SL_2-modules of small homological dimension

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Abstract

Let V_n be the SL_2-module of binary forms of degree n and let $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. We consider the algebra $R = \mathcal{O}(V)^{\mathrm{SL}_2}$ of polynomial functions on V invariant under the action of SL_2. The measure of the intricacy of these algebras is the length of their chains of syzygies, called *homological dimension* $\operatorname{hd} R$. Popov gave in 1983 a classification of the cases in which $\operatorname{hd} R \leq 10$ for a single binary form ($p = 1$) or $\operatorname{hd} R \leq 3$ for a system of two or more binary forms ($p > 1$).

We extend Popov's result and determine for $p = 1$ the cases with $\operatorname{hd} R \leq 100$, and for $p > 1$ those with $\operatorname{hd} R \leq 15$. In these cases we give a set of homogeneous parameters and a set of generators for the algebra R.

1 Introduction

This paper has two goals. First of all, following a suggestion by Popov, we extend the results of Popov [29] and determine all cases where the algebra of simultaneous invariants of a number of binary forms has low homological dimension. Secondly, we determine the minimal degrees of a homogeneous system of parameters (hsop) in these cases. We also give a minimal system of generators, confirming or correcting classical results.

Our base field is the field $\mathbb{C}$ of complex numbers. The group of all complex 2×2 matrices with determinant 1 is denoted SL_2. Let V_n be the set of binary forms (homogeneous polynomials in two variables) of degree n. If V is a rational finite-dimensional SL_2-module, then there exist $n_1, \ldots, n_p \in \mathbb{N}$ such that $V \simeq V_{n_1} \oplus \ldots \oplus V_{n_p}$ as SL_2-modules, and the algebra $R := \mathcal{O}(V)^{\mathrm{SL}_2}$ of polynomial functions on V invariant under the action of SL_2 can be identified with the algebra of joint invariants of p binary forms of degrees $n_1, \ldots, n_p$.

The algebra R is finitely generated ([23]), i.e. there exist a finite number of invariants $j_1, \ldots, j_r$ of V such that $R = \mathbb{C}[j_1, \ldots, j_r]$. Denote by r the minimal number of generators of R and by m the size of a system of parameters of R (set of algebraically independent elements $P_1, \ldots, P_m$ of R, such that R is integral over $\mathbb{C}[P_1, \ldots, P_m]$). Then m equals $\sum(n_i + 1) - 3$ when this is positive, and the homological dimension $\operatorname{hd} R$ of R equals $r - m$ ([29, Corollary 1]).

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V	$\operatorname{hd} R$
V_1, V_2, V_3, V_4, $2V_1$, $V_1 \oplus V_2$, $2V_2$, $3V_1$	0
V_5, V_6, $V_1 \oplus V_3$, $V_1 \oplus V_4$, $V_2 \oplus V_3$, $V_2 \oplus V_4$, $2V_4$ $2V_1 \oplus V_2$, $V_1 \oplus 2V_2$, $3V_2$, $4V_1$	1
$2V_3$	2
V_8, $5V_1$	3

Table 1: Popov's classification of SL_2-modules with small $\operatorname{hd} R$

Popov [29] classified the modules V with the property that $\operatorname{hd} R \leq 3$, and noticed that all of these were known classically.

In the past 25 years some progress was made and sets of generators for $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ were found in the cases $n = 7, 9, 10$ ([7, 8, 12]). The difficulty of this problem is reflected by the large homological dimensions of the algebras of invariants in these cases. For $R := \mathcal{O}(V_n)^{\mathrm{SL}_2}$ we have:

n	1	2	3	4	5	6	7	8	9	10
$\operatorname{hd} R$	0	0	0	0	1	1	25	3	85	98

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In this paper we extend Popov's classification to:

Theorem 1.1. *Let $R := \mathcal{O}(V_n)^{\mathrm{SL}_2}$ and suppose that $\operatorname{hd} R \leq 100$. Then $n \leq 10$.*

Theorem 1.2. *Let $R := \mathcal{O}(V)^{\mathrm{SL}_2}$ where $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$, and suppose that $4 \leq \operatorname{hd} R \leq 15$. Then we have one of the following:*

$n_1, \ldots, n_p$	hd	m	*hsop degrees*	r	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}
$1,1,1,2$	4	6	2 (3×), 3 (3×)	10	4	6								
$1,2,2,2$	5	8	2 (5×), 3 (3×)	13	6	4	3							
$2,2,2,2$	5	9	2 (9×)	14	10	4								
$1,1,2,2$	6	7	2 (4×), 3 (3×)	13	4	6	3							
1 (6×)	6	9	2 (9×)	15	15									
$1,1,3$	8	5	2, 4 (4×)	13	1		8		4					
$1,2,3$	9	6	$3,3,4,4,4,5$	15	1	3	4	4	2	1				
$1,1,1,1,2$	9	8	2 (4×), 3 (3×), 6	17	7	10								
1 (7×)	10	11	2 (11×)	21	21									
$1,2,4$	11	7	$2,2,3,3,4,5,6$	18	2	3	2	3	4	2	1	1		
$2,2,3$	11	7	$2,2,3,4,5,5,6$	18	3	2	2	4	3	4				
$2,2,4$	11	8	$2,2,2,3,3,3,4,4$	19	4	4	5	2	4					
$1,2,2,2,2$	13	11	2 (7×), 3 (4×)	24	10	8	6							
$2,2,2,2,2$	13	12	2 (12×)	25	15	10								
$4,4,4$	13	12	2 (6×), 3 (6×)	25	6	10	6	3						
$1,1,4$	14	6	$2,3,5,5,6,6$	20	2	1		5	5			7		
$3,4$	14	6	$2,3,4,5,6,7$	20	1	1	1	2	2	3	3	4	2	1
1 (8×)	15	13	2 (13×)	28	28									
$1,1,1,2,2$	15	9	2 (5×), 3 (4×)	24	6	12	6							

Here V has a minimal set of generators of size r, with d_i generators of degree i $(2 \leq i \leq 11)$. The size of any homogeneous system of parameters (hsop) is m, and the degrees for one particular such system are as given. The column hd *gives* $\operatorname{hd} R$.

The paper is organised as follows: In §2 we describe the classical results and correct them where needed. In §3 we find a lower bound for r given the Poincaré series. In §4 we determine V. In §5 we describe how to find a set of generators. A prerequisite is a homogeneous system of parameters, found in §6. The actual generators are constructed in §7.

2 The classical results

The table below gives the classical (that is, 19th century) results† ([3, 13–21, 28, 31, 33–36]), possibly slightly amended. Here aV_s stands for the direct sum $\oplus_{i=1}^{a} V_s$ of a copies of V_s. See also §7.7.

module	r	module	r	module	r	module	r
V_1	0	$2V_1$	1	$3V_1$	3	$4V_1$	6
V_2	1	$V_1 \oplus V_2$	2	$3V_2$	7	$V_1 \oplus 3V_2$	13
V_3	1	$V_1 \oplus V_3$	4	$3V_3$	28	$V_1 \oplus 3V_3$	97‖
V_4	2	$V_1 \oplus V_4$	5	$3V_4$	25	$V_1 \oplus 3V_4$	103¶
V_5	4	$V_1 \oplus V_5$	23	$4V_4$	80	$V_1 \oplus 4V_4$	305¶
V_6	5	$V_1 \oplus V_6$	26	$V_2 \oplus V_3$	5	$V_1 \oplus V_2 \oplus V_3$	15
V_7	30‡	$V_1 \oplus V_7$	147‡‡	$V_2 \oplus V_4$	6	$V_1 \oplus V_2 \oplus V_4$	18
V_8	9	$V_1 \oplus V_8$	69§	$V_2 \oplus V_5$	29	$V_1 \oplus V_2 \oplus V_5$	92††
$2V_2$	3	$V_1 \oplus 2V_2$	6	$V_2 \oplus V_6$	27	$V_1 \oplus V_2 \oplus V_6$	99
$2V_3$	7	$V_1 \oplus 2V_3$	26	$V_3 \oplus V_4$	20	$V_1 \oplus V_3 \oplus V_4$	63*
$2V_4$	8	$V_1 \oplus 2V_4$	28				

Table 2: The classical results

More generally, Gordan [18,19] gives for $V = pV_1$ the value $r = \binom{p}{2}$, for $V = pV_2$ the value $r = \binom{p+1}{2} + \binom{p}{3}$, and for $V = V_1 \oplus qV_2$ the value $r = q(q+1) + \binom{q}{3}$, cf. §7.8 below. From the generators in case $V \oplus V_1$ one can derive those for $V \oplus pV_1$ for all $p > 1$, cf. [9, §55], [20, §138A] and §7.7 below.

† In more complicated cases the classical techniques were not powerful enough to determine the precise values of r—the German school found upper bounds only, the English school claimed to find true values, or at least lower bounds, but the former was mistaken (cf. [22]), the latter unproved.

* Gundelfinger found 64, Sylvester 61, it is 63.

§ von Gall found 96, then 67, then 70; Sylvester 69. See also Shioda [30] and Bedratyuk [2].

‡ von Gall found 33, Sylvester 26, Hammond two more, Dixmier & Lazard 30.

‡‡ von Gall found 153, Sylvester 124, Cröni [10] and Bedratyuk [1] find 147.

‖ von Gall found 98, Sinigallia 97. (Peano [27] has partial results on pV_3, $V_1 \oplus pV_3$.)

†† Winter found 94, it is 92.

¶ Young [36] treats pV_4 and $V_1 \oplus pV_4$ for all p.

3 The number of generators

The Poincaré series of a graded k-algebra $R = \oplus_i R_i$ is defined as $P(t) = \sum_i a_i t^i$, where $a_i = \dim_k(R_i)$. Here we consider $k = \mathbb{C}$ and $R = \mathcal{O}(V)^{\mathrm{SL}_2}$, where V is an SL_2-module. Formulas for the coefficients a_i were already given by Cayley and Sylvester. A closed expression for $P(t)$ as a rational function in t was given by Springer [32] for the case of $V = V_n$, and by Brion [4] in general. The webpages [6] list some results of computations due to Bedratyuk and Brouwer that we use.

3.1 Tamisage

Suppose R has Poincaré series $P(t) = \sum a_i t^i$. (Then $a_0 = 1$ and $a_1 = 0$.) Determine numbers m_i as follows: As long as there is an $i > 0$ for which $a_i \neq 0$, find the smallest such i. If $a_i < 0$, stop. Otherwise put $m_i := a_i$ and replace $P(t)$ by $P(t)(1-t^i)^{m_i}$ and repeat. Let undefined m_i be zero. This is the process that Sylvester called 'tamisage'.

Sylvester's claim.* *The number of generators of R is at least $\sum_i m_i$. More precisely: the number of generators of R of degree i is at least m_i.*

So far this claim is unproved. We use a slightly weaker bound in the below, one that has the advantage of having an easy proof. Maintain two numbers m_i and M_i as lower and upper bounds for the number of generators of degree i in a minimal system of generators. Also maintain upper bounds M_{ij} for the dimension of the space of degree i invariants spanned by those having a factor of degree j but no factor of smaller degree, for $j \leq i$. Put $m_i = a_i - \sum_{j<i} M_{i,j}$ and

$$M_{ij} = \min\{a_{i-j}a_j,\ \sum_{t\geq 1,\, tj\leq i} \binom{m+t-1}{t} S_{i-tj,j}\}$$

where $m = M_j$ and $S_{0,j} = 1$ and $S_{a,j} = \sum_{k>j} M_{a,k}$ for $a > 0$. Finally put

$$\begin{aligned}
d_1 &= \max_{j<i,\, a_j\neq 0} a_{i-j}, \\
d_2 &= \max_{j<i,\, a_j\geq 2} (2a_{i-j} - a_{i-2j}), \\
d_3 &= \max_{j<k<i,\, a_j a_k\neq 0} (a_{i-j} + a_{i-k} - a_{i-j-k}), \\
M_i &= a_i - \max\{0, d_1, d_2, d_3\}.
\end{aligned}$$

where $a_h = 0$ for $h < 0$. This satisfies all requirements. Indeed, for M_i we need to subtract from a_i a lower bound for the number of linearly independent invariants of degree i that have a factor of some smaller degree. If $u \in R_j$, then

*"If the *fundamental postulate* were called into question, this (it may be proved) would not affect the fact of the existence of the groundforms obtained by its aid, but only the possibility of the existence of other groundforms over and above those so obtained. Thus my tables of groundforms could only err (were that possible, which I do not believe it to be) in defect; and as those found by the German method can only err in excess, it follows that, whenever the tables coincide, both must be correct." (J. J. Sylvester [33, p. 249])

$x \mapsto ux$ is an injection of R_{i-j} into R_i, so that d_1 is such a lower bound. Now consider distinct basic invariants $u \in R_j$ and $v \in R_k$. The images of $x \mapsto ux$ and $y \mapsto vy$ (for $y \in R_{i-k}$) have an intersection consisting of invariants with factor uv, so that the dimension of the intersection is a_{i-j-k}. This shows that also d_2 and d_3 are lower bounds. The value given for m_i is clear. Concerning M_{ij}, if an invariant of degree i has precisely t factors that are basic invariants of degree $j < i$, then the quotient of degree $i - tj$ can be chosen in (at most) $S_{i-tj,j}$ ways and the product of t factors can be chosen in $\binom{m+t-1}{t}$ ways.

Now the final lower bound for the number of generators is $r \geq \sum_i m_i$.

Example. The Poincaré series $P(t)$ of $\mathcal{O}(V_{12})^{\mathrm{SL}_2}$ starts

$$1 + t^2 + t^3 + 3t^4 + 3t^5 + 8t^6 + 10t^7 + 20t^8 + 28t^9 + 52t^{10} + 73t^{11} + 127t^{12} + \dots$$

We find

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
m_i	1	1	2	2	4	5	7	9	14	12	9	0	0	0	0	0
g_i	1	1	2	2	4	5	7	9	14	15	19	18	12	2	1	1
M_i	1	1	2	2	4	5	10	13	25	33	57	76				

so that $r \geq 66$, $\mathrm{hd}\, R \geq 56$. The row g_i gives the actual number of generators of degree i (known in this case, cf. [5]), so that $r \geq 113$, $\mathrm{hd}\, R \geq 103$.

3.2 Bounds

In the table below, we list modules, the Poincaré series, and lower bounds for r and $\mathrm{hd}\, R$. In many cases, better bounds are obtained by taking more terms.

module	Poincaré series	$r \geq$	$\mathrm{hd}\, R \geq$
V_{11}	$1 + 2t^4 + 13t^8 + 13t^{10} + 73t^{12} + 110t^{14} + \dots$	158	149
V_{13}	$1 + 2t^4 + 22t^8 + 33t^{10} + 181t^{12} + 375t^{14} + \dots$	502	491
V_{14}	$1 + t^2 + 3t^4 + 10t^6 + 4t^7 + 31t^8 + 27t^9 + 97t^{10} + 110t^{11} + \dots$	182	170
V_{15}	$1 + 3t^4 + t^6 + 36t^8 + 80t^{10} + 418t^{12} + \dots$	425	412
V_{16}	$1 + t^2 + t^3 + 3t^4 + 4t^5 + 13t^6 + 18t^7 + 47t^8 + 84t^9 + 177t^{10} + \dots$	198	184
V_{18}	$1 + t^2 + 4t^4 + t^5 + 16t^6 + 13t^7 + 71t^8 + 99t^9 +$	161	145
V_{20}	$1 + t^2 + t^3 + 4t^4 + 5t^5 + 20t^6 + 35t^7 + 102t^8 +$	123	105
V_{22}	$1 + t^2 + 4t^4 + t^5 + 24t^6 + 26t^7 + 144t^8 + \dots$	164	144
V_{24}	$1 + t^2 + t^3 + 5t^4 + 7t^5 + 29t^6 + 62t^7 + 201t^8 +$	242	220
V_{28}	$1 + t^2 + t^3 + 5t^4 + 8t^5 + 40t^6 + 97t^7 + 365t^8 +$	440	414
V_{32}	$1 + t^2 + t^3 + 6t^4 + 10t^5 + 54t^6 + 153t^7 + \dots$	201	171
$V_2 \oplus V_8$	$1 + 2t^2 + t^3 + 5t^4 + 5t^5 + 15t^6 + 17t^7 + 41t^8 + 54t^9 + 108t^{10} + \dots$	35	26
$V_3 \oplus V_8$	$1 + t^2 + t^3 + 3t^4 + 4t^5 + 9t^6 + 16t^7 + 30t^8 + \dots$	37	27
$V_4 \oplus V_8$	$1 + 2t^2 + 4t^3 + 8t^4 + 16t^5 + 35t^6 + 60t^7 + \dots$	42	31
$V_5 \oplus V_8$	$1 + t^2 + t^3 + 3t^4 + 6t^5 + 15t^6 + 31t^7 + \dots$	43	31
$V_6 \oplus V_8$	$1 + 2t^2 + 2t^3 + 10t^4 + 14t^5 + 46t^6 + 82t^7 + \dots$	88	75

module	Poincaré series	$r \geq$	$\operatorname{hd} R \geq$
$V_1 \oplus 2V_3$	$1 + t^2 + 13t^4 + 26t^6 + \ldots$	26	19
$V_2 \oplus 2V_3$	$1 + 2t^2 + 3t^3 + 9t^4 + 12t^5 + 26t^6 + 44t^7 + \ldots$	26	18
$V_1 \oplus 2V_2 \oplus V_3$	$1 + 3t^2 + 6t^3 + 15t^4 + 30t^5 + 65t^6 + \ldots$	34	25
$V_2 \oplus V_3 \oplus V_4$	$1 + 2t^2 + 3t^3 + 7t^4 + 14t^5 + 29t^6 + 52t^7 + \ldots$	43	34
$V_1 \oplus 2V_2 \oplus V_4$	$1 + 4t^2 + 6t^3 + 18t^4 + 33t^5 + \ldots$	27	17
$3V_2 \oplus V_4$	$1 + 7t^2 + 8t^3 + 42t^4 + 64t^5 + \ldots$	37	26
$2V_3 \oplus V_4$	$1 + 2t^2 + 2t^3 + 9t^4 + 16t^5 + 37t^6 + 71t^7 + \ldots$	69	59
$V_3 \oplus 2V_4$	$1 + 3t^2 + 4t^3 + 10t^4 + 22t^5 + 49t^6 + 96t^7 + \ldots$	45	34
$V_3 \oplus V_5$	$1 + 6t^4 + 7t^6 + 36t^8 + \ldots$	28	21
$V_4 \oplus V_5$	$1 + t^2 + t^3 + 2t^4 + 4t^5 + 8t^6 + 12t^7 + 22t^8 +$		
	$37t^9 + 56t^{10} + \ldots$	59	51
$2V_5$	$1 + t^2 + 7t^4 + 14t^6 + 72t^8 + 168t^{10} + \ldots$	105	96
$V_3 \oplus V_6$	$1 + t^2 + t^3 + 3t^4 + 4t^5 + 8t^6 + 12t^7 + 21t^8 + \ldots$	24	16
$V_4 \oplus V_6$	$1 + 2t^2 + 2t^3 + 7t^4 + 8t^5 + 24t^6 + 31t^7 + 68t^8 +$	33	24
$V_5 \oplus V_6$	$1 + t^2 + t^3 + 3t^4 + 5t^5 + 12t^6 + 22t^7 + \ldots$	31	21
$2V_6$	$1 + 3t^2 + 12t^4 + 6t^5 + 44t^6 + 40t^7 + 150t^8 + \ldots$	29	18

Table 3: Bounds from the Poincaré series

4 Determining V

Consider $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$ with $n_i \geq 1$ for all i. Let $R := \mathcal{O}(V)^{\mathrm{SL}_2}$ be the algebra of invariants of V. We want to determine V if either $p = 1$ and $\operatorname{hd} R \leq 100$, or $p > 1$ and $\operatorname{hd} R \leq 15$.

First consider the case $p = 1$, $V = V_n$. By [29, Proposition 6], if n is even and $\operatorname{hd} R \leq 100$, then $n \leq 24$ or $n \in \{28, 32\}$. By [25, p. 106], if n is odd, then $r \geq p(n-2) + \phi(n-2) - 1$, where $p()$ is the partition function and $\phi()$ is Euler's totient function. It follows that $\operatorname{hd} R \geq 168$ for odd $n \geq 17$. We know $\operatorname{hd} R$ for $n \leq 10$ (a table was given above), and $\operatorname{hd} R \geq 103$ for $n = 12$ (see the example above), and for the remaining values we found $\operatorname{hd} R \geq 105$ in Table 3. This proves Theorem 1.1.

Now consider the case $p > 1$ and assume $\operatorname{hd} R \leq 15$. By the monotony theorem [29, Theorem 2b] we have if $V = W \oplus W'$, then $\operatorname{hd} R \geq \operatorname{hd} \mathcal{O}(W)^{\mathrm{SL}_2} + \operatorname{hd} \mathcal{O}(W')^{\mathrm{SL}_2}$. Therefore, all n_i belong to $\{1, 2, 3, 4, 5, 6, 8\}$, and direct summands W have $\operatorname{hd} \mathcal{O}(W)^{\mathrm{SL}_2} \leq 15$.

If all n_i are either 1 or 2, so that $V = mV_1 \oplus nV_2$, then we have the explicit formula $r = \binom{n}{3} + \binom{m+1}{2}\binom{n+1}{2} + \binom{m}{2} + \binom{n+1}{2}$ (see §7.8 below), and $\operatorname{hd} R =$

$r-(3n+2m-3)$ for $m+n>1$. A table of $\operatorname{hd} R$ shows that V is as claimed.

$n\backslash m$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	1	3	6	10	15	21
1	0	0	1	4	9	16	25			
2	0	1	6	15	28					
3	1	5	16	34						
4	5	13	32							
5	13	26								
6	26									

Now investigate the remaining possibilities.

Case 1: One of the n_i, say n_1, is equal to 8.
By the discussion in the proof of [29, Theorem 4], $\operatorname{hd} R \geq \frac{(b-1)(b-2)}{2}$, where $b = 3 + [\frac{n_2+1}{2}] + \ldots + [\frac{n_p+1}{2}]$. If $b \geq 8$, then $\operatorname{hd} R \geq 21$. We check the cases with $b \leq 7$. By monotony it suffices to look at $V_m \oplus V_8$ for $m = 1,2,3,4,5,6$. If $V = V_1 \oplus V_8$ then R is the algebra of covariants of V_8, generated by 69 elements ([2]), and $\operatorname{hd} R = 61$. In the other cases $\operatorname{hd} R \geq 26$ by Table 3.

Case 2: One of the n_i, say n_1, is equal to 2.
By the discussion in the proof of [29, Theorem 4], $\operatorname{hd} R \geq (c-1)^2$, where $c = [\frac{n_2+1}{2}] + \ldots + [\frac{n_p+1}{2}]$. Since $n_i > 2$ for some i, we have $c \geq 2$. We have $\operatorname{hd} R \geq 16$ for $c \geq 5$. We therefore check the cases $c \in \{2,3,4\}$.

If $c = 2$, then V is $V_2 \oplus V_3$ or $V_2 \oplus V_4$ and $\operatorname{hd} R = 1$.

If $c = 3$, then V is one of $V_1 \oplus V_2 \oplus V_3$, $V_1 \oplus V_2 \oplus V_4$, $2V_2 \oplus V_3$, $2V_2 \oplus V_4$, $V_2 \oplus V_5$ or $V_2 \oplus V_6$. In these six cases one has $\operatorname{hd} R = 9, 11, 11, 11, 23, 20$, respectively. (For the first two and last two cases, see Table 2. For the other two, see §§7.3, 7.4.)

If $c = 4$, then by monotony and the above V does not have a direct summand V_5 or V_6, so that V is one of $V_2 \oplus 2V_3$, $V_2 \oplus V_3 \oplus V_4$, $V_2 \oplus 2V_4$, $2V_1 \oplus V_2 \oplus V_3$, $2V_1 \oplus V_2 \oplus V_4$, $V_1 \oplus 2V_2 \oplus V_3$, $V_1 \oplus 2V_2 \oplus V_4$, $3V_2 \oplus V_3$, $3V_2 \oplus V_4$. If V is $2V_1 \oplus V_2 \oplus V_3$ or $2V_1 \oplus V_2 \oplus V_4$, then $\operatorname{hd} R = 27$ or 48 by Table 4. Explicit generation of invariants for $V_2 \oplus 2V_4$ and $3V_2 \oplus V_3$ shows that $r \geq 29, 49$ so that $\operatorname{hd} R \geq 19, 39$ in these cases. By Table 3 $\operatorname{hd} R \geq 17$ in the remaining five cases.

Case 3: All of the n_i equal 1, 3, 4, 5 or 6.
If V is $V_1 \oplus V_3$, $V_1 \oplus V_4$, $2V_3$, $V_3 \oplus V_4$, $2V_4$, $V_1 \oplus V_5$, or $V_1 \oplus V_6$, then $\operatorname{hd} R$ equals 1, 1, 2, 14, 1, 18, 20, respectively, by Table 2. If V is $V_3 \oplus V_5$, $V_4 \oplus V_5$, $2V_5$, $V_3 \oplus V_6$, $V_4 \oplus V_6$, $V_5 \oplus V_6$, $2V_6$, $2V_3 \oplus V_4$ or $V_3 \oplus 2V_4$, then $\operatorname{hd} R \geq 16$ by Table 3. If V is $2V_1 \oplus V_3$, $2V_1 \oplus V_4$, $V_1 \oplus 2V_3$, $V_1 \oplus V_3 \oplus V_4$, $V_1 \oplus 2V_4$, $3V_3$, $3V_4$, $3V_1 \oplus V_3$, $3V_1 \oplus V_4$, $4V_4$, then $\operatorname{hd} R$ equals 8, 14, 19, 55, 19, 19, 13, 23, 55, 63, respectively, by Tables 2 and 4. By monotony we are done.

This finishes the determination of the V with $\operatorname{hd} R \leq 15$.

5 Finding generators

Let V be an SL_2-module, and $R = \mathcal{O}(V)^{\mathrm{SL}_2}$ its algebra of invariants. Finding a minimal set of generators of R is routine, only requiring computational power, if a good upper bound for the maximum degree of these generators is known. Details for V_9 and V_{10} were given in [7, 8]. (The cases considered here are much smaller.) An upper bound for the maximum degree of a basic generator follows from the Poincaré series when a hsop (homogeneous system of parameters), or at least the set of degrees of a hsop, is known.

5.1 Hilbert's Criterion

One way of computing a system of parameters of R is finding equations for the *nullcone* of V. The nullcone of V, denoted $\mathcal{N}(V)$, is the set of all elements of V on which all invariants vanish. One shows that $\mathcal{N}(V_{n_1} \oplus \ldots \oplus V_{n_p})$ is the set of all $(f_1, \ldots, f_p) \in V_{n_1} \oplus \ldots \oplus V_{n_p}$ such that $f_1, \ldots, f_p$ have a common root of multiplicity $> \frac{1}{2}n_i$ in f_i for all $i = 1, \ldots, p$. This is a consequence of the Hilbert-Mumford criterion. Let $\mathcal{V}(J)$ stand for the vanishing locus of J.

Proposition 5.1. (Hilbert [24]) *Let $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$, and $R = \mathcal{O}(V)^{\mathrm{SL}_2}$, and $m = n_1 + \ldots + n_p + p - 3 > 0$. A set $P_1, \ldots, P_m$ of homogeneous elements of R is a system of parameters of R if and only if $\mathcal{V}(P_1, \ldots, P_m) = \mathcal{N}(V)$.*

5.2 Dixmier's Criterion

Since we do not actually need the hsop but only the degrees, the following is often easier to apply than Hilbert's Criterion.

Proposition 5.2. (Dixmier [11]) *Let G be a reductive group over $\mathbb{C}$, with a rational representation in a vector space V of finite dimension over $\mathbb{C}$. Let $\mathcal{O}(V)$ be the algebra of complex polynomials on V, $R := \mathcal{O}(V)^G$ the subalgebra of G-invariants, and R_d the subset of homogeneous polynomials of degree d in R. Let X be the affine variety such that $\mathbb{C}[X] = R$. Let $m = \dim X$. Let $(d_1, \ldots, d_m)$ be a sequence of positive integers. Assume that for each subsequence $(j_1, \ldots, j_p)$ of $(d_1, \ldots, d_m)$ the subset of points of X where all elements of all R_j with $j \in \{j_1, \ldots, j_p\}$ vanish has codimension not less than p in V. Then R has a system of parameters of degrees $d_1, \ldots, d_m$.* $\square$

When applying this criterion it is convenient to have a notation for 'the codimension of the subset of X defined by the vanishing of all invariants with degree in $\{j_1, \ldots, j_p\}$'. We'll use $[j_1, \ldots, j_p]$.

Note that for $e \geq 1$ an invariant g^e vanishes if and only if g vanishes. It follows that if $j_h | j_i$, $h \neq i$, then $[j_1, \ldots, j_p] = [j_1, \ldots, j_{h-1}, j_{h+1}, \ldots, j_p]$.

5.3 From hsop degrees to generator degrees

Let $R := \mathcal{O}(V)^{\mathrm{SL}_2}$. This is a graded algebra, and formulas to compute its Poincaré series $P(t)$ are well-known. If $(P_1, \ldots, P_m)$ is a system of parameters of R, with degree sequence $(d_1, \ldots, d_m)$, then $P(t)$ can be written as

$$P(t) = \frac{t^{e_1} + \ldots + t^{e_s}}{(1 - t^{d_1}) \ldots (1 - t^{d_m})}$$

and there exist homogeneous $G_1, \ldots, G_s \in R$ with degrees $e_1, \ldots, e_s$, such that

$$R = \bigoplus_{i=1}^{s} G_i \mathbb{C}[P_1, \ldots, P_m].$$

Now $\{P_1, \ldots, P_m, G_1, \ldots, G_s\}$ is a (not necessarily minimal) system of generators of R, and $\max\{d_1, \ldots, d_m, e_1, \ldots, e_s\}$ is an upper bound for the degrees of a set of generators of R.

5.4 Polarization

Let $j = j(f)$ be an invariant of degree d defined on forms $f \in V_n$. Let $\mathbf{i} = (i_1, \ldots, i_s)$ be a sequence of nonnegative integers with $\sum i_h = d$. The $\mathbf{i}$-*polarizations* $j_{\mathbf{i}}$ of j are defined on sV_n by

$$j(\sum_i \lambda_i f_i) = \sum_{\mathbf{i}} j_{\mathbf{i}}(f_1, \ldots, f_s) \lambda_1^{i_1} \ldots \lambda_s^{i_s}.$$

Kraft & Wallach [26] showed for $n > 1$ that $\mathcal{N}(sV_n)$ is defined by the polarizations of any set of functions defining $\mathcal{N}(V_n)$.

5.5 Transvectants

The simplest examples of invariants are obtained using *transvectants*. Given $g \in V_m$ and $h \in V_n$ the expression

$$(g, h) \mapsto (g, h)_p := \frac{(m-p)!(n-p)!}{m!n!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{\partial^p g}{\partial x^{p-i} \partial y^i} \frac{\partial^p h}{\partial x^i \partial y^{p-i}}$$

defines a linear and SL_2-equivariant map $V_m \otimes V_n \to V_{m+n-2p}$, which is classically called the *p-th transvectant* (Ueberschiebung). The $(g, h)_p$ are the coefficients of the image of $g \otimes h$ under the isomorphism (Clebsch-Gordan formula)

$$V_m \otimes V_n \simeq V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{m-n}$$

(for $m \geq n$). We have $(f, g)_0 = fg$ and $(f, f)_{2i+1} = 0$ for all integers $i \geq 0$.

6 Systems of parameters

In this section we give a homogeneous system of parameters (or at least the degrees of a homogeneous system of parameters) for the algebras of invariants in the cases occurring in Theorem 1.2.

Let $\mathrm{discr}(f)$ denote the discriminant of the polynomial f. If f has degree m, then $\mathrm{discr}(f)$ is an expression of degree $2m-2$ in the coefficients of f. This expression vanishes if and only if f has a root of multiplicity greater than 1.

Let $\mathrm{res}(f,g)$ denote the resultant of the polynomials f and g. If f and g have degrees m and n, respectively, then $\mathrm{res}(f,g)$ is an expression of degree $m+n$ in the coefficients of f and g. It vanishes if and only if f and g have a common root.

Let $\sim$ denote equality up to a nonzero constant.

Lemma 6.1.
(i) Let $l,m \in V_1$. Then $\mathrm{res}(l,m) = (l,m)_1$.
(ii) Let $q \in V_2$. Then $\mathrm{discr}(q) \sim (q,q)_2$.
(iii) Let $l \in V_1$ and $q \in V_2$. Then $\mathrm{res}(l,q) = (q,l^2)_2$.
(iv) Let $q,r \in V_2$. Then $\mathrm{res}(q,r) = (q,r)_2^2 - (q,q)_2\,(r,r)_2$.
(v) Let $f \in V_3$. Then $\mathrm{discr}(f) \sim (f,(f,(f,f)_2)_1)_3$.
(vi) Let $l \in V_1$ and $f \in V_3$. Then $\mathrm{res}(l,f) \sim (f,l^3)_3$. □

Brion [4] shows for $V = V_{\mathbf{n}} = V_{n_1} \oplus \ldots \oplus V_{n_p}$ with $p > 1$ that a multihomogeneous system of parameters exists in only 13 cases, namely precisely the cases with $p \in \{2,3\}$ in Popov's classification (Table 1). We give the systems here (improving that for $2V_3$), together with those for V_2, V_3, V_4, for later use.

Proposition 6.2. *The modules V given in the table below, with generic elements as indicated, have a (multi-)homogeneous system of parameters as given.*

V	*element*	*hsop degrees*	$\mathrm{hsop}(V)$
V_2	q	2	$(q,q)_2$
V_3	c	4	$\mathrm{discr}(c)$
V_4	f	$2,3$	$(f,f)_4,\ (f,(f,f)_2)_4$
$2V_1$	(l,m)	2	$(l,m)_1$
$V_1 \oplus V_2$	(l,q)	$2,3$	$(q,q)_2,\ (q,l^2)_2$
$V_1 \oplus V_3$	(l,c)	$4,4,4$	$\mathrm{hsop}(V_3),\ (c,l^3)_3,\ (c,(c,l^2)_1)_3$
$V_1 \oplus V_4$	(l,f)	$2,3,5,6$	$\mathrm{hsop}(V_4),\ (f,l^4)_4,\ ((f,f)_2,l^4)_4$
$2V_2$	(q,r)	$2,2,2$	*polarized* $\mathrm{hsop}(V_2)$
$V_2 \oplus V_3$	(q,c)	$2,3,4,5$	$(q,q)_2,\ (c,(c,q)_1)_3,\ \mathrm{hsop}(V_3),\ \mathrm{res}(q,c)$
$V_2 \oplus V_4$	(q,f)	$2,2,3,3,4$	$(q,q)_2,\ \mathrm{hsop}(V_4),\ (f,q^2)_4,\ ((f,f)_2,q^2)_4$
$2V_3$	(c,d)	$2,4,4,4,4$	$(c,d)_3,\ \mathrm{discr}(c),\ \mathrm{discr}(d),$
			$(c,(c,(c,d)_2)_1)_3,\ (d,(d,(c,d)_2)_1)_3$
$2V_4$	(f,g)	$2,2,2,3,3,3,3$	*polarized* $\mathrm{hsop}(V_4)$
$3V_1$	(l,m,n)	$2,2,2$	$(l,m)_1,\ (l,n)_1,\ (m,n)_1$
$2V_1 \oplus V_2$	(l,m,q)	$2,2,3,3$	$(l,m)_1,\ (q,q)_2,\ (q,l^2)_2,\ (q,m^2)_2$
$V_1 \oplus 2V_2$	(l,q,r)	$2,2,2,3,3$	$\mathrm{hsop}(2V_2),\ (q,l^2)_2,\ (r,l^2)_2$
$3V_2$	(q,r,s)	$2,2,2,2,2,2$	*polarized* $\mathrm{hsop}(V_2)$

6.1 The cases $V = pV_1$ and $V = pV_2$

Let $V = pV_1$ (or $V = pV_2$), where $p > 1$. Let I be the ideal of R generated by the invariants of degree 2. By Lemma 6.1 (i) (or (ii),(iv)) we have $\mathcal{N}(V) = \mathcal{V}(I)$. Consider the proof of the Noether Normalization Lemma. In the situation of an algebra where all generators are homogeneous of the same degree, it produces a homogeneous set of parameters of this same degree. Therefore, I, and hence R, has a homogeneous system of parameters consisting of $2p-3$ (or $3p-3$) elements of degree 2. (For generators, see §7.8 below.)

6.2 The case $V = mV_1 \oplus nV_2$

Proposition 6.3. *Let $V = mV_1 \oplus nV_2$ with $m, n > 0$, and let $R = \mathcal{O}(V)^{\mathrm{SL}_2}$.*

If $2n+1 \geq m$, then R has a system of parameters consisting of $m+2n-2$ invariants of degree 2 and $m+n-1$ invariants of degree 3.

If $2n+1 < m$, then R has a system of parameters consisting of $m+2n-2$ invariants of degree 2 and $3n$ invariants of degree 3, and $m-2n-1$ invariants of degree 6.

Proof. Invoke Dixmier's Criterion. We have $\dim X = 2m+3n-3$. We have to show that $[2] \geq m+2n-2$ and $[3] \geq m+n-1$ (if $2n+1 \geq m$) and $[2,3] \geq 2m+3n-3$.

Consider $v = (l_1, \ldots, l_m, q_1, \ldots, q_n) \in V$. According to Lemma 6.1, among the invariants of degree 2 there are $\mathrm{res}(l_i, l_j)$ and $\mathrm{discr}(q_i)$ and $\mathrm{res}(q_i, q_j)$ for all i, j $(j \neq i)$, and among the invariants of degree 3 there are the $\mathrm{res}(l_i, q_j)$. If all of these vanish on v, then $v \in \mathcal{N}(V)$. Hence $[2,3] = \dim X$, as desired.

Look at [2]. The element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of SL_2 acts via $g.x = dx - by$, $g.y = -cx + ay$. Using SL_2 one can move the first nonzero linear form (if there is one) to x. Given that all linear forms have a common zero, this means that the remaining at most $m-1$ linear forms now look like cx for various constants c, zero or not. Given that all quadratic forms have a common double root, and that elements $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ preserve x and act on the q_i, we can pick orbit representatives dy^2 (for nonzero d) for the quadratic forms, unless none of them involve y, in which case we are in the nullcone. Altogether the result has dimension at most $m+n-1$, so that $[2] \geq \dim X - (m+n-1) = m+2n-2$, as desired.

Look at [3]. Now we are in a part of X where each of the quadratic forms shares a zero with each of the linear forms. Suppose first that at least one linear form and at least one quadratic form are nonzero. We can pick x as orbit-representative for one linear form. All quadratic forms now look like $x(ax+by)$ for various a, b, and we can normalize one to cx^2 or cxy with nonzero c. All linear forms now look like dx or dy for various constants d. Altogether the result has dimension at most $m-1+2(n-1)+1 = m+2n-2$, as desired.

If all linear forms are zero, then consider the invariants $(q_i, (q_j, q_k)_1)_2$ with $1 \leq i < j < k \leq n$ of degree 3. Let us compute this for the case of the three forms $ax^2 + 2bxy + cy^2$, $dx^2 + 2exy + fy^2$, $gx^2 + 2hxy + iy^2$. The result is (up to a constant) $aei - afh - bdi + bfg + cdh - ceg$, the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

If all such determinants vanish, then any three of the quadratic forms are linearly dependent, so that n quadratic forms involve at most $2n + 2 - 3 = 2n - 1$ constants, after dividing out SL_2. Since $m \geq 1$, this is not more than $m + 2n - 2$ and we also have the desired bound in this case.

If all quadratic forms are zero, then the result has dimension $2m - 3$, so codimension $3n$. We showed $[3] \geq \min(m + n - 1, 3n)$. It follows that for $mV_1 \oplus nV_2$ with $2n \geq m - 1 \geq 0$ there is a hsop with $m + 2n - 2$ elements of degree 2 and $m + n - 1$ elements of degree 3. This proves the first claim.

For the second claim we still have to show that $[6] \geq 1$ and $[2, 6] \geq m + 2n - 1$ and $[3, 6] \geq m + n - 1$, but $[6] \geq [2, 3] = 2m + 3n - 3$. $\square$

6.3 The case $V = 2V_1 \oplus V_3$

Let $V = 2V_1 \oplus V_3$. We have $\dim X = 5$. The ring R has a homogeneous system of parameters with degrees $2, 4, 4, 4, 4$.

Indeed, pick $(l, m, c) \in V$. By Proposition 6.2 we find $(l, m) \in \mathcal{N}(2V_1)$ and $(l, c), (m, c) \in \mathcal{N}(V_1 \oplus V_3)$ (and hence $(l, m, c) \in \mathcal{N}(V)$) when all invariants of degree 4 vanish on (l, m, c). This shows that there is a hsop with degrees $4, 4, 4, 4, 4$. Since $[2] \geq 1$ there is also a hsop with degrees $2, 4, 4, 4, 4$ by Dixmier's Criterion.

6.4 The case $V = V_1 \oplus V_2 \oplus V_3$

Let $V = V_1 \oplus V_2 \oplus V_3$. We have $\dim X = 6$. We show that the ring R has a system of parameters with degrees $3, 3, 4, 4, 4, 5$.

A form on which all invariants of degrees $3, 4, 5$ vanish is in the nullcone, so that $[3, 4, 5] = 6$. We have to check that $[5] \geq 1$, $[3] \geq 2$, $[4] \geq 3$, $[3, 5] \geq 3$, $[4, 5] \geq 4$, $[3, 4] \geq 5$. Let the forms be l, q, f.

If all invariants of degree 4 vanish, then f has a double root that is also a root of l, and q has a double root and only 3 variables are left, so $[4] \geq 3$. If moreover $\mathrm{res}(q, f)$ vanishes, then only pieces of dimension 2 are left, so $[4, 5] \geq 4$. Or, if moreover $(q, l^2)_2$ and $(f, (f, q)_1)_3$ vanish, then we are in the nullcone or in a piece of dimension at most 1, so that $[3, 4] \geq 5$.

If all invariants of degree 3 vanish (and in particular $\mathrm{res}(q, l)$ and $(f, lq)_3$ and $(f, (f, q)_1)_3$), and $q \neq 0$, then w.l.o.g. either $q = x^2$, $l = ax$, $f = bx^3 + 3cx^2y + 3dxy^2 + ey^3$ with $ae = 0$ and $ce - d^2 = 0$, or $q = hxy$, $l = ax$, $f = bx^3 + 3cx^2y + 3dxy^2 + ey^3$ with $ae = 0$ and $be = cd$, of dimension at most 3 (since the torus of SL_2 still acts). If $q = 0$ then we are in $V_1 \oplus V_3$ of dimension 3. Altogether $[3, 5] \geq [3] \geq 3$. (In fact $[3, 5] = 3$ because of the part with $q = 0$.)

Finally $[5] \geq 1$ is clear.

Note that in this case the denominator of $P(t)$ might suggest to look for a hsop with degrees $2, 3, 3, 4, 4, 5$, but there is none since $[2, 3, 5] = 3$.

6.5 The case $V = V_1 \oplus V_2 \oplus V_4$

Let $V = V_1 \oplus V_2 \oplus V_4$. We have $\dim X = 7$. We show that the ring R has a system of parameters with degrees $2, 2, 3, 3, 4, 5, 6$.

Since $(l, q, f) \in \mathcal{N}(V)$ if and only if $(l, q) \in \mathcal{N}(V_1 \oplus V_2)$ and $(l, f) \in \mathcal{N}(V_1 \oplus V_4)$ and $(q, f) \in \mathcal{N}(V_2 \oplus V_4)$, it follows that if the eight invariants $(q, q)_2$, $(f, f)_4$, $(f, (f, f)_2)_4$, $(q, l^2)_2$, $(f, q^2)_4$, $((f, f)_2, q^2)_4$, $(f, l^4)_4$, $((f, f)_2, l^4)_4$ (of degrees $2, 2, 3, 3, 3, 4, 5, 6$) vanish, then $(l, q, f) \in \mathcal{N}(V)$.

The above five invariants of degrees $2, 4, 5, 6$, together with two random linear combinations of the invariants of degree 3 will constitute a hsop. In particular, we find that the two combinations $(f, (f, f)_2)_4 + (f, q^2)_4$ and $(q, l^2)_2 - (f, (f, f)_2)_4$ yield such a hsop. (Using Singular one finds that the ideal generated by these seven invariants contains the sixth power of each invariant of degree 3. Now apply Proposition 5.1.)

Note that in this case the denominator of $P(t)$ suggests the existence of a hsop with degrees $2, 2, 3, 3, 3, 4, 5$. But such a hsop does not exist: all invariants of degrees 2 up to 5 vanish on $(x, 0, 4xy^3)$, which is not in $\mathcal{N}(V)$.

6.6 The case $V = 2V_2 \oplus V_3$

Let $V = 2V_2 \oplus V_3$. We have $\dim X = 7$. The ring R has a system of parameters with degrees $2, 2, 3, 4, 5, 5, 6$.

Indeed, we have to check that $[3] \geq 1$, $[2] \geq 2$, $[5] \geq 2$, $[4] \geq 3$, $[2, 3] \geq 3$, $[3, 5] \geq 3$, $[6] \geq 4$, $[2, 5] \geq 4$, $[3, 4] \geq 4$, $[4, 5] \geq 5$, $[4, 6] \geq 5$, $[2, 3, 5] \geq 5$, $[5, 6] \geq 6$, $[3, 4, 5] \geq 6$, $[4, 5, 6] \geq 7$.

Let the forms be q, r, f. If $f \neq 0$, then we can normalize f to one of x^3, x^2y or $axy(x + y)$. If $f = 0$ but $q \neq 0$, then we can normalize q to x^2 or xy.

If all invariants of degree 2 vanish, then $\mathrm{discr}(q) = \mathrm{discr}(r) = \mathrm{res}(q, r) = 0$ so that q and r have a common double zero. Now if $q \neq 0$, then r is determined by a single constant. So dimensions are at most 1 larger than for the corresponding case for $V_2 \oplus V_3$. Hence $[2] \geq 3$, $[4] \geq 4$, $[6] \geq [2, 3] \geq 4$, $[2, 5] \geq 4$, $[4, 6] \geq [3, 4] \geq 5$, $[4, 5] \geq 5$, $[2, 3, 5] \geq 5$, $[3, 4, 5] \geq 6$.

If all invariants of degree 3 vanish, then $(f, (f, q)_1)_3 = (f, (f, r)_1)_3 = 0$. That $(f, (f, q)_1)_3 = 0$ says that the three quadratic forms $\frac{df}{dx}$, $\frac{df}{dy}$ and q are linearly dependent. So, here either f has a triple root, or f_x and f_y are linearly independent and q, r are determined by two coefficients each. Hence $[3] \geq 2$.

If all invariants of degree 5 vanish, then $\mathrm{res}(f, q) = \mathrm{res}(f, r) = 0$ restrict q, r when $f \neq 0$. Hence $[5] \geq 2$.

If all invariants of degrees $3, 5$ vanish, then if f does not have a double root, then q, r are determined by one coefficient each. If $f = x^2y$ or $f = x^3$, then q, r are determined by two coefficients each. If $f = 0$, then we are in $2V_2$ which has dimension 3. Hence $[3, 5] \geq 3$.

If all invariants of degrees $2, 3, 5$ vanish, then either $f = axy(x+y)$, $q = r = 0$, or ($f = x^2y$ or $f = x^3$), $q = bx^2$, $r = cx^2$ and $(q, r, f) \in \mathcal{N}(V)$, or $f = 0$ and again $(q, r, f) \in \mathcal{N}(V)$. Hence $[5, 6] \geq [2, 3, 5] \geq 6$ and $[3, 4, 5] \geq [2, 3, 5] \geq 6$.

Finally, if all invariants of degrees $2, 3, 4, 5$ vanish, then $\mathrm{discr}(f) = 0$ and we are in the nullcone.

Note that in this case the denominator of $P(t)$ might suggest to look for a hsop with degrees $2, 2, 3, 3, 4, 5, 5$, but there is none since $[3, 5] = 3$.

6.7 The case $V = 2V_2 \oplus V_4$

Let $V = 2V_2 \oplus V_4$. We have $\dim X = 8$. We show that the ring R has a system of parameters with degrees $2, 2, 2, 3, 3, 3, 4, 4$.

Since $(q, r, f) \in \mathcal{N}(V)$ if and only if $(q, r) \in \mathcal{N}(2V_2)$ and $(q, f), (r, f) \in \mathcal{N}(V_2 \oplus V_4)$, it follows that if the nine invariants $(q, q)_2$, $(q, r)_2$, $(r, r)_2$, $(f, f)_4$, $(f, (f, f)_2)_4$, $(f, q^2)_4$, $(f, r^2)_4$, $((f, f)_2, q^2)_4$, and $((f, f)_2, r^2)_4$ (of degrees $2, 2, 2, 2, 3, 3, 3, 4, 4$) vanish, then $(q, r, f) \in \mathcal{N}(V)$.

The above five invariants of degrees $3, 4$, together with three random linear combinations of the four invariants of degree 2 will constitute a hsop. In particular, we find that the three combinations $(q, q)_2 + (q, r)_2$, $(r, r)_2 + (f, f)_4$, and $(q, q)_2 - (f, f)_4$ yield such a hsop. (Using Singular one finds that the ideal generated by these eight invariants contains the 7th power of each invariant of degree 2. Now apply Proposition 5.1.)

Note that in this case the denominator of $P(t)$ might suggest to look for a hsop with degrees $2, 2, 2, 2, 3, 3, 3, 4$, but no such hsop exists since all invariants of degree 2 or 3 vanish on (ax^2, bx^2, xy^3), so that $[2, 3] \leq 6$.

6.8 The case $V = 3V_4$

Let $V = 3V_4$. We have $\dim X = 12$. We show that the ring R has a system of parameters consisting of 6 invariants of degree 2 and 6 of degree 3.

Indeed, since $2V_4$ has a hsop consisting of elements of degrees 2 and 3 only, $\mathcal{N}(3V_4)$ is determined by elements of degrees $2, 3$, so that $[2, 3] = 12$. We have $\dim R_2 = 6$, and using Singular we find that $[2] \geq 6$. Since $[2, 3] = 12$, and the six invariants of degree 2 can decrease dimensions by not more than 6, we must have $[3] \geq 6$. By Dixmier's Criterion there is a hsop as claimed.

On the other hand, using Singular we also find that $[3] \geq 7$. It follows that there is also a hsop consisting of 5 invariants of degree 2 and 7 of degree 3.

6.9 The case $V = 2V_1 + V_4$

Let $V = 2V_1 + V_4$. We have $\dim X = 6$. We show that the ring R has a system of parameters with degrees $2, 3, 5, 5, 6, 6$.

Indeed, let the forms be l, m, f. The vanishing of $(f, f)_4$, $(f, (f, f)_2)_4$, $(f, l^4)_4$, $((f, f)_2, l^4)_4$, $(f, m^4)_4$, $((f, f)_2, m^4)_4$ implies that l and f, and also m and f, have a common zero of multiplicity at least 3 for f. If $f \neq 0$ then

$(l,m,f) \in \mathcal{N}(V)$. If $f=0$ then the same conclusion follows if also $(l,m)_1 = 0$. By Dixmier's Criterion there is a hsop with the above degrees.

6.10 The case $V = V_3 + V_4$

Let $V = V_3 + V_4$. We have $\dim X = 6$. We show that the ring R has a system of parameters with degrees $2,3,4,5,6,7$.

Indeed, let $(f,g) \in V$. Let $i_2 = (g,g)_4$, $i_3 = (g,(g,g)_2)_4$, $i_4 = \mathrm{discr}(f)$. The vanishing of i_2, i_3, i_4 forces f to have a double zero, and g to have a triple zero. If these zeros occur at different places, then w.l.o.g. $f = x^2(ax+by)$ and $g = y^3(cx+dy)$. Consider the four invariants $i_5' = (f,(f,(f,(f,g)_1)_3)_1)_3$, $i_5'' = (f,(f,(g,(g,g)_2)_1)_3)_3$, $i_6 = (f,(f,(f,(f,(g,g)_2)_1)_3)_1)_3$, $i_7 = (f,(f,g)_3^3)_3$. They have values b^4d, a^2c^3, b^4c^2 and $ab^3c^3 - 10a^2b^2c^2d + 32a^3bcd^2 - 32a^4d^3$, all up to some nonzero constant. Let $i_5 = i_5' + i_5''$. Suppose i_5, i_6, i_7 vanish on (f,g). If the form f has a triple zero (i.e., $b=0$) then i_5, i_6, i_7 take the values a^2c^3, 0 and a^4d^3, so that either $f=0$ or $g=0$. Otherwise we have w.l.o.g. $b=1$, the vanishing of i_6 and i_5 forces $c=0$ and $d=0$, so that $g=0$. This shows that $i_2, i_3, i_4, i_5, i_6, i_7$ determine the nullcone and hence form a hsop.

7 Generators

In this section we give a set of generators for the algebras of invariants in the cases occurring in Theorem 1.2.

7.1 The case $V = V_1 \oplus V_2 \oplus V_3$

Let $V = V_1 \oplus V_2 \oplus V_3$. Here $r = 15$, cf. [18], [33], [20], §140. Let the forms be l, q, c of degrees 1, 2, 3, respectively. The table below gives the 15 basic generators with degree and multidegree. Abbreviations are as above.

dg	mdeg	form	dg	mdeg	form	dg	mdeg	form
2	020	$(q,q)_2$	4	121	$(\mathrm{lq}_1,\mathrm{qc}_2)_1$	5	212	$(\mathrm{lq}_1,(l,\mathrm{cc}_2)_1)_1$
3	012	$(q,\mathrm{cc}_2)_2$	4	202	$(\mathrm{lc}_1,\mathrm{lc}_1)_2$	5	311	$(l^2,(\mathrm{lc}_1,q)_1)_2$
3	111	$(q,\mathrm{lc}_1)_2$	4	301	$(l^2,\mathrm{lc}_1)_2$	6	123	$((l,\mathrm{cc}_2)_1,(q,\mathrm{qc}_1)_2)_1$
3	210	$(l^2,q)_2$	5	032	$(\mathrm{qc}_2,(q,\mathrm{qc}_1)_2)_1$	6	303	$((l,\mathrm{cc}_2)_1,(l^2,c)_2)_1$
4	004	$(\mathrm{cc}_2,\mathrm{cc}_2)_2$	5	113	$((\mathrm{lc}_1,q)_1,\mathrm{cc}_2)_2$	7	034	$(\mathrm{qc}_2^2,(\mathrm{qc}_1,c)_2)_2$

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate R_d for $d \le 14$, and they do.

7.2 The case $V = V_1 \oplus V_2 \oplus V_4$

Let $V = V_1 \oplus V_2 \oplus V_4$. Here $r = 18$, cf. [18], [33]. Let the forms be l, q, f of degrees 1, 2, 4, respectively. The table below gives the 18 basic generators with degree and multidegree.

dg	mdeg	form	dg	mdeg	form	dg	mdeg	form
2	020	$(q,q)_2$	4	022	$((f,f)_2,q^2)_4$	6	222	$(f,l.(l,q)_1.(f,q)_2)_4$
2	002	$(f,f)_4$	5	401	$(f,l^4)_4$	6	033	$(f,(f,q)_2.(f,q^2)_3)_4$
3	210	$(q,l^2)_2$	5	221	$(f,(lq.(l,q)_1)_4$	7	412	$(f,lq.(f,l^3)_3)_4$
3	021	$(f,q^2)_4$	5	212	$(f,l^2.(q,f)_2)_4$	7	223	$(f,lq.(f,l.(f,q)_1)_4)_4$
3	003	$(f,(f,f)_2)_4$	6	411	$(f,l^3.(l,q)_1)_4$	8	413	$(f,l^2.(f,lq.(f,l)_1)_4)_4$
4	211	$(f,l^2q)_4$	6	402	$((f,f)_2,l^4)_4$	9	603	$(f,l^3.(f,l.(f,l^2)_2)_3)_4$

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate R_d for $d \leq 15$, and they do.

7.3 The case $V = 2V_2 \oplus V_3$

Let $V = 2V_2 \oplus V_3$. We find $r = 18$. Let the forms be q, r, c, of degrees 2, 2, 3, respectively. Let $u = (c,q^2)_3$ and $v = (c,qr)_3$. The table below gives the 18 basic generators with degree and multidegree. (For multidegree $i.j.k$ only the entries with $i \geq j$ are given.)

dg	mdeg	form	dg	mdeg	form	dg	mdeg	form
2	200	$(q,q)_2$	4	004	$(c,(c,(c,c)_2)_1)_3$	6	222	$(c,(q,r)_1v)_3$
2	110	$(q,r)_2$	5	302	$(c,qu)_3$	7	304	$(c,u.(c,(c,q)_1)_2)_3$
3	102	$(c,(c,q)_1)_3$	5	212	$(c,qv)_3$	7	214	$(c,u.(c,(c,r)_1)_2)_3$
4	112	$(c,q.(c,r)_2)_3$	6	312	$(c,(q,r)_1u)_3$			

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate R_d for $d \leq 17$, and they do.

7.4 The case $V = 2V_2 \oplus V_4$

Let $V = 2V_2 \oplus V_4$. We find $r = 19$. Let the forms be q, r, f, of degrees 2, 2, 4, respectively. The table below gives the 19 basic generators with degree and multidegree. (For multidegree $i.j.k$ only the entries with $i \geq j$ are given.)

dg	mdeg	form	dg	mdeg	form	dg	mdeg	form
2	200	$(q,q)_2$	3	111	$(f,qr)_4$	4	211	$(f,q.(q,r)_1)_4$
2	110	$(q,r)_2$	3	003	$(f,(f,f)_2)_4$	5	212	$(f,(f,q)_2.(q,r)_1)_4$
2	002	$(f,f)_4$	4	202	$(f,q.(f,q)_2)_4$	6	303	$(f,(f,q)_2.(f,q^2)_3)_4$
3	201	$(f,q^2)_4$	4	112	$(f,q.(f,r)_2)_4$	6	213	$(f,(f,q)_2.(f,qr)_3)_4$

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate R_d for $d \leq 12$, and they do.

7.5 The case $V = 3V_4$

Let $V = 3V_4$. We find $r = 25$. Let the forms be f, g, h, all of degree 4. The table below gives the 25 basic generators with degree and multidegree. (For multidegree $i.j.k$ only the entries with $i \geq j \geq k$ are given.)

dg	mdeg	form	dg	mdeg	form	dg	mdeg	form
2	200	$(f,f)_4$	3	210	$(f,(f,g)_2)_4$	4	211	$(f,(f,(g,h)_2)_2)_4$
2	110	$(f,g)_4$	3	111	$(f,(g,h)_2)_4$	5	221	$(h,(f,g)_3.(f,g)_3)_4$
3	300	$(f,(f,f)_2)_4$	4	220	$(f,(f,(g,g)_2)_2)_4$			

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate R_d for $d \leq 15$, and they do. For $d = 15$ this required computing the rank (34734) of a matrix with 10^{10} entries.

7.6 The case $V = V_3 \oplus V_4$

Let $V = V_3 \oplus V_4$. We find $r = 20$. Let the forms be c, e. Omit parentheses where that does not introduce ambiguity, so that $(c,(c,e)_3(c,(c,e)_1)_3)_3$ is written $(c, ce_3cce_{13})_3$. Write $l = ce_3$. The table below gives the 20 basic generators with degree and multidegree.

dg	mdeg	form	dg	mdeg	form	dg	mdeg	form
2	02	ee_4	7	43	$ccccee e_{213213}$	9	63	$(c, cccce_{1231}cee_{23})_3$
3	03	eee_{24}	7	43	$(c, cee_{23}cce_{13})_3$	9	63	$(c, cccee_{2123}cce_{13})_3$
4	40	$cccc_{213}$	7	43	$(c, l^3)_3$	9	63	$(c, ccce_{123}l^2)_3$
5	41	$cccce_{1313}$	8	62	$(c, cccce_{1231}l)_3$	9	45	$(c, cee_{23}cee_{23}l)_3$
5	23	$cceee_{2133}$	8	44	$(c, cceee_{2132}l)_3$	10	64	$(c, cccee_{2123}ccee_{222})_3$
6	42	$ccccee_{21313}$	8	44	$(c, cee_{23}l^2)_3$	10	64	$(c, cccee_{2123}l^2)_3$
6	42	$(c, cce_{13}l)_3$				11	65	$(c, ccceee_{21313}l^2)_3$

Given the Poincaré series and the hsop degrees, it suffices to check that these generators generate R_d for $d \leq 18$, and they do.

7.7 The case $V = W \oplus mV_1$

Given a basic system of covariants for a module W, one finds a basic system of covariants for $W \oplus V_1$ by replacing each covariant j (of order o) of the system for W by the $o+1$ covariants $(j, l^i)_i$ ($0 \leq i \leq o$) and adding the covariant l, where l is the linear form corresponding to V_1. (This is classical. See [9, §55], [20, §138A].) Equivalently, given the invariants of $W \oplus V_1$, one finds the invariants of $W \oplus mV_1$ by polarization.

One finds further results that should be regarded classical:

module	r	module	r	module	r
$2V_1 \oplus V_2$	5	$2V_1 \oplus V_3$	13	$2V_1 \oplus V_4$	20
$2V_1 \oplus 2V_2$	13	$3V_1 \oplus 2V_2$	24	$3V_1 \oplus V_3$	30
$2V_1 \oplus V_2 \oplus V_3$	35	$2V_1 \oplus 2V_4$	103	$3V_1 \oplus V_4$	63
$2V_1 \oplus V_2 \oplus V_4$	57				

Table 4: Some classical results involving multiple V_1

7.8 The case $V = mV_1 \oplus nV_2$

Let $V = mV_1 + nV_2$. Then $r = \binom{n}{3} + \binom{m+1}{2}\binom{n+1}{2} + \binom{m}{2} + \binom{n+1}{2}$.

If the forms are ℓ_i $(1 \le i \le m)$ and q_j $(1 \le j \le n)$, then there are $\binom{m}{2}+\binom{n+1}{2}$ basic invariants of degree 2, namely $(\ell_i, \ell_j)_1$ for $i < j$ and $(q_i, q_j)_2$ for $i \le j$, and $n\binom{m+1}{2} + \binom{n}{3}$ basic invariants of degree 3, namely $(q_k, \ell_i\ell_j)_2$ for $i \le j$ and $(q_i, (q_j, q_k)_1)_2$ for $i < j < k$, and $\binom{m+1}{2}\binom{n}{2}$ basic invariants of degree 4, namely $((q_i, q_j)_1, \ell_k\ell_m)_2$ for $i < j$, $k \le m$.

In order to show this, we quote the following result [9, §54]:

Proposition 7.1. *Let $\mathcal{R}$ and $\mathcal{S}$ be two SL_2-algebras whose covariants are finitely generated. Then the covariants of $\mathcal{R}\oplus\mathcal{S}$ are alsofinitely generated. If $P_1, \ldots, P_r$ are the generators of the covariants of $\mathcal{R}$, and $Q_1, \ldots, Q_s$ are the generators of the covariants of $\mathcal{S}$, then a finite generating system can be chosen from the set of transvectants $[P, Q]_l$, $l \ge 0$, where P is a monomial in the P_i's and Q a monomial in the Q_j's.*

Apply this with $\mathcal{R} = mV_1$ and $\mathcal{S} = nV_2$, with forms as above. The covariants of mV_1 are generated by the ℓ_i themselves, and the invariants $(\ell_i, \ell_j)_1$ for $i < j$. The covariants of nV_2 are generated by the q_i themselves, the covariants $(q_i, q_j)_1$ for $i \le j$, and the invariants $(q_i, q_j)_2$ for $i \le j$ and $(q_i, (q_j, q_k)_1)_2$ for $i < j < k$.

We add to the set of generators of R the invariants of degrees 3 $(q_k, \ell_i\ell_j)_2$ for $i \le j$, and the invariants of degree 4 $((q_i, q_j)_1, \ell_k\ell_m)_2$ for $i < j$, $k \le m$. Given that

$$(r_1 \ldots r_p, \ell_1 \ldots \ell_{2p})_{2p} \sim \sum (r_1, \ell_{i_1}\ell_{i_2})_2 \ldots (r_p, \ell_{i_{2p-1}}\ell_{i_{2p}})_2,$$

there are no other irreducible invariants.

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