RESOLUTIONS AND BETTI DIAGRAMS OF ALGEBRAS OF SL_2-INVARIANTS

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ABSTRACT. For the algebras of SL_2-invariants of small homological dimension the free graded resolutions and graded Betti diagrams are calculated.

1. Let V_d be the complex $(d+1)$-dimensional SL_2-module of binary forms of degree d and let $V_{\boldsymbol{d}} = V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_n}$, $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$. Denote by $\mathcal{I} := \mathbb{C}[V_{\boldsymbol{d}}]^{SL_2}$ the algebra of polynomial SL_2-invariant functions on the module $V_{\boldsymbol{d}}$. It is well known that the algebra $\mathcal{I}$ is finitely generated. Let $f_1, f_2, \ldots, f_m$ be a minimal generating set. The measure of intricacy of the algebra $\mathcal{I}$ is the length of its chains of syzygies, called homological (or projective) dimension $\operatorname{hd} \mathcal{I}$. In [1] Popov gave a classification of the cases in which $\operatorname{hd} \mathcal{I} \leq 10$ for a single binary form ($n = 1$) or $\operatorname{hd} \mathcal{I} \leq 3$ for $n > 1$. Recently Brouwer and Popoviciu [2] extended these results and determined for $n = 1$ the cases with $\operatorname{hd} \mathcal{I} \leq 100$, and for $n > 1$ those with $\operatorname{hd} \mathcal{I} \leq 15$.

In this short note we present the results of calculations of the finite graded free resolutions and graded Betti diagrams for the algebras of SL_2-invariants $\mathcal{I}$ in the cases $\operatorname{hd} \mathcal{I} \leq 10$.

2. Let $R := \mathbb{C}[x_1, \ldots, x_m]$ be positively graded by $\deg(x_h) = \deg(f_h)$, $h = 1, \ldots, m$, and put $e_h := \deg(x_h)$. Recall that a finite graded free resolution of $\mathcal{I}$ of length $l = \operatorname{hd} \mathcal{I}$ is an exact sequence of R-modules

$$0 \longrightarrow F_l \longrightarrow F_{l-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{I} \longrightarrow 0,$$

where $F_i = \oplus_j R(-j)^{\beta_{i,j}}$, $F_0 = R$ are finitely generated graded free R-modules. The image F_i is called the i-th module of syzygies of $\mathcal{I}$. Since the algebra $\mathcal{I}$ is Cohen-Macaulay, the Auslander-Buchsbaum theorem implies that $l = \dim \mathcal{I} - \operatorname{trdeg}_{\mathbb{C}} \mathcal{I}$, see also [1].

The numbers $\beta_{i,j}$ are called the graded Betti numbers. They can be arranged into the graded Betti diagram $\beta(\mathcal{I})$. Our Betti diagrams $\beta(\mathcal{I})$ have entries $\beta_{i,j}$, where such an entry indicates that F_i has summand $R(-j)$ with multiplicity $\beta_{i,j}$. The row index i runs from 0 to l. We only give the columns that have at least one nonzero entry.

The Hilbert-Poincaré series of the algebra $\mathcal{I}$ can be recovered from the Betti numbers by

$$\mathcal{P}(\mathcal{I}, z) = \frac{\displaystyle\sum_{i=0}^{l} \sum_{j \in \mathbb{Z}} (-1)^i \beta_{i,j} z^j}{\displaystyle\prod_{i=1}^{m} (1 - z^{e_i})}$$

The inverse statement is false, see below the case $2V_1 \oplus V_3$.

Theorem. *The free graded resolution and the graded Betti diagrams of the algebras of SL_2-invariants $\mathcal{I}$ in the cases* $\operatorname{hd}\mathcal{I} \leq 10$ *are as described below.*

hd $\mathcal{I} = 0$.

Set of modules: $V_1,\ V_2,\ V_3,\ V_4,\ 2V_1,\ V_1 \oplus V_2,\ 2V_2,\ 3V_1$.

Minimal free resolution:

$$0 \longrightarrow R \longrightarrow \mathcal{I} \longrightarrow 0.$$

module	e_h
V_1	
V_2	2
V_3	4
V_4	2, 3
$2V_1$	2
$V_1 \oplus V_2$	2, 3
$2V_2$	2, 2, 2
$3V_1$	2, 2, 2

hd $\mathcal{I} = 1$.

Set of modules: V_5, V_6, $V_1 \oplus V_3$, $V_1 \oplus V_4$, $V_2 \oplus V_3$, $V_2 \oplus V_4$, $2V_4$, $2V_1 \oplus V_2$, $V_1 \oplus 2V_2$, $3V_2$, $4V_1$.

Minimal free resolutions: $0 \to F_1 \to R \to I \to 0$.

module	F_1	e_h
V_5	$R(-36)$	4, 8, 12, 18
V_6	$R(-30)$	2, 4, 6, 10, 15
$V_1 \oplus V_3$	$R(-12)$	4, 4, 4, 6
$V_1 \oplus V_4$	$R(-18)$	2, 3, 5, 6, 9
$V_2 \oplus V_3$	$R(-14)$	2, 3, 4, 5, 7
$V_2 \oplus V_4$	$R(-12)$	2×2, 2×3, 4, 6
$2V_4$	$R(-12)$	3×2, 4×3, 4
$2V_1 \oplus V_2$	$R(-6)$	2×2, 3×3
$V_1 \oplus 2V_2$	$R(-8)$	3×2, 2×3, 4
$3V_2$	$R(-6)$	6×2, 3
$4V_1$	$R(-4)$	6×2

The cases V_5, V_6 are well-known classical results, see [3].

hd $\mathcal{I} = 2$.

Set of modules: $V_3 \oplus V_3$.

Minimal free resolutions: $0 \to F_2 \to F_1 \to R \to I \to 0$.

module	e_h
$2V_3$	2, 5×4, 6

Betti diagram:

$2V_3$	0	8	12	20
0	1	-	-	-
1	-	1	1	-
2	-	-	-	1

This Betti diagram says that $F_1 = R(-8) \oplus R(-12)$ and $F_2 = R(-20)$. The entries of the Betti diagram are multiplicities, so that an entry m in column β indicates a summand $R(-\beta)^m$.

hd $\mathcal{I}$ = 3.

Set of modules: $V_8,\ 5V_1$.

Minimal free resolutions: $0 \to F_3 \to F_2 \to F_1 \to R \to I \to 0$.

module	e_h
V_8	$2, 3, 4, 5, 6, 7, 8, 9, 10$
$5V_1$	10×2

Betti diagrams:

V_8	0	16	17	18	19	20	25	26	27	28	29	45
0	1	-	-	-	-	-	-	-	-	-	-	-
1	-	1	1	1	1	1	-	-	-	-	-	-
2	-	-	-	-	-	-	1	1	1	1	1	-
3	-	-	-	-	-	-	-	-	-	-	-	1

$5V_1$	0	4	6	10
0	1	-	-	-
1	-	5	-	-
2	-	-	5	-
3	-	-	-	1

The results for V_8 are due to Shioda [4].

hd $\mathcal{I}$ = 4.

Set of modules: $3V_1 \oplus V_2$.

Minimal free resolutions: $0 \to F_4 \to F_3 \to F_2 \to F_1 \to R \to I \to 0$.

module	e_h
$3V_1 \oplus V_2$	$4 \times 2, 6 \times 3$

Betti diagram:

$i\backslash j$	0	5	6	8	9	11	12	17
0	1	-	-	-	-	-	-	-
1	-	3	6	-	-	-	-	-
2	-	-	-	8	8	-	-	-
3	-	-	-	-	-	6	3	-
4	-	-	-	-	-	-	-	1

hd $\mathcal{I}$ = 5.

Set of modules: $V_1 \oplus 3V_2,\ 4V_2$.

Minimal free resolutions: $0 \to F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to R \to I \to 0$.

module	e_h
$3V_1 \oplus V_2$	$6 \times 2, 4 \times 3, 3 \times 4$
$4V_2$	$10 \times 2, 4 \times 3$

Betti diagrams:

$V_1 \oplus 3V_2$:	0	6	7	8	9	10	11	12	13	14	15	16	17	18	19	25
0	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	4	4	6	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	3	12	12	8	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	8	12	12	3	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-	-	-	6	4	4	-
5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1

$4V_2$	0	5	6	8	9	11	12	14	15	20
0	1	-	-	-	-	-	-	-	-	-
1	-	4	10	-	-	-	-	-	-	-
2	-	-	-	15	20	-	-	-	-	-
3	-	-	-	-	-	20	15	-	-	-
4	-	-	-	-	-	-	-	10	4	-
5	-	-	-	-	-	-	-	-	-	1

hd $\mathcal{I} = 6$.

Set of modules: $2V_1 \oplus 2V_2,\ 6V_1$.

Minimal free resolutions: $0 \to F_6 \to F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to R \to I \to 0$.

module	e_h
$2V_1 \oplus 2V_2$	$4 \times 2, 6 \times 3, 3 \times 4$
$6V_1$	15×2

Betti diagrams:

$2V_1 \oplus 2V_2$	0	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	28
0	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	6	8	6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	8	24	24	8	-	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	3	24	36	24	3	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-	-	8	24	24	8	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	6	8	6	-
6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1

$6V_1$	0	4	6	8	10	12	14	18
0	1	-	-	-	-	-	-	-
1	-	15	-	-	-	-	-	-
2	-	-	35	-	-	-	-	-
3	-	-	-	21	21	-	-	-
4	-	-	-	-	-	35	-	-
5	-	-	-	-	-	-	15	-
6	-	-	-	-	-	-	-	1

hd $\mathcal{I} = 7$.

There are no such modules.

hd $\mathcal{I} = 8$.

Set of modules: $2V_1 \oplus V_3$.

Minimal free resolutions: $0 \to F_8 \to F_7 \to F_6 \to F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to R \to I \to 0$.

module	e_h
$2V_1 \oplus 2V_2$	$2, 8 \times 4, 4 \times 6$

Betti diagram

$i\backslash j$	0	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	50
0	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	10	15	10	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	20	60	60	20	-	-	-	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	15	90	140	90	15	-	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	4	60	160	160	60	4	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-	15	90	140	90	15	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-	-	-	-	20	60	60	20	-	-	-
7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	10	15	10	-
8	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1

hd $\mathcal{I} = 9$.

Set of modules: $V_1 \oplus V_2 \oplus V_3,\ 4V_1 \oplus V_2$.

Minimal free resolutions:

$$0 \to F_9 \to F_8 \to F_7 \to F_6 \to F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to R \to I \to 0.$$

module	e_h
$V_1 \oplus V_2 \oplus V_3$	$2, 3 \times 3, 4 \times 4, 4 \times 5, 2 \times 6, 7$
$4V_1 \oplus V_2$	$7 \times 2, 10 \times 3$

Betti diagram for $V_1 \oplus V_2 \oplus V_3$:

$V_1 \oplus V_2 \oplus V_3$	0	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
0	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	4	8	13	10	6	2	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	4	19	40	53	52	36	18	7	2	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	1	14	44	90	123	128	99	60	26	8	1	-	-
4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3	20	62	122	178	194	165	108
5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3	16	53
6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

$V_1 \oplus V_2 \oplus V_3$	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	57
0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	53	16	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
5	108	165	194	178	122	62	20	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-
6	-	-	1	8	26	60	99	128	123	90	44	14	1	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	2	7	18	36	52	53	40	19	4	-	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	2	6	10	13	8	4	-
9	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1

Betti diagram for $4V_1 \oplus V_2$:

$i\backslash j$	0	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	33
0	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	1	15	20	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
2	-	-	-	-	16	80	64	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	-	-	-	-	-	-	1	60	164	90	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	-	-	-	-	-	-	-	-	-	-	80	144	84	45	35	-	-	-	-	-	-	-	-	-	-	-	-	-
5	-	-	-	-	-	-	-	-	-	-	-	-	-	35	45	84	144	80	-	-	-	-	-	-	-	-	-	-
6	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	90	164	60	1	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	64	80	16	-	-	-	-
8	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	20	15	1	-
9	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1

hd $\mathcal{I} = 10$.

Set of modules: $7V_1$.

Minimal free resolutions:

$$0 \to F_{10} \to F_9 \to F_8 \to ... \to F_3 \to F_2 \to F_1 \to R \to I \to 0$$

module	e_h
$7V_1$	21×2

Betti diagram:

$7V_1$	0	4	6	8	10	12	14	16	18	20	22	24	28
0	1	-	-	-	-	-	-	-	-	-	-	-	-
1	-	35	-	-	-	-	-	-	-	-	-	-	-
2	-	-	140	-	-	-	-	-	-	-	-	-	-
3	-	-	-	189	196	-	-	-	-	-	-	-	-
4	-	-	-	-	84	735	-	-	-	-	-	-	-
5	-	-	-	-	-	-	1080	-	-	-	-	-	-
6	-	-	-	-	-	-	-	735	84	-	-	-	-
7	-	-	-	-	-	-	-	-	196	189	-	-	-
8	-	-	-	-	-	-	-	-	-	-	140	-	-
9	-	-	-	-	-	-	-	-	-	-	-	35	-
10	-	-	-	-	-	-	-	-	-	-	-	-	1

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