))/(\delta-h(L))$ rich planes on L.*

Proof. Choose non-holes P, Q on L. Each of these lies on δ poor planes (since dually each rich plane contains δ holes) and therefore on $\delta-h(L)$ poor planes not containing L. Fix such a plane Π on P and let Π' vary over the $\delta-h(L)$ such planes Π' on Q. Then we see $\delta-h(L)$ lines $M:=\Pi\cap\Pi'$, all skew to L, and we are done if we show that each of the $q+1-h(L)$ rich planes Π'' on L is met by at least one of the lines M. But for each Π'' the line $\Pi''\cap\Pi$ contains a hole R since Π is poor, and the line (Q,R) is on one of the planes Π' (indeed, it is on a poor plane, and this plane cannot contain L since otherwise it would be the plane Π'', which is not poor), and we are done. □

Proposition (Heden's Proposition 11.1). *Let L be Q a line such that $h(L)<\delta$ and $h(L)<q+l-\delta$. Then*

$$h(L) \leq \delta - \tfrac{1}{2} - \sqrt{q+\tfrac{5}{4}-\delta}. \qquad (1)$$

Proof. Let M be a line skew to L as found in Lemma 2. If $H(M)$ meets all $q+1-h(L)$ rich planes on L, then $q+1-h(L)=\delta=h(M)$; now if Π is any rich plane on M, then $H(\Pi)=H(M)$ also meets all poor planes on L, so that $H(M)$ meets all planes on L, and $h(M)=q+1$, contradiction. Now Lemma 1 yields

$$\frac{q+l-h(L)}{\delta-h(L)} \leq \delta - h(L)$$

and (1) follows. □

3 Application of Rédei's theorem

Let Π be an arbitrary fixed poor plane, and put $S=H(\Pi)$. As we have seen, S is a blocking set in Π of size $q+\delta$. Let x_i be the number of lines in Π meeting S in $i+1$ points. By the usual counting arguments we find

$$\sum_{i=1}^{\delta-l} i x_i = \delta(q+1)-1 \tag{2}$$

(count poor planes distinct from Π) and

$$\sum_{i=1}^{\delta-1} i(i+l)x_i = (\delta+q)(\delta+q-1). \tag{3}$$

Suppose $x_{\delta-1}>0$, i.e. suppose that some line L of H has δ holes. Then $\Sigma=\Pi\setminus L$ is an affine plane (with L as line at infinity), $h(\Sigma)=q$, and any line meeting $H(\Sigma)$ in more than one point must meet $H(L)$ (otherwise a parallel line would not meet $H(\Sigma)$ and contradict the fact that $H(\Pi)$ is a blocking set in Π). Now Rédei [7], p. 215 (Hilfssatz 42) proves that if the secants of a subset X of cardinality q of the desarguesian affine plane $\mathrm{AG}(2,q)$ have not more than $\frac{1}{2}q$ distinct directions, then each secant meets X in a number of points divisible by p. In our case this means that if $\delta \leq \frac{1}{2}q$, then for any line M on P distinct from L we have $p \mid h(M)-1$. In particular, either $p \mid \delta-1$ or $x_{\delta-1}=1$. Now assume $p \nmid \delta-1$ and $\delta \leq \frac{1}{2}q$. Then Rédei tells us that $x_{\delta-l} \leq 1$, and in the previous section we saw that $x_i=0$ for $a<i+1<\delta$, where $a=\delta=\frac{1}{2}-\sqrt{q+\frac{5}{4}-\delta}$.

Subtracting (3) from a times (2), we get, using these estimates,

$$(\delta-1)(a-\delta) \leq \sum_{i=1}^{\delta-1}(a-i-1)ix_i = a(\delta(q+1)-1)-(\delta+q)(\delta+q-1)$$

and, substituting a,

$$\delta-\tfrac{5}{2}-\frac{q-1}{\delta}-\sqrt{q+\tfrac{5}{4}-\delta} \geq 0$$

the left hand side of this inequality is an increasing function of δ. For $\delta = 1+\frac{1}{2}(1+\sqrt{5})\sqrt{q}$, the left hand side is negative, and hence Heden's theorem follows.

4 An improvement

Another result by Rédei states that the secants of a subset X of cardinality q of the desarguesian affine plane $\mathrm{AG}(2,q)$ have at least $1+(q-1)/(p^{[n/2]}+1)$ distinct directions (Rédei [7], p. 237). Thus, if $x_{\delta-1}>0$, then $\delta \geq 1+(q-1)/(p^{[n/2]}+1)$. If q is not a square, this implies that $\delta > \sqrt{pq}-p+1$, and hence

$$\delta \geq \sqrt{pq}-p+2. \tag{4}$$

Now suppose that $x_{\delta-l}=0$. If $\delta < \sqrt{4q+1}$, then $2a-1<\delta$, and it follows that in each poor plane Π each non-hole is on at most $q-2$ tangents. On the other hand, since a blocking set in $\mathrm{AG}(2,q)$ has size at least $2q-1$ (Brouwer and Schrijver [2], Jamison [6]) it follows that any point in $\Pi \setminus S$ lies on at most $q-\delta+1$ tangents to S. Counting incident pairs (tangent to S, point in $\Pi \setminus S$) in two ways, one gets

$$q(q+\delta)(q-\delta+1) \leq (q^2-\delta+1)(q-2)$$

and it follows that

$$\delta \geq \sqrt{3q}+1. \tag{5}$$

Thus we have proved Theorem 2 (for $q \leq 11$ a few ad hoc arguments are required).

Note that we have the additional geometric information that if $\delta < \sqrt{3q}+1$, then each poor plane contains a line with δ holes, and dually each hole lies on such a line.

5 Groups

We needed Rédei in order to show that $x_{\delta-1}$ is small, but this required some unfortunate hypotheses (q not a square, or p does not divide $\delta-1$). Now suppose that we cannot find a plane in which $x_{\delta-1}=0$, i.e. suppose that each plane contains a line with δ holes—let us call such a line a δ-line. Using the geometry of $\mathrm{PG}(3,q)$ and the classification of subgroups of $\mathrm{PSL}(2,q)$, we can say a little about the number δ.

Lemma. *Let K be a δ-line. The δ-set $H(K)$ is an orbit of some subgroup H of the* $\mathrm{PSL}(2,q)$ *acting on K.*

Proof. If M is a δ-line, and p is a hole not on M, then the plane $\langle M,P\rangle$ is poor. Consequently, if M, N are two skew δ-lines, then the δ poor planes on M are the δ planes $\langle M,P\rangle$, where P runs over $H(N)$. Thus, if L, M, N are three mutually skew δ-lines, and we define a map $\pi_{LMN}: M \to N$ by $\pi_{LMN}(P)=\langle L,P\rangle \cap N$ for P on M, then π_{LMN} maps $H(M)$ onto $H(N)$. In this way, any point of $H(M)$ can be mapped to any point of $H(N)$: if $P \in H(M)$ and $Q \in H(N)$, then as we saw in Section 3 the number of holes on the line $\langle P,Q\rangle$ is congruent to 1 (mod p), so this line is on at least three poor planes, and hence is on a poor plane Π not containing M or N. Let L be a δ-line in Π. Then $\pi_{LMN}(P)=Q$. Now let K, L be two skew δ-lines, and let M, N be δ-lines skew to both K and L. Composing two maps π_{MKL} and π_{NLK} we find a map from K to K; the subgroup H generated by all such maps is a subgroup of the $\mathrm{PSL}(2,q)$ acting on K, and has $H(K)$ as orbit. □

References

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