A strongly regular graph on 336 vertices

aeb

September 12, 2014

Recently, Thomas Jenrich [1] constructed a strongly regular graph with parameters $(v, k, \lambda, \mu) = (336, 80, 28, 16)$ starting from the $G_2(4)$ graph. Graphs with these parameters were known already: examples are the block graphs of Steiner systems $S(2, 4, 64)$, and in particular the line graph of the affine space $AG(3, 4)$. This new graph is different.

0.1 The Suzuki tower

There exists a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (1782, 416, 100, 96)$, that has as full group of automorphisms Suz.2, acting rank 3. Fix a vertex ∞ of Γ. The graph Δ induced by Γ on the set of neighbours of ∞ is strongly regular with parameters $(v, k, \lambda, \mu) = (416, 100, 36, 20)$, and has as full group of automorphisms $G_2(4).2$, acting rank 3.

Fix a vertex p of Γ, nonadjacent to ∞. Then the set B of common neighbours of ∞ and p has size 96, and the set C of remaining vertices of Δ has size 320. In this way we find 1365 splits of the vertex set of Δ, in a single orbit of $\mathrm{Aut}\,\Delta$. The subgraphs induced on B and C have valencies 20 and 76, respectively.

The graph E constructed by Jenrich arises from C by adjoining a set D of 16 new vertices, where D is a coclique and each vertex of C is adjacent to 4 vertices of D. (For a strongly regular graph with parameters (336,80,28,16) the Hoffman bound for cocliques is 16, and necessarily every vertex outside a 16-coclique is adjacent to 4 vertices inside.)

The block graph of a Steiner system $S(2, 4, 64)$, and in particular the line graph of the affine space $AG(3, 4)$, contains cliques of size 21 formed by all blocks on a given point. On the other hand, maximal cliques in Δ have size 5, so the graph E has maximal cliques of size at most 6, and cannot be a block graph of some $S(2, 4, 64)$.

0.2 Construction of Δ

The graph Δ can be constructed as follows (cf. [2, 3]). Consider the projective plane $\mathrm{PG}(2, 16)$ provided with a nondegenerate Hermitean form. There are 273 points, 65 isotropic and 208 nonisotropic. There are $208 \cdot 12 \cdot 1/6 = 416$ orthogonal bases. These 416 orthogonal bases will be the vertices of Δ. Each nonisotropic point a is orthogonal to 5 isotropic points; call the set of these 5 points S_a. Then an orthogonal base $u = \{a, b, c\}$ determines a 15-set $T_u := S_a \cup S_b \cup S_c$. Now two vertices u, v of Δ are adjacent when $|T_u \cap T_v| = 3$.

(More in detail: The group $U_3(4){:}4$ of semilinear transformations preserving the form acts transitively on the 416 bases, with rank 5. The suborbit sizes

(sizes of the orbits of the stabilizer of a fixed base u) are 1, 15, 100, 150, 150. One has $|T_u \cap T_v|$ equal to 15, 5, 3, 2, 5 (respectively) for v in one of these suborbits. The suborbit of size 15 consists of the bases that have an element in common with $u = \{a, b, c\}$. The first suborbit of size 150 consists of the bases that are disjoint from $\{a, b, c\}$ but contain a point orthogonal to one of a, b, c.)

0.3 Construction of E

Let p be a fixed isotropic point. Let B be the set of vertices u with $p \in T_u$, and let C be the set of remaining vertices of Δ. Then $|B| = 96$ and $|C| = 320$. Each $u \in B$ has a unique element a with $p \in S_a$. Let L_u be the set S_a (a hyperbolic line on p).

Each $v \in C$ is adjacent to 76 vertices in C, and to 24 in B. These 24 neighbours $u \in B$ determine 24 lines L_u, and it turns out that each of these occurs twice. So we see 12 of the 16 hyperbolic lines on p. Let W_v be the set of 4 hyperbolic lines not seen.

Take for the 16-set D the set of 16 hyperbolic lines on p, and join v to the elements of W_v. The resulting graph E is strongly regular with parameters $(336, 80, 28, 16)$.

0.4 The affine plane locally at p

Above we found for each $v \in C$ a 4-subset W_v of D. These 320 sets W_v coincide in groups of 16, so that only 20 distinct such sets occur. We find an affine plane $\mathrm{AG}(2, 4)$ with point set D and as lines the distinct sets W_v.

There is a different way to find W_v from $v = \{a, b, c\}$. The isotropic parts of the (distinct) lines $p + a$, $p + b$, $p + c$ are in W_v, and this gives a triple in W_v. Since giving the collinear triples in $\mathrm{AG}(2, 4)$ suffices to determine the lines, and since any two points determine a unique line, this provides a more direct way to go from v to W_v. No detour over B is needed.

0.5 Groups

The graph on the lines of $\mathrm{AG}(3, 4)$, adjacent when they meet, is strongly regular with parameters $(v, k, \lambda, \mu) = (336, 80, 28, 16)$. Its full group of automorphisms is $2^6.\Gamma L_3(4)$ of order 23224320. The subgraph of all lines not in some fixed direction has 320 vertices and full group of order 1105920.

Compare this to the full group $2^9 : S_5 \times S_3$ of order 368640 of the 320-point graph on C, and the full group of order 3840 of Jenrich's new graph E.

References

[1] T. Jenrich, `arXiv:1409.3520v1`, 11 Sept. 2014.

[2] D. Crnković & V. Mikulić, *Block designs and strongly regular graphs constructed from the group U(3,4)*, Glasnik Matematicki **41** (2006) 189–194.

[3] A. E. Brouwer, N. Horiguchi, M. Kitazume & H. Nakasora, *A construction of the sporadic Suzuki graph from* $U_3(4)$, J. Comb. Th. (A) **116** (2009) 1056–1062.