A Hilton-Milner theorem for vector spaces

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Abstract

We show for $k \geq 3$ that if $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+3$, then any intersecting family $\mathcal{F}$ of k-subspaces of an n-dimensional vector space over $GF(q)$ with $\bigcap_{F \in \mathcal{F}} F = 0$ has size at most $\left[{n-1 \atop k-1}\right] - q^{k(k-1)}\left[{n-k-1 \atop k-1}\right] + q^k$. This bound is sharp as is shown by Hilton-Milner type families. As an application of this result, we determine the chromatic number of the corresponding q-Kneser graphs.

1 Introduction

In 1961, Erdős, Ko and Rado [4] proved that if $\mathcal{F}$ is a k-uniform intersecting family of subsets of an n-element set X, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ when $2k \leq n$. Furthermore they proved that if $2k+1 \leq n$, then equality holds if and only if $\mathcal{F}$ is the family of all subsets containing a fixed element $x \in X$.

For any family $\mathcal{F}$ of sets the *covering number* $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. The result of Erdős, Ko and Rado states that to obtain an intersecting family of maximum size, one has to consider a family with $\tau(\mathcal{F}) = 1$ when $2k+1 \leq n$.

Hilton and Milner [13] determined the maximum size of an intersecting family with $\tau(\mathcal{F}) \geq 2$.

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Theorem 1.1 (Hilton & Milner [13]) *Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $k \geq 3$, $n \geq 2k+1$ and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. The families achieving that size are*

(i) for any k-subset F and $x \in X \setminus F$ the family

$$\mathcal{F}_{HM} = \{F\} \cup \{G \in \tbinom{X}{k} : x \in G,\ F \cap G \neq \emptyset\},$$

(ii) if $k = 3$, then for any 3-subset S the family

$$\mathcal{F}_3 = \{F \in \tbinom{X}{3} : |F \cap S| \geq 2\}.$$

In this paper we will be interested in the q-analogue of Theorem 1.1. If q is a prime power, then a family $\mathcal{F}$ of k-subspaces of an n-dimensional vector space V over $GF(q)$ (in notation $\mathcal{F} \subseteq \left[{V \atop k}\right]$) is *$t$-intersecting* if for any $F_1, F_2 \in \mathcal{F}$ we have $\dim(F_1 \cap F_2) \geq t$. Intersecting means 1-intersecting. We will say that two subspaces U_1, U_2 of V are *disjoint* if $U_1 \cap U_2 = 0$.

In 1975, Hsieh [14] proved the q-analogue of the theorem of Erdős, Ko and Rado for $2k+1 \leq n$. Greene and Kleitman [12] found an elegant proof for the case where $k \mid n$, hence proving the missing $n = 2k$ case. In 1986, Frankl and Wilson [9] proved the following result giving the maximum size of a t-intersecting family of k-spaces for $2k - t \leq n$.

Theorem 1.2 (Frankl & Wilson [9]) *Let V be a vector space over $GF(q)$ of dimension n. For any t-intersecting family $\mathcal{F} \subseteq \left[{V \atop k}\right]$ we have*

$|\mathcal{F}| \leq \left[{n-t \atop k-t}\right]$ *if* $2k \leq n$,

and

$|\mathcal{F}| \leq \left[{2k-t \atop k}\right]$ *if* $2k - t \leq n \leq 2k$.

These bounds are best possible.

Let the *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F}$ of subspaces of V be defined as the minimal dimension of a subspace of V meeting all elements of $\mathcal{F}$ nontrivially.

Already Hsieh's proof showed that if $t = 1$ and $n \geq 2k+1$ then only *point-pencils*, that is, families $\mathcal{F}$ with $\tau(\mathcal{F}) = 1$, can achieve the bound in Theorem 1.2. We will prove a q-analogue of Theorem 1.1 for intersecting families of subspaces with $\tau(\mathcal{F}) \geq 2$.

Let us first remark that for a fixed 1-subspace $E \leqslant V$ and a k-subspace U with $E \not\leqslant U$ the family $\mathcal{F}_{E,U} = \{U\} \cup \{W \in \left[{V \atop k}\right] : E \leqslant W,\ \dim(W \cap U) \geq 1\}$

is not maximal as we can add all subspaces in $\left[{E+U \atop k}\right]$. We will say that $\mathcal{F}$ is an *HM-type family* if

$$\mathcal{F} = \left\{W \in \left[{V \atop k}\right] : E \leqslant W,\ \dim(W \cap U) \geq 1\right\} \cup \left[{E+U \atop k}\right]$$

for some fixed $E \in \left[{V \atop 1}\right]$ and $U \in \left[{V \atop k}\right]$ with $E \not\leqslant U$. Note that the size of an HM-type family is $\left[{n-1 \atop k-1}\right] - q^{k(k-1)}\left[{n-k-1 \atop k-1}\right] + q^k$.

The main result of the paper is the following theorem.

Theorem 1.3 *Let V be an n-dimensional vector space over $GF(q)$, where $q \geq 3$ and $n \geq 2k+1$, $k \geq 3$. Then for any intersecting family $\mathcal{F} \subseteq \left[{V \atop k}\right]$ with $\tau(\mathcal{F}) \geq 2$ we have*

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + q^k.$$

When equality holds, either $\mathcal{F}$ is an HM-type family, or $k = 3$ and $\mathcal{F} = \mathcal{F}_3 = \{F \in \left[{V \atop k}\right] : \dim(S \cap F) \geq 2\}$ for some $S \in \left[{V \atop 3}\right]$.

Furthermore, if $k \geq 4$, then there exists an $\epsilon > 0$ (independent of n, q, k) such that if $|\mathcal{F}| \geq (1-\epsilon)\left(\left[{n-1 \atop k-1}\right] - q^{k(k-1)}\left[{n-k-1 \atop k-1}\right] + q^k\right)$, then $\mathcal{F}$ is a subfamily of an HM-type family.

We have the same result for $q = 2$, $n \geq 2k+3$ (see Proposition 3.4).

If $n \geq 3k$, and $\mathcal{F}$ is large enough (see Proposition 3.2), then we can describe the essential part of the intersecting system, see Proposition 2.6. This is a more general stability theorem than the one indicated in Theorem 1.3 and our remarks on the stability of relatively large systems will be given in Section 3.

After proving the above theorem in Section 2, we apply this result to determine the chromatic number of q-Kneser graphs. The vertex set of the q-Kneser graph $qK_{n:k}$ is $\left[{V \atop k}\right]$, where V is an n-dimensional vector space over $GF(q)$. Two vertices of $qK_{n:k}$ are adjacent if and only if the corresponding k-subspaces are disjoint. Section 4 contains the proof of the following theorem.

Theorem 1.4 *If $q \geq 3$, $n \geq 2k+1$, $k \geq 3$ or $q = 2$, $n \geq 2k+3$, $k \geq 3$, then for the chromatic number of the q-Kneser graph we have $\chi(qK_{n:k}) = \left[{n-k+1 \atop 1}\right]$. Moreover, each color class of a minimum coloring is a point-pencil and the points determining a color are the points of an $(n-k+1)$-dimensional subspace.*

In Section 5 we prove the non-uniform version of the Erdős-Ko-Rado theorem.

Theorem 1.5 *Let $\mathcal{F}$ be an intersecting family of subspaces of a vector space V of dimension n. Then*

(i) if n is odd, then

$$|\mathcal{F}| \leq \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix},$$

(ii) if n is even, then

$$|\mathcal{F}| \leq \begin{bmatrix} n-1 \\ n/2-1 \end{bmatrix} + \sum_{i>n/2} \begin{bmatrix} n \\ i \end{bmatrix}.$$

For odd n equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix}$. For even n equality holds only if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \{F \in \begin{bmatrix} V \\ n/2 \end{bmatrix} : E \leqslant F\}$ for some $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$, or if $\mathcal{F} = \begin{bmatrix} V \\ >n/2 \end{bmatrix} \cup \begin{bmatrix} U \\ n/2 \end{bmatrix}$ for some $U \in \begin{bmatrix} V \\ n-1 \end{bmatrix}$.

Note that Theorem 1.5 follows from the profile polytope of intersecting families which was determined implicitly by Bey [1] and explicitly by Gerbner and Patkós [10], but the proof we present in Section 4 is direct and very simple.

2 Proof of Theorem 1.3

For any $A \leqslant V$ and $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ let $\mathcal{F}_A = \{F \in \mathcal{F} : A \leqslant F\}$.

Before starting with the proof let us state an easy technical lemma for q-binomial coefficients that will simplify our computations.

Lemma 2.1 *Let $a \geq 0$ and $n \geq k \geq a+1$ and $q \geq 2$. Then*

$$\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-a-1 \\ k-a-1 \end{bmatrix} < \frac{1}{(q-1)q^{n-2k}} \begin{bmatrix} n-a \\ k-a \end{bmatrix}.$$

Proof. The inequality to be proved simplifies to

$$(q^{k-a}-1)(q^k-1)q^{n-2k} < q^{n-a} - 1. \qquad \square$$

Lemma 2.2 *Let $E \in \begin{bmatrix} V \\ 1 \end{bmatrix}$. If $E \not\leq L \leq V$, where L is an l-subspace, then the number of k-subspaces of V containing E and intersecting L is at least $\begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} l \\ 2 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ (with equality for $l = 2$), and at most $\begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.*

Proof. The k-spaces containing E and intersecting L in a 1-dimensional space are counted exactly once in the first term. Those subspaces that intersect L in a 2-dimensional space are counted $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = q + 1$ times in the first term and $-q$ times in the second term, thus once overall. If a subspace intersects L in a subspace of dimension $i \geq 3$, then it is counted $\begin{bmatrix} i \\ 1 \end{bmatrix}$ times in the first term and $-q\begin{bmatrix} i \\ 2 \end{bmatrix}$ times in the second term, thus a negative number of times overall. □

Our next lemma gives bounds on the size of a HM-type family that are easier to work with than the precise formula mentioned in the introduction.

Lemma 2.3 *Let $n \geq 2k + 1$, $k \geq 3$ and $q \geq 2$. If $\mathcal{F}$ is a HM-type family, then $(1 - \frac{1}{q^3-q})\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} k \\ 2 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \leq |\mathcal{F}| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.*

Proof. The first inequality follows immediately from Lemma 2.1 by noting that $q\begin{bmatrix} k \\ 2 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}(\begin{bmatrix} k \\ 1 \end{bmatrix} - 1)/(q + 1)$ and $n \geq 2k + 1$. □

Lemma 2.4 *If a subspace S does not intersect each element of $\mathcal{F}$, then there is a subspace $T > S$ with $\dim T = \dim S + 1$ and $|\mathcal{F}_T| \geq |\mathcal{F}_S|/\begin{bmatrix} k \\ 1 \end{bmatrix}$.*

Proof. There is an $F \in \mathcal{F}$ such that $S \cap F = 0$. Average over all $T = S + E$ where E is a 1-subspace of F. □

Lemma 2.5 *If an s-dimensional subspace S does not intersect each element of $\mathcal{F}$, then $|\mathcal{F}_S| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-s-1 \\ k-s-1 \end{bmatrix}$.*

Proof. There is an $(s + 1)$-space T with $\begin{bmatrix} n-s-1 \\ k-s-1 \end{bmatrix} \geq |\mathcal{F}_T| \geq |\mathcal{F}_S|/\begin{bmatrix} k \\ 1 \end{bmatrix}$. □

Before proving the q-analogue of the theorem of Hilton-Milner we describe the essential part of maximal intersecting families with $\tau(\mathcal{F}) = 2$. Let us define $\mathcal{T}$ to be the family of 2-spaces of V that intersect all subspaces in $\mathcal{F}$.

Proposition 2.6 *Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Then $\mathcal{F}$ contains all k-spaces containing an element of $\mathcal{T}$ and we have one of the following three possibilities:*

(i) $|\mathcal{T}| = 1$ *and* $\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < |\mathcal{F}| < \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + (q + 1)\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - 1\right)\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$;

(ii) $|\mathcal{T}| > 1$, $\tau(\mathcal{T}) = 1$, *and there is an* $(l+1)$*-space* W *(with* $2 \le l \le k$*) and a* 1*-space* $E \leqslant W$ *so that* $\mathcal{T} = \{M : E \leqslant M \leqslant W,\ \dim M = 2\}$*. In this case*
$$\begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} l \\ 2 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} \le |\mathcal{F}| \le \begin{bmatrix} l \\ 1 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} l \\ 1 \end{bmatrix}\right)\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} + q^l\begin{bmatrix} n-l \\ k-l \end{bmatrix}.$$
For $l = 2$ *the upper bound here can be strengthened to*
$$|\mathcal{F}| \le (q+1)\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}\left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} n-3 \\ k-3 \end{bmatrix} + q^2\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix};$$

(iii) $\mathcal{T} = \begin{bmatrix} A \\ 2 \end{bmatrix}$ *for some* 3*-subspace* A *and* $\mathcal{F} = \{U \in \begin{bmatrix} V \\ k \end{bmatrix} : \dim(U \cap A) \ge 2\}$ *and* $|\mathcal{F}| = (q^2+q+1)(\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}) + \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$.

In case (ii) *there is a* 1*-space* E *and an* l*-space* L *such that* $\mathcal{F}$ *contains the set* $\mathcal{F}_{E,L}$ *of all* k*-spaces containing* E *and intersecting* L*. The last two terms of the upper bound for* $|\mathcal{F}|$ *in* (ii) *give an upper bound on* $|\mathcal{F} \setminus \mathcal{F}_{E,L}|$*.*

Proof. Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Since $\mathcal{F}$ is maximal, it contains all k-spaces containing a $T \in \mathcal{T}$. Since $n \ge 2k$ and $k \ge 2$ two disjoint elements of $\mathcal{T}$ would be contained in disjoint elements of $\mathcal{F}$, which is impossible. So $\mathcal{T}$ is intersecting.

The following observation is immediate: if $A, B \in \mathcal{T}$ and $A \cap B < C < A+B$, then $C \in \mathcal{T}$. As an intersecting family of 2-spaces is either a family of 2-spaces containing some fixed 1-space E or a set of 2-subspaces of a 3-space, we get the following:

$(*)$: $\mathcal{T}$ is either a family of all 2-subspaces in a given $(l+1)$-space containing some fixed 1-space E (and $k \ge l \ge 1$), or $\mathcal{T}$ is the set of all 2-subspaces of a 3-space.

(i) : If $|\mathcal{T}| = 1$, then let S denote the only 2-space in $\mathcal{T}$ and let $E \leqslant S$ be any 1-space. Since $\tau(\mathcal{F}) > 1$ there exists an $F \in \mathcal{F}$ with $E \nleqslant F$, for which we must have $\dim(F \cap S) = 1$. Since S is the only element of $\mathcal{T}$, for any 1-subspace E' of F different from $F \cap S$, $\mathcal{F}_{E+E'} \le \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ by Lemma 2.5, hence the number of subspaces containing E but not containing S is at most $(\begin{bmatrix} k \\ 1 \end{bmatrix} - 1)\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$. This gives the upper bound.

(ii) : Assume that $\tau(\mathcal{T}) = 1$ and $|\mathcal{T}| > 1$. By $(*)$, $\mathcal{T}$ is the set of 2-spaces in an $(l+1)$-space W (with $l \ge 2$) containing some fixed 1-space E. Every $F \in \mathcal{F} \setminus \mathcal{F}_E$ intersects W in a hyperplane. Let L be a hyperplane in W not on E. Then $\mathcal{F}$ contains all k-spaces on E that intersect L. Hence the lower bound and the first term in the upper bound come from Lemma 2.2. The second term comes from counting the k-spaces of $\mathcal{F}$ that contain E and intersect a given $F \in \mathcal{F}$ (not containing E) in a point of $F \setminus W$. Here Lemma

2.5 is used. If $l \geq 3$, then there are q^l hyperplanes in W not containing E and there are $\begin{bmatrix} n-l \\ k-l \end{bmatrix}$ k-spaces through such a hyperplane. For $l = 2$ there are q^2 hyperplanes in W and they cannot be in $\mathcal{T}$. Using Lemma 2.5 gives the bound.

(iii) is immediate. $\square$

Corollary 2.7 *Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. If $\mathcal{F}$ is at least as large as an HM-type family, and either $q \geq 3$, $n \geq 2k+1$, $k \geq 3$ or $q = 2$, $n \geq 2k+3$, $k \geq 3$, then $\mathcal{F}$ is an HM-type family, or, in case $k = 3$, an $\mathcal{F}_3$-type family.*

There exists an $\epsilon > 0$ (independent of n, q, k) such that if $k \geq 4$ and either $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+3$, and $|\mathcal{F}|$ is at least $(1-\epsilon)$ times the size of an HM-type family, then $\mathcal{F}$ is an HM-type family.

Proof. Apply Proposition 2.6. Note that the Hilton-Milner families are precisely those from case (ii) with $k = l$. Now we can easily prove that for $\tau(\mathcal{F}) = 2$ the size of $\mathcal{F}$ cannot exceed the size of an HM-type family, and if $k \geq 4$, then we will prove an inequality of the form

$$|\mathcal{F}| \leq C(q) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix},$$

such that the inequality $C(q) < (1-\epsilon)(1 - \frac{1}{q^3-q})$ will trivially hold for some $\epsilon > 0$ that does not depend on n, k, q and thus we will be done by Lemma 2.3.

First assume $q \geq 3$. In case (i) of Proposition 2.6 we have $|\mathcal{F}|/\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < 1 + \frac{q+1}{q^2-q}\begin{bmatrix} k \\ 1 \end{bmatrix}$ by Lemma 2.1. In case (ii) we find for $l < k$ that $|\mathcal{F}|/\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < (\frac{1}{q} + \frac{1}{q^2-q})\begin{bmatrix} k \\ 1 \end{bmatrix} + \frac{q^2}{q^2-q}$. In both cases, for $q \geq 3$, $k \geq 3$, this is less than $(1-\epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3.

In case (iii) $|\mathcal{F}_3| = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \frac{q^3-q}{q-1}\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$. For $k \geq 4$, this is much smaller than the size of the HM-type families. For $k = 3$, the two families have the same size.

Next suppose $q = 2$. Since we are assuming $n \geq 2k+3$, the factor q^{n-2k} in the estimate of Lemma 2.1 now gives an extra factor q^2. In case (i) we have $|\mathcal{F}|/\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < 1 + \frac{q+1}{(q-1)q^3}\begin{bmatrix} k \\ 1 \end{bmatrix}$. In case (ii) we find for $l < k$ that $|\mathcal{F}|/\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < (\frac{1}{q} + \frac{1}{(q-1)q^3})\begin{bmatrix} k \\ 1 \end{bmatrix} + \frac{q^2}{(q-1)q^3}$. In both cases, for $k \geq 3$, this is less than $(1-\epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3. In case (iii) the conclusion is as before. $\square$

Proof of Theorem 1.3. Let $\mathcal{F}$ be a maximal intersecting family with $\tau(\mathcal{F}) \geq 2$. If $\tau(\mathcal{F}) = 2$ then we are done by the above corollary. This leaves the case $\tau(\mathcal{F}) > 2$. First we consider the case $k = 3$ separately.
In this case the size of the HM-type family is

$$N := (q^2 + q + 1)\left(\begin{bmatrix} n-2 \\ 1 \end{bmatrix} - 1\right) + 1.$$

Let A be an i-space, $0 \leq i \leq 3$. Then $|\mathcal{F}_A| \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}^{3-i}$. Indeed, if $i < \tau(\mathcal{F})$ there exists an $F \in \mathcal{F}$ with $A \cap F = 0$ so that for some $(i+1)$-space B on A we have $|\mathcal{F}_B| \geq |\mathcal{F}_A| / \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. And if $\dim C = 3$ then $|\mathcal{F}_C| \leq 1$. Assume that $|\mathcal{F}| \geq N$. Then $N \leq |\mathcal{F}| = |\mathcal{F}_0| \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}^3$ implies $n = 7$ (since $n \geq 2k+1$).

Pick a 1-space E such that $|\mathcal{F}_E| \geq |\mathcal{F}| / \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and a 2-space S on E such that $|\mathcal{F}_S| \geq |\mathcal{F}_E| / \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Then $|\mathcal{F}_S| > q + 1$ since $|\mathcal{F}| > \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}^2$. Pick $F' \in \mathcal{F}$ disjoint from S. Put $H = S + F'$. Then all $F \in \mathcal{F}_S$ are contained in the 5-space H. But $|\mathcal{F}| > \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ so there is an $F_0 \in \mathcal{F}$ not contained in H. If $F_0 \cap S = 0$, then each $F \in \mathcal{F}_S$ is contained in $S + (H \cap F_0)$, so $|\mathcal{F}_S| \leq q + 1$, contradiction. Thus, all elements of $\mathcal{F}$ disjoint from S are in H.

Now F_0 must meet F' and S, so F_0 meets H in a 2-space S_0. Since $|\mathcal{F}_S| > q + 1$, we can find two elements F_1, F_2 of $\mathcal{F}_S$ with the property that S_0 is not contained in the 4-space $F_1 + F_2$. Since any $F \in \mathcal{F}$ disjoint from S is contained in H and meets F_0, it must meet S_0 and also F_1 and F_2. Hence the number of such F's is at most q^5. Altogether $|\mathcal{F}| \leq q^5 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}^2$ (counting F disjoint from S or on a given $E < S$), contradiction.

Now let $k \geq 4$ (and $\tau(\mathcal{F}) > 2$).

Lemma 2.8 *Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geq s$. Suppose that for some z-space Z we know that for any s-space S containing Z we have $|\mathcal{F}_S| \leq f(s)$, then we have $|\mathcal{F}_Z| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{s-z} f(s)$.*

Proof. By $\tau(\mathcal{F}) \geq s$ we know that for any $(s-i)$-space A, $1 \leq i \leq z$, there exists an $F \in \mathcal{F}$ disjoint from A. Now apply Lemma 2.4 $s - z$ times. $\square$

Corollary 2.9 *Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geq s$. Then for any 1-space $E \leqslant V$ we have $|\mathcal{F}_E| \leq \begin{bmatrix} k \\ 1 \end{bmatrix}^{s-1} \begin{bmatrix} n-s \\ k-s \end{bmatrix}$.* $\square$

For any integer s with $3 \leq s \leq k$ let $A(s, \mathcal{F})$ denote the statement that for any $(s-1)$-subspace S of V there exist $F_1^S, F_2^S \in \mathcal{F}$ such that $\dim(F_1^S \cap F_2^S) \leq s-1$ and $S \cap (F_1^S + F_2^S) = 0$.

Lemma 2.10 *Let $\mathcal{F}$ be an intersecting family such that $A(s,\mathcal{F})$ holds. Then for any 1-subspace E of V we have*

$$|\mathcal{F}_E| \le \begin{bmatrix}k\\1\end{bmatrix}^{s-2}(\begin{bmatrix}s-1\\1\end{bmatrix}\begin{bmatrix}n-s\\k-s\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix}).$$

Proof. For any $(s-1)$-space S we have $|\mathcal{F}_S| \le \begin{bmatrix}s-1\\1\end{bmatrix}\begin{bmatrix}n-s\\k-s\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix}$ as any $F \in \mathcal{F}_S$ must intersect F_1^S, F_2^S. By Lemma 2.8 we get the statement of the lemma. $\square$

Lemma 2.11 *Let $\mathcal{F}$ be an intersecting family such that $\tau(\mathcal{F}) = s > 2$ and $A(s,\mathcal{F})$ holds. Then we have $|\mathcal{F}| \le (1-\epsilon_1)(1-\frac{1}{q^3-q})\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix}$.*

Proof. By Lemma 2.10 and $\tau(\mathcal{F}) = s$ we have

$$|\mathcal{F}| \le \begin{bmatrix}s\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{s-2}(\begin{bmatrix}s-1\\1\end{bmatrix}\begin{bmatrix}n-s\\k-s\end{bmatrix} + \begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix}).$$

Now observe that the following inequalities hold:

$$\begin{bmatrix}s\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{s}\begin{bmatrix}n-s-1\\k-s-1\end{bmatrix}/\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} < \begin{bmatrix}s\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{s-1}\frac{1}{q^{(s-1)(k+1)}} < \frac{q}{(q-1)^s},$$

$$\begin{bmatrix}s\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{s-2}\begin{bmatrix}s-1\\1\end{bmatrix}\begin{bmatrix}n-s\\k-s\end{bmatrix}/\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix} < \begin{bmatrix}s-1\\1\end{bmatrix}\begin{bmatrix}s\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^{s-3}\frac{1}{q^{(s-2)(k+1)}} < \frac{q^{s+1-k}}{(q-1)^{s-1}}.$$

Since $q \ge 3$ and $3 \le s \le k$, $k \ne 3$, we have $\frac{q+q^{s+1-k}(q-1)}{(q-1)^s} < 1 - \frac{1}{q^3-q}$. $\square$

Lemma 2.12 *Let $\mathcal{F}$ be an intersecting family with $\tau(\mathcal{F}) \ge 3$. Then either we have $|\mathcal{F}| < (1-\epsilon_2)(1-\frac{1}{q^3-q})\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix}$ or $A(3,\mathcal{F})$ holds.*

Proof. Suppose that $A(3,\mathcal{F})$ does not hold and let S be a 2-space witnessing this, i.e for any $(n-2)$-space U with $S \cap U = 0$ the family $\mathcal{F}^U = \{F \in \mathcal{F} : F \leqslant U\}$ is 3-intersecting. By Theorem 1.2 we have $|\mathcal{F}^U| \le \max\{\begin{bmatrix}2k-3\\k-3\end{bmatrix}, \begin{bmatrix}n-5\\k-3\end{bmatrix}\} \le \begin{bmatrix}n-4\\k-3\end{bmatrix}$ where we used $n \ge 2k+1$. There are $q^{2(n-2)}$ such U's and every $F \in \mathcal{F}$ with $F \cap S = 0$ is contained in $q^{2(n-k-2)}$ such $(n-2)$-spaces. Therefore the number of subspaces in $\mathcal{F}$ disjoint from S is at most $q^{2k}\begin{bmatrix}n-4\\k-3\end{bmatrix}$.

By Corollary 2.9 $\tau(\mathcal{F}) \ge 3$ implies $|\mathcal{F}_E| \le \begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix}$ for any 1-space E. Summing over all 1-spaces in S and adding the number of subspaces in $\mathcal{F}$ disjoint from S we obtain

$$|\mathcal{F}| \le \begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} + q^{2k}\begin{bmatrix}n-4\\k-3\end{bmatrix}.$$

Observe

$$\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}k\\1\end{bmatrix}^2\begin{bmatrix}n-3\\k-3\end{bmatrix} < \frac{2}{q}\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-2\\k-2\end{bmatrix}$$

since $q \geq 3$, and
$q^{2k} \begin{bmatrix} n-4 \\ k-3 \end{bmatrix} = q^{2k} \frac{(q^{k-2}-1)(q^{n-k}-1)}{(q^{n-2}-1)(q^{n-3}-1)} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \frac{q^{2k}(q-1)}{(q^k-1)(q^{n-3}-1)} \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \frac{6561}{29120} \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$
since $q \geq 3, k \geq 4, n \geq 2k+1$. The proof of the lemma is complete, as $\frac{2}{q} + \frac{6561}{29120} < 1 - \frac{1}{q^3-q}$ for $q \geq 3$. □

Lemma 2.13 *Let $\mathcal{F}$ be an intersecting family such that $\tau(\mathcal{F}) \geq s+1$ and $A(s, \mathcal{F})$ holds for some $s \geq 3$. Then either we have $|\mathcal{F}| < (1-\epsilon_3)(1-\frac{1}{q^3-q}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$ or $A(s+1, \mathcal{F})$ holds.*

Proof. Suppose $A(s+1, \mathcal{F})$ does not hold and let S be an s-space witnessing this, i.e for any $(n-s)$-space U disjoint from S the family $\mathcal{F}^U = \{F \in \mathcal{F} : F \leqslant U\}$ is $(s+1)$-intersecting. By Theorem 1.2 we have $|\mathcal{F}^U| \leq \max\{\begin{bmatrix} n-2s-1 \\ k-s-1 \end{bmatrix}, \begin{bmatrix} 2k-s-1 \\ k-s-1 \end{bmatrix}\} \leq \begin{bmatrix} n-s-2 \\ k-s-1 \end{bmatrix}$. There are $q^{s(n-s)}$ such U's and every $F \in \mathcal{F}$ disjoint from S is contained in $q^{s(n-k-s)}$ such $(n-s)$-spaces. Therefore the number of elements of $\mathcal{F}$ disjoint from S is at most $a_s = q^{ks} \begin{bmatrix} n-s-2 \\ k-s-1 \end{bmatrix}$. Since $a_{s+1}/a_s < 1$ this number is largest when $s = 3$ and

$$a_3 = q^{3k} \begin{bmatrix} n-5 \\ k-4 \end{bmatrix} < \frac{1}{q} \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$$

using $k \geq 4$. Since $A(s, \mathcal{F})$ holds, by Lemma 2.10 we find that the number of elements of $\mathcal{F}$ meeting S is at most

$$\begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{s-2} \left(\begin{bmatrix} s-1 \\ 1 \end{bmatrix} \begin{bmatrix} n-s \\ k-s \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-s-1 \\ k-s-1 \end{bmatrix} \right),$$

which by the calculation in Lemma 2.11 is not more than

$$\frac{q + q^{s+1-k}(q-1)}{(q-1)^s} \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}.$$

Altogether this yields

$$|\mathcal{F}| / \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} \leq \frac{1}{q} + \frac{q + q^{s+1-k}(q-1)}{(q-1)^s} \leq 1 - \frac{1}{q^3-q}$$

since $q \geq 3$ and $3 \leq s \leq k-1$. This is almost good enough, but we want an additional factor $(1-\epsilon)$, while as it is, equality holds for $q = 3$, $s = 3$, $k = 4$. However, the term $q/(q-1)^s$ arose estimating $\begin{bmatrix} s \\ 1 \end{bmatrix} < q^s/(q-1)$, but for $q = s = 3$ one wins a factor $26/27$ here, and we are done. □

By Lemma 2.12 and Lemma 2.13 we obtain that either $\mathcal{F}$ is smaller than the HM-type family, or $A(k, \mathcal{F})$ holds. But since for any intersecting family we have $\tau(\mathcal{F}) \leq k$, we are done by Lemma 2.11. This completes the proof of Theorem 1.3. □

3 Stability

In this section we prove that if k is fixed, and n and $|\mathcal{F}|$, the size of an intersecting family $\mathcal{F}$ of k-spaces, are large enough, then $\tau(\mathcal{F}) = 2$, and the examples are described in Proposition 2.6.

A subspace will be called a *hitting subspace* (and we shall say that the subspace intersects $\mathcal{F}$), it it intersects each element of $\mathcal{F}$.

When n gets larger than $2k+1$ (or rather when it is at least $3k$) we can prove more than just the q-analogue of the theorem of Hilton-Milner, particularly when q is also large. For the sake of simplicity, we will impose a bound on $|\mathcal{F}|$ which is weaker than the Hilton-Milner bound if n is close to $2k+1$ (and q is very small). If q is large then for $n = 2k+1$ it has the same order of magnitude as the size of the Hilton-Milner family, for $n > 2k+1$ it has a smaller order of magnitude. Our first proposition shows that with our bound only the $\tau = 2$ case of the proof has to be considered.

Proposition 3.1 *Suppose that $k \geq 3$ and $n \geq 2k$. Let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. Let $3 \leq l \leq k$. If $\mathcal{F}$ has an intersecting l-space, and*

$$|\mathcal{F}| > \frac{(q^l - 1)(q^k - 1)}{(q-1)^l q^{(n-2k)(l-2)}} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} \tag{3.1}$$

then $\mathcal{F}$ has an intersecting $(l-1)$-space.

Proof. Assume $\tau(\mathcal{F}) = l$. By averaging there is a 1-space P with $|\mathcal{F}_P| \geq |\mathcal{F}|/\begin{bmatrix} l \\ 1 \end{bmatrix}$. By Corollary 2.9 we have $|\mathcal{F}| \leq \begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1} \begin{bmatrix} n-l \\ k-l \end{bmatrix}$. Applying Lemma 2.1 $l-2$ times we see that

$$\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} > \begin{bmatrix} n-l \\ n-l \end{bmatrix} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} (q-1) q^{n-2k} \right)^{l-2},$$

so that the lower bound on $|\mathcal{F}|$ contradicts the upper bound. □

Corollary 3.2 *Let $k \geq 3$ and $n \geq 2k+1$ and $n \geq 2k+2$ if $q=2$. If $|\mathcal{F}| \geq \frac{q^2+q+1}{(q-1)q^{n-2k}} \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$, then $\tau(\mathcal{F})=2$, that is, $\mathcal{F}$ is contained in one of the systems described in Proposition 2.6 satisfying the bound on $|\mathcal{F}|$.*

Proof. Since the right hand side of (3.1) is decreasing in l for $3 \leq l \leq k$ (this uses $n \geq 2k+1$ and $n \geq 2k+2$ for $q=2$), we can find a hitting 2-space if the condition (3.1) holds for $l=3$, and it does by the assumption on $|\mathcal{F}|$. □

Remark 3.3 For $n \geq 3k$ all the systems described in Proposition 2.6 occur.

Proposition 3.4 *The q-analogue of Hilton-Milner theorem (Theorem 1.3) is also true if $q=2$ and $n \geq 2k+3$ and $k \geq 3$.*

Proof. The bound in Corollary 3.2 is smaller than the size of an HM-type family. (The weaker bound in Lemma 2.3 suffices to see this for $n \geq 2k+4$, the stronger bound also for $n=2k+3$.) Now Corollary 2.7 yields the required conclusion. □

Proposition 3.1 can also be used to bound $\tau(\mathcal{F})$ if a lower bound on $|\mathcal{F}|$, for example the Hilton-Milner bound, is given.

Corollary 3.5 *If $|\mathcal{F}| \geq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} (1-\frac{1}{q^3-q})$, then $\tau(\mathcal{F}) \leq 2$ if $n \geq 2k+2$ and $q \geq 3$, and $\tau(\mathcal{F}) \leq 3$ if $n=2k+1$ and $q \geq 4$.*

Proof. We have to check (3.1) for $l=3$ or 4, respectively. We need

$$\frac{q^2(q-1)^2}{q^3-1}\left(1-\frac{1}{q^3-q}\right) > 1$$

for $n \geq 2k+2$, which is true for $q \geq 3$. For $n=2k+1$, $l=4$ we need

$$\frac{q^2(q-1)^3}{q^4-1}\left(1-\frac{1}{q^3-q}\right) > 1,$$

which is true for $q \geq 4$. □

Using the existence of 3-spaces intersecting $\mathcal{F}$ we can actually prove that $\tau(\mathcal{F}) \leq 2$ even for $n=2k+1$ (and $q \geq 4$), so this gives a *second proof of the q-Hilton-Milner theorem* for k, $q \geq 4$.

Assume $|\mathcal{F}| \geq \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} (1-\frac{1}{q^3-q})$. We shall repeat the proof of Proposition 3.1. If there is a hitting 3-space and no hitting 2-space then there is a 1-space

P with $|\mathcal{F}_P| \geq |\mathcal{F}|/\left[{3 \atop 1}\right]$, a 2-space $L > P$ with $|\mathcal{F}_L| \geq |\mathcal{F}_P|/\left[{k \atop 1}\right] \geq |\mathcal{F}|/\left[{3 \atop 1}\right]\left[{k \atop 1}\right]$, and finally a 3-space $W > L$ with $|\mathcal{F}_W| \geq |\mathcal{F}|/\left[{3 \atop 1}\right]\left[{k \atop 1}\right]^2$. This W intersects $\mathcal{F}$, since $|\mathcal{F}|/\left[{3 \atop 1}\right]\left[{k \atop 1}\right]^3 > \left[{n-4 \atop k-4}\right]$. Indeed, for $n = 2k+1$ and $q \geq 3$ we find (applying Lemma 2.1 twice)

$$\frac{\left[{k \atop 1}\right]\left[{n-2 \atop k-2}\right]\left(1 - \frac{1}{q^3-q}\right)}{\left[{n-4 \atop k-4}\right]\left[{3 \atop 1}\right]\left[{k \atop 1}\right]^3} > \frac{q^2(q-1)^3}{q^3-1}\left(1 - \frac{1}{q^3-q}\right) > 1.$$

If $\mathcal{F}^* = \mathcal{F} \setminus \mathcal{F}_W$, then for the previous P and L we have $|\mathcal{F}^*_L| \geq |\mathcal{F}_L| - \left[{n-3 \atop k-3}\right] \geq |\mathcal{F}|/\left[{3 \atop 1}\right]\left[{k \atop 1}\right] - \left[{n-3 \atop k-3}\right]$ and we find a $W^* \neq W$ for which $|\mathcal{F}^*_{W^*}| \geq |\mathcal{F}^*_L|/\left[{k \atop 1}\right]$. The previous computation shows that W^* is also a hitting 3-space.

The k-spaces $F \in \mathcal{F}$ not meeting L meet each of W and W^* in a single point, hence there are at most $q^4\left[{k \atop 1}\right]\left[{n-3 \atop k-3}\right]$ such elements of $\mathcal{F}$, by Lemma 2.5. Since $\left[{n-2 \atop k-2}\right]/\left[{n-3 \atop k-3}\right] > q^5$ for $n = 2k+1$, $k \geq 4$, this means that at least $\left[{k \atop 1}\right]\left[{n-2 \atop k-2}\right](1 - \frac{1}{q} - \frac{1}{q^3-q})$ elements of $\mathcal{F}$ intersect L. Hence in the beginning of the proof we can choose the 1-space P with $|\mathcal{F}_P| \geq \left[{k \atop 1}\right]\left[{n-2 \atop k-2}\right](1 - \frac{1}{q} - \frac{1}{q^3-q})/(q+1)$ and find a 2-space L with $|\mathcal{F}_L| \geq \left[{n-2 \atop k-2}\right](1 - \frac{1}{q} - \frac{1}{q^3-q})/(q+1)$. This L is a hitting 2-space since $\left[{n-2 \atop k-2}\right](1 - \frac{1}{q} - \frac{1}{q^3-q})/(q+1) > \left[{k \atop 1}\right]\left[{n-3 \atop k-3}\right]$. Indeed, by Lemma 2.1 it suffices to check $\frac{q(q-1)}{q+1}(1 - \frac{1}{q} - \frac{1}{q^3-q}) > 1$, which is true for $q \geq 4$. □

The previous results just used the parameter τ, so only the hitting subspaces of smallest dimension were taken into account. A more precise description is possible if we make the intersecting system of subspaces critical.

Definition 3.6 An intersecting family $\mathcal{F}$ of subspaces of V is *critical* if for any two distinct $F, F' \in \mathcal{F}$ we have $F \not\subset F'$, and moreover for any hitting subspace G there is a $F \in \mathcal{F}$ with $F \subset G$.

Lemma 3.7 *For every non-extendable intersecting family $\mathcal{F}$ of k-spaces there exists some critical family $\mathcal{G}$ such that*

$$\mathcal{F} = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : \exists G \in \mathcal{G},\ G \subseteq F\}.$$

Proof. Extend $\mathcal{F}$ to a maximal intersecting family $\mathcal{H}$ of subspaces of V, and take for $\mathcal{G}$ the minimal elements of $\mathcal{H}$. □

The following construction and result are an adaptation of the corresponding results from Erdős and Lovász [5]:

Construction 3.8 *Let* $A_1, \ldots, A_k$ *be subspaces of* V *such that* $\dim A_i = i$ *and* $\dim(A_1 + \cdots + A_k) = \binom{k+1}{2}$. *Define*

$$\mathcal{F}_i = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : A_i \subseteq F, \ \ \dim A_j \cap F = 1 \textit{ for } j > i\}.$$

Then $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ *is a critical, non-extendable, intersecting family of* k*-spaces, and* $|\mathcal{F}_i| = \begin{bmatrix} i+1 \\ 1 \end{bmatrix} \begin{bmatrix} i+2 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} k \\ 1 \end{bmatrix}$ *for* $1 \le i \le k$.

For subsets Erdős and Lovász proved that a critical, non-extendable, intersecting family of k-sets cannot have more than k^k members. They conjectured that the above construction is best possible but this was disproved by Frankl, Ota and Tokushige [8]. Here we prove the following analogous result.

Theorem 3.9 *Let* $\mathcal{F}$ *be a critical, intersecting family of subspaces of* V *of dimension at most* k. *Then* $|\mathcal{F}| \le \begin{bmatrix} k \\ 1 \end{bmatrix}^k$.

Proof. Suppose that $|\mathcal{F}| > \begin{bmatrix} k \\ 1 \end{bmatrix}^k$. By induction on i, $0 \le i \le k$, we find an i-dimensional subspace A_i of V such that $|\mathcal{F}_{A_i}| > \begin{bmatrix} k \\ 1 \end{bmatrix}^{k-i}$. Indeed, since by induction $|\mathcal{F}_{A_i}| > 1$ and $\mathcal{F}$ is critical, the subspace A_i is not hitting, and there is an $F \in \mathcal{F}$ disjoint from A_i. Now all elements of $\mathcal{F}_{A_i}$ meet F, and we find $A_{i+1} > A_i$ with $|\mathcal{F}_{A_{i+1}}| > |\mathcal{F}_{A_i}| / \begin{bmatrix} k \\ 1 \end{bmatrix}$. For $i = k$ this is a contradiction. $\square$

Remark 3.10 For $l \le k$ this argument shows that there are not more than $\begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}^{l-1}$ l-spaces in $\mathcal{F}$.

If $l = 3$ and $\tau > 2$ then for the size of $\mathcal{F}$ the previous remark essentially gives $(q^2 + q + 1) \begin{bmatrix} k \\ 1 \end{bmatrix}^2 \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$, which is basically the bound in Proposition 3.2.

Modifying the Erdős-Lovász construction (see Frankl [6]), one can get intersecting families with many l-spaces in the corresponding critical family.

Construction 3.11 *Let* $A_1, \ldots, A_l$ *be subspaces with* $\dim A_1 = 1$, $\dim A_i = k+i-l$ *for* $i \ge 2$. *Define* $\mathcal{F}_i = \{F \in \begin{bmatrix} V \\ k \end{bmatrix} : A_i \le F, \dim(F \cap A_j) \ge 1$ *for* $j > i\}$. *Then* $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l$ *is intersecting and the corresponding critical family has at least* $\begin{bmatrix} k-l+2 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} k \\ 1 \end{bmatrix}$ l*-spaces.*

For n large enough the Erdős-Ko-Rado theorem for vector spaces follows from the obvious fact that no critical, intersecting family can contain more than one 1-dimensional member. The Hilton-Milner theorem and the stability of the systems follow from $(*)$ which was used to describe the intersecting systems with $\tau = 2$. As remarked above, the fact that the critical family has to contain only spaces of dimension 3 or more limits its size to $O(\left[{n \atop k-3}\right])$, if k is fixed and n is large enough. Stronger and more general stability theorems can be found in Frankl [7] for the subset case.

4 Coloring q-Kneser graphs

In this section, we prove Theorem 1.4, that is, we show that $\chi(qK_{n:k}) = \left[{n-k+1 \atop 1}\right]$. The case $k = 2$ was proven in [3] and the general case for $q > q_k$ in [15]. We will need the following result of Bose and Burton [2].

Theorem 4.1 (Bose & Burton [2]) *If V is an n-dimensional vector space over $GF(q)$ and $\mathcal{E}$ is a family of 1-subspaces of V such that any k-subspace of V contains at least one element of $\mathcal{E}$, then $|\mathcal{E}| \geq \left[{n-k+1 \atop 1}\right]$. Furthermore, equality holds if and only if $\mathcal{E} = \left[{H \atop 1}\right]$ for some $(n-k+1)$-subspace H of V.*

Before starting the proof of Theorem 1.4, we first give two natural extensions of the Bose-Burton result, each of which can be used in the proof.

Proposition 4.2 *If V is an n-dimensional vector space over $GF(q)$ and $\mathcal{E}$ is a family of $\left[{n-k+1 \atop 1}\right] - \varepsilon$ 1-subspaces of V, then the number of k-subspaces of V that are disjoint from all $E \in \mathcal{E}$ is at least $\varepsilon q^{(k-1)(n-k+1)} / \left[{k \atop 1}\right]$.*

Proof. Induction on k. For $k = 1$ there is nothing to prove. Next, let $k > 1$ and count incident pairs (1-space, k-space), where the k-space is disjoint from all $E \in \mathcal{E}$:

$$N \begin{bmatrix} k \\ 1 \end{bmatrix} \geq \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix} + \varepsilon \right) \varepsilon q^{(k-2)(n-k+1)} \Big/ \begin{bmatrix} k-1 \\ 1 \end{bmatrix} \geq \varepsilon q^{(k-1)(n-k+1)}.$$

□

Of course the true value is $\varepsilon q^{(k-1)(n-k)}$, and probably the proof is an exercise as well, but this is good enough for our purpose.

Proposition 4.3 *If V is an n-dimensional vector space over $GF(q)$ and $\mathcal{E}$ is a family of $\begin{bmatrix} m \\ 1 \end{bmatrix}$ 1-spaces, then the number of l-spaces disjoint from all $E \in \mathcal{E}$ is at least $N(m, l, n) = q^{lm} \begin{bmatrix} n-m \\ l \end{bmatrix}$, the number of l-spaces disjoint from an m-space, with equality for $l = 1$ if and only if the elements of $\mathcal{E}$ are all different, and for $l > 1$ if and only if $\mathcal{E}$ is the set of 1-subspaces in an m-space.*

Proof. Induction on l. For $l = 1$ there is nothing to prove. For $l > 1$ take a 1-space $P \notin \mathcal{E}$. By induction, the number of l-spaces on P disjoint from all $E \in \mathcal{E}$ is at least $N(m, l-1, n-1)$, and varying P we find at least $N(m, l-1, n-1)(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} m \\ 1 \end{bmatrix})/\begin{bmatrix} l \\ 1 \end{bmatrix} = N(m, l, n)$ l-spaces. If we have equality, then the elements of $\mathcal{E}$ are all different in the local space at P, for every $P \notin \mathcal{E}$, and we have a subspace (of dimension m). $\square$

Proof of Theorem 1.4. Suppose that we have a coloring with at most $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$ colors. Let G (the good colors) be the set of colors that are point-pencils and let B (the bad colors) be the remaining set of colors. Then $|G| + |B| \leq \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}$. Suppose $|B| = \varepsilon > 0$. By Proposition 4.2, the number of k-spaces with a color in B is at least $\varepsilon q^{(k-1)(n-k+1)}/\begin{bmatrix} k \\ 1 \end{bmatrix}$, so that the average size of a bad color class is at least $q^{(k-1)(n-k+1)}/\begin{bmatrix} k \\ 1 \end{bmatrix}$. This must be smaller than the size of a HM-type family. Thus, by Lemma 2.3,

$$q^{(k-1)(n-k+1)} < \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} \begin{bmatrix} k \\ 1 \end{bmatrix}.$$

For $k \geq 3$ and $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+3$, this is a contradiction.

If $|B| = 0$, then all color classes are point-pencils, and we are done by Theorem 4.1. $\square$

A similar proof can be based on Proposition 4.3.

5 Proof of Theorem 1.5

Let $a + b = n, a < b$ and let $\mathcal{F}_a = \mathcal{F} \cap \begin{bmatrix} V \\ a \end{bmatrix}$ and $\mathcal{F}_b = \mathcal{F} \cap \begin{bmatrix} V \\ b \end{bmatrix}$. We prove

$$|\mathcal{F}_a| + |\mathcal{F}_b| \leq \begin{bmatrix} n \\ b \end{bmatrix} \tag{5.2}$$

with equality only if $\mathcal{F}_a = \emptyset$, $\mathcal{F}_b = \begin{bmatrix} V \\ b \end{bmatrix}$.

Adding up (5.2) for $n/2 < b \leq n$ gives the bound on $|\mathcal{F}|$ in Theorem 1.5 if n is odd and adding the result of Greene and Kleitman [12] that states

$|\mathcal{F}_{n/2}| \leq \left[{n-1 \atop n/2-1}\right]$ proves it for n even. For the uniqueness part of Theorem 1.5 we only have to note that if n is even and $|\mathcal{F}_{n/2}| = \left[{n-1 \atop n/2-1}\right]$, then by results of Frankl and Wilson [9] and Godsil and Newman [11] we must have $\mathcal{F}_{n/2} = \{F \in \left[{V \atop n/2}\right] : E \leqslant F\}$ for some $E \in \left[{V \atop 1}\right]$ or $\mathcal{F}_{n/2} = \left[{U \atop n/2}\right]$ for some $U \in \left[{V \atop n-1}\right]$.

Now let us prove (5.2). Consider the bipartite graph with vertex set $\left[{V \atop a}\right] \cup \left[{V \atop b}\right]$ and join $A \in \left[{V \atop a}\right]$ and $B \in \left[{V \atop b}\right]$ if and only if $A \cap B = 0$. Clearly this graph is regular (with degree q^{ab}) and therefore any independent set (that corresponds to an intersecting subfamily of $\left[{V \atop a}\right] \cup \left[{V \atop b}\right]$) has size at most $\left[{n \atop b}\right]$. Moreover, independent sets of that size can only be $\left[{V \atop a}\right]$ or $\left[{V \atop b}\right]$ but the former is not an intersecting family. This proves (5.2). $\square$

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