Classification of small (0,2)-graphs

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Abstract

We find the graphs of valency at most 7 with the property that any two nonadjacent vertices have either 0 or 2 common neighbours. In particular, we find all semibiplanes of block size at most 7.

1 Introduction

A $(0, 2)$-*graph* is a connected graph such that any two vertices have either 0 or 2 common neighbours. A *rectagraph* is a $(0, 2)$-graph without triangles. A *semibiplane* is a connected point-block incidence structure such that any two points are in either 0 or 2 common blocks, and any two blocks have either 0 or 2 common points. Clearly, the incidence graph of a semibiplane is a bipartite $(0, 2)$-graph (and conversely).

These topics have been studied by various authors. Semibiplanes were introduced by Dan Hughes [2] in the study of projective planes with involution, and are the topic of the thesis [8] of Peter Wild. Rectagraphs were introduced by Arnold Neumaier [7] in the study of diagram geometries. Semibiplanes provide examples of geometries with c.c* diagram (where the objects are the points, the pairs, and the blocks), and these have been studied under various assumptions on the group of automorphisms. Martyn Mulder [6] showed that $(0, 2)$-graphs are regular, say of valency k, and that the number v of vertices and the diameter d satisfy $v \le 2^k$ and $d \le k$ with in both cases equality only for the hypercube.

During GAC 3 (Oisterwijk, 2005) Kris Coolsaet asked whether it is possible to completely classify all rectagraphs with valency 6. That is indeed possible, and one can go a bit further.

There is earlier classification work. Peter Wild [9] determined all semibiplanes of block size at most 6. Michel Mollard [4] determined all $(0, 2)$-graphs on at most 31 vertices (and finds that these have valency at most 7).

2 Parameter restriction

The number of closed walks of length 4 without repeated vertices is $vk(k-1)/8$ and this must be integral, so for $k = 2, 3 \pmod 4$ we have $4|v$, and for $k = 4, 5 \pmod 8$ we have $2|v$.

3 Doubling

The *bipartite double* Δ of a graph Γ is the (bipartite) graph with vertices x^+ and x^- for each vertex x of Γ, and edges x^-y^+ and x^+y^- for each edge xy of Γ. It is connected if and only if Γ is connected and nonbipartite. It is a $(0, 2)$-graph if and only if Γ is a nonbipartite $(0, 2)$-graph, and then has the same valency as Γ.

This shows that non-bipartite $(0,2)$-graphs of valency k satisfy $v \le 2^{k-1}$.

The bipartite double has an involutory automorphism without fixed edges, namely the one interchanging x^+ and x^- for each x. Conversely, given any bipartite $(0,2)$-graph Δ with involutory automorphism σ without fixed edges that interchanges the two classes of the bipartition, one can construct a $(0,2)$-graph $\Gamma = \Delta/\sigma$ that has the σ-orbits as vertices, where two σ-orbits are adjacent when there are edges between them.

In terms of semibiplanes, σ is a polarity without absolute points. This shows that there is a 1-1 correspondence between nonbipartite $(0,2)$-graphs of valency k and semibiplanes of block size k with given polarity without absolute points.

Classifying $(0,2)$-graphs of small valency is thus reduced to classifying semibiplanes of small block size, and finding their polarities.

The *extended bipartite double* of a graph Γ is obtained from the bipartite double by adding edges x^-x^+ for each x. The extended bipartite double of a nonbipartite rectagraph is a bipartite rectagraph. The extended bipartite double of a bipartite graph Γ is just the product $K_2 \times \Gamma$.

The *universal* $(0,2)$*-cover* of a $(0,2)$-graph is its universal cover modulo 4-cycles. It is a $(0,2)$-graph again, and is bipartite.

4 Products

Let us use the symbol $\sim$ to denote adjacency.

For graphs Γ and Δ, let $\Gamma \times \Delta$ be the graph of which the vertex set is the Cartesian product of the vertex sets of Γ and Δ, where $(x,y) \sim (x',y')$ whenever either $x = x'$ and $y \sim y'$ or $x \sim x'$ and $y = y'$.

If Γ and Δ are $(0,2)$-graphs, then also $\Gamma \times \Delta$ is one. And if both are bipartite, then so is $\Gamma \times \Delta$. The valency of $\Gamma \times \Delta$ is the sum of the valencies of Γ and Δ. The diameter of $\Gamma \times \Delta$ is the sum of the diameters of Γ and Δ.

The hypercube 2^k (also called Q_k) is $\times_{i=1}^{k} K_2$.

5 Quotients

Let Γ be a graph and G a group of automorphisms of Γ. The quotient Γ/G is the graph that has as vertices the G-orbits on the vertex set of Γ, where two G-orbits are adjacent when they contain adjacent elements.

If Γ is a $(0,2)$-graph and no two elements of any G-orbit are joined by a path of length 1, 2 or 4, then also Γ/G is a $(0,2)$-graph.

In case $G = \langle\sigma\rangle$ we write Γ/σ instead of Γ/G.

For example, let Δ be the bipartitite double of the icosahedron ($\Delta_{5.2}$ in the tables below), and let $\Gamma = 2^2 \times \Delta$. Then Γ has 96 vertices and valency 7 and group of order 3840. (It is $\Delta_{7.38}$.) Let σ be the automorphism that sends each vertex to the unique vertex at distance 6. Then Γ/σ is bipartite, with 48 vertices, valency 7 and group of order 1920. (It is $\Delta_{7.2}$.)

Let δ be the automorphism of Γ that interchanges (u,x^+) and (u,x^-) for all u and x. The $\tau = \sigma\delta$ is an automorphism of Γ that sends each vertex to a vertex at distance 5, and Γ/τ is a nonbipartite rectagraph with 48 vertices, valency 7 and group of order 960. (It is $\Gamma_{7.44}$.)

6 Quotients of hypercubes

Let C be a linear code in 2^k viewed as binary vector space. If no distances 1, 2 or 4 occur between two code words, then the coset graph $2^k/C$ is a $(0,2)$-graph of valency k. If moreover no distance 3 occurs (so that C has minimum distance at least 5), then $2^k/C$ is a rectagraph. If no odd weights occur, then $2^k/C$ is bipartite.

For example, with $C = \langle 1110000, 0111111 \rangle$, the code C has weights 0, 3, 5, 6 and the quotient $2^7/C$ is a $(0,2)$-graph of valency 7 on 32 vertices, with automorphism group of order $1536 = 2!.4!.2^5$.

Conversely, every rectagraph in which any 3-claw determines a unique 3-cube is a quotient of a hypercube (not necessarily a coset graph), and in particular has a number of vertices that is a power of two. For details, see [1], 4.3.6 and 4.3.8.

7 Nonbipartite quotients of a hypercube

Let Δ be the hypercube 2^k, with as vertices the binary vectors of length k. Its group of automorphisms is $2^k : \mathrm{Sym}(k)$.

The involutions without fixed edges that interchange the two classes of the bipartition are the maps $x \mapsto \pi(x) + u$, where $\pi \in \mathrm{Sym}(k)$ is a coordinate permutation of order 1 or 2 and $u \in 2^k$ is a vector of odd weight, where $\pi(u) = u$, and π fixes at least three coordinates in the support of u.

(Indeed, if $x \mapsto \pi(x) + u$ has order 2, then $x = \pi(\pi(x)) + \pi(u) + u$ for all x, so that $\pi(u) = u$ and $\pi^2 = 1$. The vector u must have odd weight to interchange the two classes of the bipartition. If there is a fixed edge $(x, \pi(x) + u)$, then $x + \pi(x) + u$ has weight 1, and π fixes a unique element of the support of u.)

The conjugacy classes of these involutions have representatives given by such maps $x \mapsto \pi(x) + u$, where the support of u has odd weight at least 3, and is fixed pointwise by π.

(Indeed, choose the vector a so that the support of $a + \pi(a) + u$ is fixed pointwise by π. Then $x \mapsto \pi(x + a) + u + a$ is the required conjugate.)

If π fixes a coordinate outside the support of u, then the resulting quotient is of the form $2 \times E$ for some $(0,2)$-graph E of valency $k-1$. Hence, we may restrict ourselves to the case where π moves all positions outside the support of u. Now k is odd.

8 Nonbipartite quotients of a folded hypercube

If k is even, then the quotient $2^k/\mathbf{1}$, the folded k-cube, is still bipartite. The involutions without fixed edges that interchange the two classes of the bipartition are the maps $x \mapsto \pi(x) + u$ where $\pi \in \mathrm{Sym}(k)$ is a coordinate permutation of order 1 or 2 and $u \in 2^k$ is a vector of odd weight, and either $\pi(u) = u + \mathbf{1}$, or $\pi(u) = u$ where π fixes at least three coordinates inside and three coordinate outside the support of u.

In particular, for $k = 6$ we can use either $\pi = 1$, $u = (000111)$ and find $K_4 \times K_4$, or $\pi = (12)(34)(56)$, $u = (010101)$ and find the Shrikhande graph.

9 Quotients of projective planes

Given a projective plane Π with involution σ, construct a semibiplane of which the points and blocks are the σ-orbits of size 2 on the points and blocks of Π (Hughes [2]). If Π has parameters $PG(2,q)$ then for the incidence graph of the semibiplane one has $k = q$ and one of: (i) $v = q^2$ (elation, q even), (ii) $v = q^2 - 1$ (homology, q odd), (iii) $v = q^2 - \sqrt{q}$ (Baer involution, q square). There is work by Moorhouse [5] on reconstructing Π given the semibiplane Π/σ.

10 The half-double of a locally bipartite graph

Let Γ be a graph, and assume that for each vertex x a partition Π_x of the set of neighbours of Γ is given. Then we can define a graph Δ by taking as vertices the pairs (x, π) where $\pi \in \Pi_x$, and letting (x, π) be adjacent to (y, ρ) when $y \in \pi$ and $x \in \rho$.

The resulting graph Δ has the same number as edges as Γ, and the covering map $(x, \pi) \mapsto x$ sends edges to edges.

Now let Γ be locally connected and locally bipartite, without vertices of valency 1. Then for each vertex x, the set of neighbours of x has a unique partition into two cocliques, and the above construction yields a graph Δ, called the *half-double* of Γ (with twice as many vertices and the same number of edges). Note that Δ need not be connected, even when Γ is.

Apply this construction to the strongly regular graph with parameters $(v, k, \lambda) = (36, 14, 4)$ (with automorphism group $U_3(3).2$ and point stabiliser $L_2(7).2$) that is the first subconstituent of the Hall-Janko graph on 100 points. The result is a bipartite (0,2)-graph of diameter 5 and valency 7 on 72 vertices. Each vertex has distance 5 to a unique other point. Interchanging antipodes is not an automorphism, but identifying antipodes yields the graph Γ again. This graph has automorphism group $U_3(3).2$ with point stabilizer $L_2(7)$. (It is $\Delta_{7.29}$.)

11 A cover

Let us describe one more explicit construction of a bipartite $(0, 2)$-graph. It is a 2-cover of the extended bipartite double of $K_4 \times K_4$ on 64 vertices with $k = 7$. Let the vertex set be $A \times A \times B \times B$, where $A = \mathbf{Z}_3 \cup \{\infty\}$ and $B = \mathbf{Z}_2$. Let the adjacencies be $(a, b, 0, j) \sim (a, c, 1, j + f(a, b, c))$ and $(a, b, 0, j) \sim (c, b, 1, j + f(b, a, c))$ for $a, b, c \in A$ and $j \in B$, where $f(a, b, c) = 1$ for $a = \infty$, $b \neq \infty$, $c = b - 1$ or $c = b$, and for $a \neq \infty$, $b \neq \infty$, $c = b + 1$, and $f(a, b, c) = 0$ otherwise. The resulting graph has the stated properties. Its group is vertex transitive of order 1152. (It is $\Delta_{7.22}$.)

12 A $(0, 2)$-graph from the dodecahedron

Consider the graph on the 20 vertices of the dodecahedron, adjacent when either adjacent on the dodecahedron, or at distance three joined by a path ”step, turn left, step, turn right, step”. (This latter relation is an equivalence relation, inducing $5K_4$. Thus, the graph is the edge-disjoint union of the dodecahedron graph and $5K_4$, each of valency 3.) The resulting graph is $\Gamma_{6.8}$.

13 Semibiplanes with $k \leq 7$

We find unique bipartite $(0, 2)$-graphs of valency k for $k \leq 3$, and 2, 4, 13, 40 nonisomorphic ones for $k = 4, 5, 6, 7$, respectively.

(A referee asks: ‘How?’. The short answer is: by computer search. Patric Östergård independently verified these numbers, and went on to do $k = 8$.)

In the table below, # gives a serial number (given k), d denotes the diameter, ‘bipd’ stands for bipartite double, ‘drg’ means distance-regular graph, ‘ico’ means icosahedron. The ‘distribution’ column gives the distance distribution: the number of points at each distance from a given point. If the group is nontransitive, there may be several different distributions, depending on the choice of the given point.

Viewed as semibiplanes, each of these graphs is self-dual, that is, all bipartite $(0, 2)$-graphs with $1 \leq k \leq 7$ have an automorphism that interchanges the two classes of the bipartition. In particular, each has a nontrivial group of automorphisms G. The ‘gsz’ column gives $|G|$.

The bipartite $(0,2)$-graph of valency k with serial number i is called $\Delta_{k.i}$.

#	k	v	d	distribution	gsz	orbits	graph
1	0	1	0	1	1	tra	2^0
1	1	2	1	1+1	2	tra	2^1
1	2	4	2	1+2+1	8	tra	2^2
1	3	8	3	1+3+3+1	48	tra	2^3
1	4	14	3	1+4+6+3	336	tra	2-(7,4,2)
2	4	16	4	1+4+6+4+1	384	tra	2^4
1	5	22	3	1+5+10+6	1320	tra	2-(11,5,2)
2	5	24	4	1+5+10+7+1	480	tra	bipd(ico)
3	5	28	4	1+5+10+9+3	672	tra	$2 \times \Delta_{4.1}$
4	5	32	5	1+5+10+10+5+1	3840	tra	2^5
1	6	32	3	1+6+15+10	1536	tra	2-(16,6,2)
2	6	32	3	1+6+15+10	768	tra	2-(16,6,2)
3	6	32	3	1+6+15+10	23040	tra	2-(16,6,2), $2^6/\mathbf{1}$
4	6	36	4	1+6+15+12+2	96	12+24	
5	6	36	4	1+6+15+12+2	4320	tra	drg [3],[1, p.399]
6	6	36	4	1+6+15+12+2	48	12+24	
7	6	40	4	1+6+15+14+4	48	8+8+24	
8	6	40	4	1+6+15+14+4	120	tra	
9	6	44	4	1+6+15+16+6	2640	tra	$2 \times \Delta_{5.1}$
10	6	48	5	1+6+15+18+8 (32x)	256	16+32	
				1+6+15+17+8+1 (16x)			
11	6	48	5	1+6+15+17+8+1	960	tra	$2 \times \Delta_{5.2}$
12	6	56	5	1+6+15+19+12+3	2688	tra	$2^2 \times \Delta_{4.1}$
13	6	64	6	1+6+15+20+15+6+1	46080	tra	2^6
1	7	48	4	1+7+21+17+2	384	tra	
2	7	48	4	1+7+21+17+2	1920	tra	$\Delta_{7.38}/\sigma$, see §5
3	7	48	4	1+7+21+17+2	48	tra	
4	7	48	4	1+7+21+17+2	64	16+32	
5	7	48	4	1+7+21+17+2	64	16+32	
6	7	48	4	1+7+21+17+2	48	tra	
7	7	48	4	1+7+21+17+2	96	tra	
8	7	48	4	1+7+21+17+2	72	12+36	
9	7	48	4	1+7+21+17+2	96	tra	
10	7	48	4	1+7+21+17+2	2016	tra	Π/σ
11	7	56	4	1+7+21+21+6	16	2*4+2*8+2*16	
12	7	56	4	1+7+21+21+6	8	4*4+5*8	
13	7	56	4	1+7+21+21+6	16	2*4+2*8+2*16	
14	7	56	4	1+7+21+21+6	24	2+6+4*12	
15	7	56	4	1+7+21+21+6	8	4*4+5*8	
16	7	56	4	1+7+21+21+6	4	14*4	
17	7	56	4	1+7+21+21+6	16	3*8+2*16	
18	7	64	4	1+7+21+25+10	48	16+48	
19	7	64	4	1+7+21+25+10	120	24+40	
20	7	64	4	1+7+21+25+10	3072	tra	$2 \times \Delta_{6.1}$
21	7	64	4	1+7+21+25+10	1536	tra	$2 \times \Delta_{6.2}$
22	7	64	4	1+7+21+25+10	1152	tra	see §11
23	7	64	4	1+7+21+25+10	46080	tra	$2 \times \Delta_{6.3}$
24	7	64	5	1+7+21+25+10 (60x)	12	4+5*12	
				1+7+21+24+10+1 (4x)			
25	7	64	5	1+7+21+24+10+1 (36x)	24	4+5*12	
				1+7+21+25+10 (28x)			
26	7	64	5	1+7+21+24+10+1 (24x)	8	8*4+4*8	
				1+7+21+25+10 (40x)			
27	7	64	5	1+7+21+24+10+1 (14x)	12	2*2+2*6+4*12	
				1+7+21+25+10 (50x)			
28	7	64	5	1+7+21+24+10+1 (8x)	336	8+56	

#	k	v	d	distribution	gsz	orbits	graph
				1+7+21+25+10 (56x)			
29	7	72	5	1+7+21+28+14+1	12096	tra	$U_3(3).2/L_2(7)$
30	7	72	5	1+7+21+27+14+2	192	24+48	$2 \times \Delta_{6.4}$
31	7	72	5	1+7+21+27+14+2	8640	tra	$2 \times \Delta_{6.5}$
32	7	72	5	1+7+21+27+14+2	96	24+48	$2 \times \Delta_{6.6}$
33	7	80	5	1+7+21+30+18+3 (32x)	48	8+24+48	
				1+7+21+31+18+2 (48x)			
34	7	80	5	1+7+21+29+18+4	96	16+16+48	$2 \times \Delta_{6.7}$
35	7	80	5	1+7+21+29+18+4	240	tra	$2 \times \Delta_{6.8}$
36	7	88	5	1+7+21+31+22+6	10560	tra	$2^2 \times \Delta_{5.1}$
37	7	96	6	1+7+21+32+25+9+1 (32x)	512	32+64	$2 \times \Delta_{6.10}$
				1+7+21+33+26+8 (64x)			
38	7	96	6	1+7+21+32+25+9+1	3840	tra	$2^2 \times \Delta_{5.2}$
39	7	112	6	1+7+21+34+31+15+3	16128	tra	$2^3 \times \Delta_{4.1}$
40	7	128	7	1+7+21+35+35+21+7+1	645120	tra	2^7

The above table contains five pairs of graphs for which identical data is given. Given a vertex x of a $(0,2)$-graph Γ with set of neighbours A, we can identify a vertex y at distance 2 from x with a pair of vertices ("edge") on A. Given a vertex z at distance 3 from x, the union of the geodesics from z to x determines a graph on a subset of A that is regular of valency 2, that is, a union of polygons.

For graphs $\Delta_{7.i}$ with $1 \le i \le 10$ of valency 7 on 48 vertices, for each vertex x the number of vertices z at distance 3 from x that determine $2K_3$ equals 2, 10, 5, 6, 2, 7, 4, 6 or 7, 8, 7, respectively. This suffices to distinguish them. (They can also be distinguished by looking at their nonbipartite quotients.)

For graphs $\Delta_{7.i}$ with $11 \le i \le 17$ of valency 7 on 56 vertices, for each vertex x the number of vertices z at distance 3 from x that determine K_3 equals 0/1, 0/1/2/3, 0/1/2, 0/1/3, 0/1, 0/1/2, 0/1, respectively, where / abbreviates "or" (and all alternatives do occur). This suffices to distinguish them.

Of the graphs given here only $\Delta_{6.3}$, $\Delta_{7.2}$, and $\Delta_{7.23}$ are not their own universal $(0,2)$-cover. (Their universal $(0,2)$-covers are 2^6, $\Delta_{7.38}$ and 2^7, respectively.)

14 Nonbipartite $(0,2)$-graphs with $k \le 7$

The nonbipartite $(0,2)$-graphs with $k \le 7$ have bipartite doubles found above. We find 1, 1, 4, 11, 56 nonisomorphic solutions for $k = 3,4,5,6,7$, respectively.

The column 'locally' describes the local graph by a string of digits giving the component sizes (each component is either a single point or a cycle). For example, the C31 for $K_2 \times K_4$ means that all local graphs are the union of a triangle and an isolated vertex. (The C is just to remind the reader that what follows is a digit string, not a number.) A rectagraph does not have triangles, so all components of local graphs are single points, and we write 'recta' instead of C111..1. In cases where this is an automorphism, ϕ denotes the interchange of antipodal vertices.

The nonbipartite $(0,2)$-graph of valency k with serial number i is called $\Gamma_{k.i}$.

#	bipd	k	v	d	distribution	locally	gsz	orbits	graph
1	3.1	3	4	1	1+3	C3	24	tra	$2^3/(111) \cong K_4$
1	4.2	4	8	2	1+4+3	C31	48	tra	$2^4/(0111) \cong K_2 \times K_4$
1	5.2	5	12	3	1+5+5+1	C5	120	tra	icosahedron
2	5.4	5	16	2	1+5+10	recta	1920	tra	$2^5/(11111)$
3	5.4	5	16	3	1+5+7+3	C311	192	tra	$2^5/(00111) \cong 2^2 \times K_4$
4	5.4	5	16	3	1+5+7+3 (8x)	C311	96	8+8	$2^5/\pi_{(12)}(00111)$
					1+5+10 (8x)	C11111			
1	6.3	6	16	2	1+6+9	C33	1152	tra	$K_4 \times K_4$
2	6.3	6	16	2	1+6+9	C6	192	tra	Shrikhande

#	bipd	k	v	d	distribution	locally	gsz	orbits	graph
3	6.8	6	20	3	1+6+12+1	C3111	60	tra	§12
4	6.7	6	20	3	1+6+10+3 (12x)	C51	24	4+4+12	
					1+6+9+4 (4x)	C6			
					1+6+12+1 (4x)	C3111			
5	6.11	6	24	3	1+6+12+5 (16x)	C3111	32	8+16	
					1+6+15+2 (8x)	C111111			
6	6.11	6	24	3	1+6+15+2	recta	480	tra	$(2 \times \text{bipd(ico)})/\phi$
7	6.11	6	24	4	1+6+10+6+1	C51	240	tra	$2 \times \Gamma_{5.1}$
8	6.12	6	28	3	1+6+12+9 (16x)	C3111	48	4+12+12	
					1+6+15+6 (12x)	C111111			
9	6.13	6	32	3	1+6+15+10	recta	3840	tra	$2 \times \Gamma_{5.2}$
10	6.13	6	32	4	1+6+12+10+3	C3111	1152	tra	$2^3 \times K_4$
11	6.13	6	32	4	1+6+12+10+3 (16x)	C3111	192	16+16	$2 \times \Gamma_{5.4}$
					1+6+15+10 (16x)	C111111			
1	7.1	7	24	2	1+7+16	C511	96	tra	
2	7.1	7	24	2	1+7+16	C511	96	tra	
3	7.2	7	24	2	1+7+16	C511	480	tra	
4	7.2	7	24	2	1+7+16	C511	480	tra	
5	7.1	7	24	3	1+7+15+1	C331	96	tra	
6	7.2	7	24	3	1+7+15+1	C61	48	tra	
7	7.4	7	24	3	1+7+14+2 (4x)	C7	8	4*4+8	
					1+7+15+1 (16x)	C61			
					1+7+16 (4x)	C511			
8	7.5	7	24	3	1+7+14+2 (8x)	C7	16	8+8+8	
					1+7+15+1 (8x)	C61			
					1+7+16 (8x)	C511			
9	7.6	7	24	3	1+7+16 (4x)	C511	4	6*4	
					1+7+15+1 (16x)	C61			
					1+7+14+2 (4x)	C7			
10	7.8	7	24	3	1+7+14+2 (12x)	C7	12	4*6	
					1+7+15+1 (12x)	C61			
11	7.9	7	24	3	1+7+16 (12x)	C511	12	4*6	
					1+7+15+1 (12x)	C61			
12	7.10	7	24	3	1+7+14+2	C7	336	tra	drg [1, p.386]
13	7.24	7	32	3	1+7+18+6 (12x)	C31111	6	2+5*6	
					1+7+16+8 (12x)	C511			
					1+7+15+9 (6x)	C61			
					1+7+21+3 (2x)	C1111111			
14	7.26	7	32	3	1+7+18+6 (8x)	C31111	4	8*2+4*4	
					1+7+21+3 (4x)	C1111111			
					1+7+16+8 (14x)	C511			
					1+7+15+9 (2x)	C331			
					1+7+15+9 (2x)	C61			
					1+7+14+10 (2x)	C7			
15	7.27	7	32	3	1+7+21+3 (3x)	C1111111	2	8*1+12*2	
					1+7+16+8 (13x)	C511			
					1+7+18+6 (10x)	C31111			
					1+7+14+10 (1x)	C7			
					1+7+15+9 (4x)	C61			
					1+7+15+9 (1x)	C331			
16	7.18	7	32	3	1+7+16+8 (24x)	C511	8	4*8	
					1+7+15+9 (8x)	C61			
17	7.18	7	32	3	1+7+18+6	C31111	24	8+24	
18	7.20	7	32	3	1+7+18+6	C31111	512	tra	
19	7.20	7	32	3	1+7+18+6	C31111	256	tra	
20	7.21	7	32	3	1+7+18+6	C31111	96	tra	
21	7.21	7	32	3	1+7+18+6	C31111	768	tra	
22	7.28	7	32	3	1+7+21+3 (4x)	C1111111	24	4+4+24	

#	bipd	k	v	d	distribution	locally	gsz	orbits	graph
					1+7+18+6 (24x)	C31111			
					1+7+15+9 (4x)	C331			
23	7.28	7	32	3	1+7+21+3 (4x)	C1111111	168	4+28	
					1+7+15+9 (28x)	C61			
24	7.22	7	32	3	1+7+16+8 (24x)	C511	48	8+24	
					1+7+15+9 (8x)	C61			
25	7.23	7	32	3	1+7+18+6	C31111	1536	tra	$2^7/C$, see §6
26	7.23	7	32	3	1+7+18+6	C31111	256	tra	
27	7.23	7	32	3	1+7+15+9	C331	2304	tra	$2 \times \Gamma_{6.1}$
28	7.23	7	32	3	1+7+15+9	C61	384	tra	$2 \times \Gamma_{6.2}$
29	7.25	7	32	4	1+7+16+7+1 (6x)	C511	12	2+5*6	
					1+7+16+8 (12x)	C511			
					1+7+14+9+1 (6x)	C7			
					1+7+18+6 (8x)	C31111			
30	7.28	7	32	4	1+7+14+9+1 (4x)	C7	56	4+28	
					1+7+16+8 (28x)	C511			
31	7.30	7	36	3	1+7+18+10 (20x)	C31111	32	4+2*8+16	
					1+7+21+7 (16x)	C1111111			
32	7.32	7	36	3	1+7+18+10 (20x)	C31111	8	3*4+3*8	
					1+7+21+7 (16x)	C1111111			
33	7.34	7	40	4	1+7+16+13+3 (24x)	C511	48	8+8+24	$2 \times \Gamma_{6.4}$
					1+7+15+13+4 (8x)	C61			
					1+7+18+13+1 (8x)	C31111			
34	7.34	7	40	4	1+7+18+12+2 (8x)	C31111	16	3*8+16	
					1+7+18+14 (16x)	C31111			
					1+7+21+11 (16x)	C1111111			
35	7.35	7	40	4	1+7+21+11 (16x)	C1111111	8	5*8	
					1+7+18+14 (16x)	C31111			
					1+7+18+12+2 (8x)	C31111			
36	7.35	7	40	4	1+7+18+13+1	C31111	120	tra	$2 \times \Gamma_{6.3}$
37	7.36	7	44	3	1+7+18+18 (20x)	C31111	80	4+20+20	
					1+7+21+15 (24x)	C1111111			
38	7.37	7	48	4	1+7+21+19 (16x)	C1111111	32	3*16	
					1+7+18+20+2 (16x)	C31111			
					1+7+21+17+2 (16x)	C1111111			
39	7.37	7	48	4	1+7+18+17+5 (8x)	C31111	32	4*8+16	
					1+7+21+16+3 (8x)	C1111111			
					1+7+18+18+4 (8x)	C31111			
					1+7+21+17+2 (24x)	C1111111			
40	7.37	7	48	4	1+7+18+17+5 (16x)	C31111	128	16+32	
					1+7+21+17+2 (32x)	C1111111			
41	7.38	7	48	4	1+7+18+17+5 (32x)	C31111	64	16+32	$2 \times \Gamma_{6.5}$
					1+7+21+17+2 (16x)	C1111111			
42	7.38	7	48	4	1+7+18+19+3 (24x)	C31111	96	24+24	
					1+7+21+19 (24x)	C1111111			
43	7.38	7	48	4	1+7+18+19+3 (16x)	C31111	64	3*16	
					1+7+21+16+3 (16x)	C1111111			
					1+7+21+19 (16x)	C1111111			
44	7.38	7	48	4	1+7+21+16+3	recta	960	tra	$\Delta_{7.38}/\tau$, see §5
45	7.38	7	48	4	1+7+21+17+2	recta	960	tra	$2 \times \Gamma_{6.6}$
46	7.38	7	48	5	1+7+16+16+7+1	C511	960	tra	$2^2 \times \Gamma_{5.1}$
47	7.38	7	48	5	1+7+16+16+7+1 (24x)	C511	480	24+24	
					1+7+21+16+3 (24x)	C1111111			
48	7.39	7	56	4	1+7+18+21+9 (32x)	C31111	96	8+24+24	$2 \times \Gamma_{6.8}$
					1+7+21+21+6 (24x)	C1111111			
49	7.39	7	56	4	1+7+18+21+9	C31111	8064	tra	$K_4 \times \Delta_{4.1}$
50	7.39	7	56	4	1+7+18+21+9 (24x)	C31111	384	8+16+32	
					1+7+21+24+3 (32x)	C1111111			

#	bipd	k	v	d	distribution	locally	gsz	orbits	graph
51	7.40	7	64	3	1+7+21+35	recta	322560	tra	$2^7/(1111111)$
52	7.40	7	64	4	1+7+21+25+10	recta	15360	tra	$2^2 \times \Gamma_{5.2}$
53	7.40	7	64	4	1+7+21+25+10 (32x)	recta	7680	32+32	$2^7/\pi_{(12)}\cdot$
					1+7+21+35 (32x)				$\cdot(0011111)$
54	7.40	7	64	5	1+7+18+22+13+3	C31111	9216	tra	$2^4 \times K_4$
55	7.40	7	64	5	1+7+18+22+13+3 (32x)	C31111	768	32+32	$2^2 \times \Gamma_{5.4}$
					1+7+21+25+10 (32x)	C1111111			
56	7.40	7	64	5	1+7+18+22+13+3 (16x)	C31111	768	2*16+32	$2^7/\pi_{(12)(34)}\cdot$
					1+7+21+25+10 (32x)	C1111111			$\cdot(0000111)$
					1+7+21+35 (16x)	C1111111			

The above table contains two pairs of graphs on 24 points for which identical data is given. All four have local graphs C511, so that in each the edges that occur in a triangle form two disjoint icosahedra, and the remaining edges, two on each point, join that point to two antipodal vertices in the other icosahedron. The structure is that of two icosahedra together with a matching of the six antipodal pairs of one with the six antipodal pairs of the other. One sees that there are four possibilities for the number of induced $2 \times K_3$ on each edge across, namely 3, 2, 5, 0 for the for cases $\Gamma_{7.i}$, $i = 1, 2, 3, 4$, respectively.

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