The invariants of the binary decimic

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July 31, 2009

Abstract

We consider the algebra of invariants of binary forms of degree 10 with complex coefficients, construct a system of parameters with degrees 2, 4, 6, 6, 8, 9, 10, 14 and find the 106 basic invariants.

1 Introduction

Invariants

Let $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ denote the algebra of invariants of binary forms (forms in two variables) of degree n with complex coefficients. This algebra was extensively studied in the nineteenth century, and for $n \leq 6$ the structure was clear and a finite basis was known. Gordan [10] proved in 1868 that $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ has a finite basis for all n. For $n = 7$ the invariants were determined by von Gall [8] and Dixmier & Lazard [7] (see also Bedratyuk [1]). The invariants for $n = 8$ were found by von Gall [9] and Shioda [15]. The case $n = 9$ was done by Cröni [4] and the present authors [2]. Here we consider the case $n = 10$, and show that $\mathcal{O}(V_{10})^{\mathrm{SL}_2}$ is generated by 106 (explicitly known) basic invariants, and give the degrees.

Proposition 1.1. *The algebra I of invariants of the binary decimic (form of degree* 10*) is generated by* 106 *invariants. The nonzero numbers d_m of basic invariants of degree m are*

m	2	4	6	8	9	10	11	12	13	14	15	16	17	18	19	21
d_m	1	1	4	5	5	8	8	12	15	13	19	5	5	1	2	2

This list agrees with Sylvester & Franklin [18] for degrees less than 17. Sylvester predicted 3 basic invariants of degree 17 and none of degree higher than 17 for a total of 99 basic invariants. Tom Hagedorn (unpublished) found 104 invariants, cf. Olver [13] (p. 40). The existence of basic invariants of degree 21 seems to be new. That the list is complete follows as a corollary from the construction of a homogeneous system of parameters (hsop), see below.

Systems of parameters

A (homogeneous) *system of parameters* for a graded algebra A is an algebraically

*The second author is partially supported by the Swiss National Science Foundation.

independent set S of homogeneous elements of A such that A is module-finite over the subalgebra generated by the set S. Hilbert [12] showed the existence of a system of parameters for algebras of invariants, cf. Proposition 3.2 below.

Here we find an explicit system of parameters for $\mathcal{O}(V_{10})^{\mathrm{SL}_2}$.

Proposition 1.2. *The algebra I of invariants of the binary decimic has a system of parameters of degrees* 2, 4, 6, 6, 8, 9, 10, 14.

This is useful, since it provides an upper bound for the degrees of basic invariants that is sufficiently low, so that a simple computer search can find a basis for the invariants up to that degree.

2 Finding the basic invariants

A set of basic invariants of the algebra I of invariants is a minimal set of generators. The individual generators are not uniquely determined, but their degrees are.

The ring I is graded: $I = \oplus_m I_m$, where I_m is the subspace of invariants, homogeneous of degree m. If $y_1, \ldots, y_{n-2}$ is a system of parameters, where y_i has degree d_i, then the Poincaré series $P(t)$, defined by $P(t) = \sum_m \dim I_m t^m$, can be written as a rational function in t with denominator $\prod(1-t^{d_i})$. (Throughout this note, dim is vector space dimension, that is, is $\dim_{\mathbb{C}}$.)

Now $P(t)$ is known: it was given as a series by Cayley & Sylvester (cf. [17]) and as a rational function by Springer [16]. For $n = 10$ we have

$$\begin{aligned}P(t) = 1 &+ t^2 + 2t^4 + 6t^6 + 12t^8 + 5t^9 + 24t^{10} + 13t^{11} + 52t^{12} + 33t^{13} + 97t^{14} + \\ &80t^{15} + 177t^{16} + 160t^{17} + 319t^{18} + 301t^{19} + 540t^{20} + 547t^{21} + 887t^{22} + \\ &926t^{23} + 1429t^{24} + 1512t^{25} + 2219t^{26} + 2402t^{27} + 3367t^{28} + 3681t^{29} + \\ &5015t^{30} + 5502t^{31} + 7294t^{32} + 8064t^{33} + 10419t^{34} + 11550t^{35} + \\ &14664t^{36} + 16253t^{37} + 20287t^{38} + 22531t^{39} + 27682t^{40} + 30738t^{41} + \\ &37319t^{42} + 41378t^{43} + 49671t^{44} + 55060t^{45} + 65390t^{46} + 72391t^{47} + \\ &85250t^{48} + \cdots\end{aligned}$$

If we generate invariants of degree m, and have found $\dim I_m$ independent ones, then we have found all in degree m. If we know that there is a hsop with degrees 2, 4, 6, 6, 8, 9, 10, 14, then

$$P(t) = a(t)/(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)(1-t^9)(1-t^{10})(1-t^{14})$$

where

$$\begin{aligned}a(t) = 1 &+ 2t^6 + 4t^8 + 4t^9 + 7t^{10} + 8t^{11} + 15t^{12} + 15t^{13} + 20t^{14} + 27t^{15} + \\ &29t^{16} + 35t^{17} + 40t^{18} + 44t^{19} + 47t^{20} + 55t^{21} + 52t^{22} + 57t^{23} + 56t^{24} + \\ &57t^{25} + 52t^{26} + 55t^{27} + 47t^{28} + 44t^{29} + 40t^{30} + 35t^{31} + 29t^{32} + 27t^{33} + \\ &20t^{34} + 15t^{35} + 15t^{36} + 8t^{37} + 7t^{38} + 4t^{39} + 4t^{40} + 2t^{42} + t^{48}.\end{aligned}$$

This means that all basic invariants have degree at most 48, and we never have to consider subspaces of dimension larger than 85250, which is doable.

So, the procedure is to find basic invariants in some way, and multiply them together so as to construct for each m the invariants in I_m that are known already. Compute a basis for the subspace of I_m spanned by these known invariants, and if this subspace has the same dimension as I_m itself, it is all of I_m and we can go to the next m. Since any invariant can be written as a linear combination of bracket monomials, it seems reasonable to expect that one can find a spanning set for I_m by just randomly generating some bracket monomials. This worked fine for the nonic, and for the decimic in degrees below 21, but in degree 21 where $\dim I_{21} = 547$ and we quickly generated a subspace of dimension 546, a few dozen attempts to randomly generate an invariant outside this hyperplane failed. Therefore, we reverted to the procedure with guaranteed success: Gordan proved that a basis for the invariants can be found effectively by computing transvectants, and this indeed yielded the 106th invariant. (Immediately afterwards the random process also succeeded.)

Various reductions simplify the calculations. First of all, we did the computations modulo a small prime p, e.g. $p = 109$ worked. If the images of the invariants under reduction mod p are independent, then the invariants are independent. Secondly, if the form is $\sum_{i=0}^{10} \binom{10}{i} a_i x^{10-i} y^i$, we took $a_4 = a_7 = a_9 = 0$ and $a_{10} = 1$. Again: if the images of the invariants under this substitution are independent, then the invariants are independent. Similar things work for the nonic. But here we have the invariant $j_2 = a_0a_{10} - 10a_1a_9 + 45a_2a_8 - 120a_3a_7 + 210a_4a_6 - 126a_5^2$ of degree 2. After the substitutions this becomes $a_0 + 45a_2a_8 - 126a_5^2$, and the substitution $a_0 = -45a_2a_8 + 126a_5^2$ maps I_m onto I_m/j_2I_{m-2}, and $\dim I_m/j_2I_{m-2} = \dim I_m - \dim I_{m-2}$. Now six variables $(a_1, a_2, a_3, a_5, a_6, a_8)$ are left, and the largest dimension occurring is $\dim I_{48}/j_2I_{46} = 19860$, comparatively small. (Compared to $\dim I_{48} = 85250$, this saves almost a factor 80 in computation time when an $O(N^3)$ rank algorithm is used.)

The computation was done, and the result is: for $m \leq 48$ the values of d_m are as listed in Proposition 1.1. Consequently, if there is a system of parameters with degrees 2, 4, 6, 6, 8, 9, 10, 14, so that no basic invariant has degree larger than 48, then Proposition 1.1 follows.

3 A system of parameters for $\mathcal{O}(V_{10})^{\mathrm{SL}_2}$

Let V_n be the space of forms of degree n (in the variables x, y). A *covariant of order m and degree d* of V_n is an SL_2-equivariant homogeneous polynomial map $\phi : V_n \to V_m$ of degree d. The invariants of V_n are the covariants of order 0. The identity map is a covariant of order n and degree 1. Customarily, one indicates such a covariant ϕ by giving its image of a generic element $f \in V_n$. (In particular, the identity map is noted f.) Let $V_{m,d}$ be the space of covariants of order m and degree d.

Consider $f \in V_{10}$,

$$f = a_0x^{10} + 10a_1x^9y + \ldots + 10a_9xy^9 + a_{10}y^{10},$$

and the following covariants

$$\begin{aligned}
k &= (f,f)_8 \ \in V_{4,2}, & m &= (f,k)_4 \ \in V_{6,3},\\
q &= (f,f)_6 \ \in V_{8,2}, & r &= (f,q)_8 \ \in V_{2,3},\\
k_q &= (q,q)_6 \ \in V_{4,4}, & k_m &= (m,m)_4 \in V_{4,6},\\
m_q &= (q,k_q)_4 \in V_{4,6},
\end{aligned}$$

and invariants (the suffix indicates the degree)

$$\begin{aligned}
j_2 \ &= (f,f)_{10}, & A_6 \ &= (m,m)_6,\\
j_4 \ &= (k,k)_4, & C_6 \ &= (r,r)_2,\\
j_8 \ &= (k,k_m)_4, & j_{14} &= ((k_q,k_q)_2,m_q)_4,\\
j_9 \ &= ((m,k)_1,k^2)_8, & A_{14} &= ({(k,k)_2}^2,(m,m)_2)_8,\\
j_{10} &= ((m,m)_2,k^2)_8.
\end{aligned}$$

Theorem 3.1. *The eight invariants j_2, j_4, A_6, C_6, j_8, j_9, j_{10}, $j_{14} + A_{14}$ form a homogeneous system of parameters for the ring $\mathcal{O}(V_{10})^{\mathrm{SL}_2}$ of invariants of the binary decimic.*

This is proved by invoking Hilbert's characterization of homogeneous systems of parameters as sets that define the nullcone.

3.1 The nullcone

The *nullcone* of V_n, denoted $\mathcal{N}(V_n)$, is the set of binary forms of degree n on which all invariants vanish. It turns out that this is precisely the set of binary forms of degree n with a root of multiplicity $> \frac{n}{2}$. The elements of $\mathcal{N}(V_n)$ are called *nullforms*. The nullcone $\mathcal{N}(V_n \oplus V_m)$ is the set of pairs $(g,h) \in V_n \oplus V_m$ such that g and h have a common root of multiplicity $> \frac{n}{2}$ in g and of multiplicity $> \frac{m}{2}$ in h.

We have the following result, due to Hilbert [12], formulated for the particular case of binary forms:

Proposition 3.2. *For $n \geq 3$, consider $i_1, \ldots, i_{n-2} \in \mathcal{O}(V_n)^{\mathrm{SL}_2}$ homogeneous invariants of V_n. The following two conditions are equivalent:*

(i) $\mathcal{N}(V_n) = \mathcal{V}(i_1, \ldots, i_{n-2})$,

(ii) $\{i_1, \ldots, i_{n-2}\}$ *is a homogeneous system of parameters of* $\mathcal{O}(V_n)^{\mathrm{SL}_2}$.

We prove the above theorem by first finding a defining set for the nullcone that is still too large, and then showing that some elements are superfluous.

We need information on the invariants of V_n for $n = 2$, 4, 6, 8:

Lemma 3.3. *The following are systems of parameters of $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ for $n = 2, 4, 6, 8$.*

(i) If $n = 2$: $(f,f)_2$ of degree 2.

(ii) If $n = 4$: $(f,f)_4$ and $((f,f)_2,f)_4$ of degrees 2 and 3.

(iii) If $n = 6$: $(f,f)_6$, $(k,k)_4$, $((k,k)_2,k)_4$, and $(m^2,(k,k)_2)_4$ of degrees 2, 4, 6 and 10, where $k = (f,f)_4$ and $m = (f,k)_4$.

(iv) If $n = 8$: $(f,f)_8$, $((f,f)_4,f)_8$, $(k,k)_4$, $(m,k)_4$, $((k,k)_2,k)_4$, $((k,k)_2,m)_4$ of degrees 2, 3, 4, 5, 6 and 7, where $k = (f,f)_6$ and $m = (f,k)_4$.

Proof. This is classical for $n = 2, 4, 6$, see, e.g., [3, 11, 14], and due to von Gall [9] and Shioda [15] for $n = 8$. □

Lemma 3.4. (Weyman [19]) *Let $f \in V_d$. If $d > 4k-4$ and all $(f,f)_{2k}$, $(f,f)_{2k+2}$, ... vanish, then f has a root of multiplicity $d-k+1$. If $d = 4k-4$ and $((f,f)_{2k-2},f)_d$, $(f,f)_{2k}$, $(f,f)_{2k+2}$, ... vanish, then f has a root of multiplicity $d-k+1$.* □

Lemma 3.5. *Let $f \in V_{10}$ and $j_2 = (f,f)_{10}$, $k = (f,f)_8 \in V_4$, $m = (f,k)_4 \in V_6$, $q = (f,f)_6 \in V_8$. We have:*

(i) If $j_2 = 0$, $k \neq 0$ and $(k,m) \in \mathcal{N}_{V_4 \oplus V_6}$, then f has a root of multiplicity 6.

(ii) If $j_2 = 0$, $k = 0$ and $0 \neq q \in \mathcal{N}_{V_8}$, then f has a root of multiplicity 7.

(iii) If $j_2 = 0$, $k = 0$ and $q = 0$, then f has a root of multiplicity 8.

Proof. The covariants k, q and the invariant j_2 are:

$$
\begin{aligned}
j_2 &= -252a_5^2 + 420a_4a_6 - 240a_3a_7 + 90a_2a_8 - 20a_1a_9 + 2a_0a_{10},\\
k &= (70a_6^2 - 112a_5a_7 + 56a_4a_8 - 16a_3a_9 + 2a_2a_{10})y^4+\\
&\quad (56a_5a_6 - 112a_4a_7 + 80a_3a_8 - 28a_2a_9 + 4a_1a_{10})xy^3+\\
&\quad (168a_5^2 - 252a_4a_6 + 96a_3a_7 - 6a_2a_8 - 8a_1a_9 + 2a_0a_{10})x^2y^2+\\
&\quad (56a_4a_5 - 112a_3a_6 + 80a_2a_7 - 28a_1a_8 + 4a_0a_9)x^3y+\\
&\quad (70a_4^2 - 112a_3a_5 + 56a_2a_6 - 16a_1a_7 + 2a_0a_8)x^4,\\
q &= (-20a_7^2 + 30a_6a_8 - 12a_5a_9 + 2a_4a_{10})y^8+\\
&\quad (-40a_6a_7 + 72a_5a_8 - 40a_4a_9 + 8a_3a_{10})y^7x+\\
&\quad (-140a_6^2 + 168a_5a_7 - 40a_3a_9 + 12a_2a_{10})y^6x^2+\\
&\quad (-168a_5a_6 + 280a_4a_7 - 120a_3a_8 + 8a_1a_{10})y^5x^3+\\
&\quad (-252a_5^2 + 280a_4a_6 + 40a_3a_7 - 90a_2a_8 + 20a_1a_9 + 2a_0a_{10})y^4x^4+\\
&\quad (-168a_4a_5 + 280a_3a_6 - 120a_2a_7 + 8a_0a_9)y^3x^5+\\
&\quad (-140a_4^2 + 168a_3a_5 - 40a_1a_7 + 12a_0a_8)y^2x^6+\\
&\quad (-40a_3a_4 + 72a_2a_5 - 40a_1a_6 + 8a_0a_7)yx^7+\\
&\quad (-20a_3^2 + 30a_2a_4 - 12a_1a_5 + 2a_0a_6)x^8.
\end{aligned}
$$

(i). If $(k,m) \in \mathcal{N}_{V_4 \oplus V_6}$ then k and m have a common root, of multiplicity 3 in k and of multiplicity 4 in m. Without loss of generality we consider the cases $k = x^4$, $x^4 \mid m$ and $k = x^3y$, $x^4 \mid m$.

Case 1: $k = x^4$. Then m becomes:

$$m = (f, x^4)_4 = a_4x^6 + 6a_5x^5y + 15a_6x^4y^2 + 20a_7x^3y^3 + 15a_8x^2y^4 + 6a_9xy^5 + a_{10}y^6.$$

From $x^4 \mid m$ it follows $a_7 = \ldots = a_{10} = 0$. We replace this in k and because we supposed $k = x^4$ we obtain also $a_6 = a_5 = 0$. But then $x^6 \mid f$, hence f will have a root of multiplicity 6.

Case 2: $k = x^3y$. Then m becomes:

$$m = (f, x^3y)_4 = -a_3x^6 - 6a_4x^5y - 15a_5x^4y^2 - 20a_6x^3y^3 - 15a_7x^2y^4 - 6a_8xy^5 - a_9y^6.$$

From $x^4 \mid m$ it follows $a_6 = \ldots = a_9 = 0$. We replace this in k and j_2 and as we supposed $k = x^3y$ we obtain

$$168a_5^2 + 2a_0a_{10} = 0,$$
$$-252a_5^2 + 2a_0a_{10} = 0,$$

which implies $a_5 = 0$. But then the coefficient of x^3 in k becomes 0. Contradiction with our assumption.

(ii). Without loss of generality we suppose $x^5 \mid q$. We denote by J the ideal generated by j_2, the coefficients of k and the coefficients of $x^4y^4, x^3y^5, \ldots, y^8$ in q. Denote also by p_1, p_2 and p_3 the coefficients of x^7y, x^6y^2 and x^5y^3, respectively, in q. We have

$$p_1^4, p_2^3, p_3^2 \in J,$$

which implies that $x^8 \mid q$.

Consider now the ideal J generated by j_2, the coefficients of k and the coefficients of $x^7y, x^6y^2, \ldots, y^8$ in q. Denote by p_0 the coefficient of x^8 in q. We have $a_ip_0 \in J$ for $i = 10, 9, 8, 7, 6, 5, 4$. Because $q \neq 0$ we find $a_{10} = \ldots = a_4 = 0$. This means that $x^7 \mid f$, so f will have a root of multiplicity 7.

(iii). This follows from Lemma 3.4. □

Lemma 3.6. *Let $k \in V_4$ and $m \in V_6$, $k \neq 0$, $m \neq 0$, both of them nullforms. If the transvectants $((m,m)_4, k)_4$, $((m,m)_2, k^2)_8$, $(m^2, k^3)_{12}$, $((m,k)_1, k^2)_8$, and $((k,k)_2^2, (m,m)_2)_8$ vanish, then $(k,m) \in \mathcal{N}_{V_4 \oplus V_6}$.*

Proof. Suppose $(k,m) \notin \mathcal{N}_{V_4 \oplus V_6}$. Without loss of generality we suppose

$$k = x^3(a_1x + a_2y),$$
$$m = y^4(b_1x^2 + b_2xy + b_3y^2).$$

We have

$$0 = ((m,m)_4, k)_4 \sim a_1b_1^2$$

Case 1: $a_1 = 0$. Then

$$\begin{aligned}
0 &= ((m,m)_2, k^2)_8 &&\sim a_2^2 b_1^2, \\
0 &= ((m,k)_1, k^2)_8 &&\sim a_2^3 b_3, \\
0 &= ((k,k)_2^2, (m,m)_2)_8 &&\sim a_2^4(5b_2^2 - 12b_1 b_3)
\end{aligned}$$

Because $k \neq 0$ we have $a_2 \neq 0$, but then it follows that $b_1 = b_3 = b_2 = 0$. Contradiction with $m \neq 0$.

Case 2: $a_1 \neq 0$, $b_1 = 0$. Then

$$\begin{aligned}
0 &= ((m,m)_2, k^2)_8 &&\sim a_1^2 b_2^2, \\
0 &= ((m,k)_1, k^2)_8 &&\sim a_2^3 b_3, \\
0 &= ((k,k)_2^2, (m,m)_2)_8 &&\sim a_2^4 b_2^2, \\
0 &= (m^2, k^3)_{12} &&\sim a_1(a_2^2 b_2^2 - 11a_1 a_2 b_2 b_3 + 22a_1^2 b_3^2)
\end{aligned}$$

If $a_2 \neq 0$ then $b_2 = b_3 = 0$. And if $a_2 = 0$ then $a_1^2 b_2^2 = a_1^3 b_3^2 = 0$, and again $b_2 = b_3 = 0$. Contradiction with $m \neq 0$. $\square$

After this preparation we can write down a defining set for the nullcone. Define k, m, q, j_2, j_4, A_6, j_8, j_9, j_{10}, j_{14}, A_{14} as above (before Theorem 3.1), and moreover

$$\begin{aligned}
j_6 &= ((k,k)_2, k)_4, \qquad & A_{12} &= (m^2, k^3)_{12}, \\
B_6 &= ((q,q)_4, q)_8.
\end{aligned}$$

Proposition 3.7. *With notations as above, the nullcone $\mathcal{N}_{V_{10}}$ is defined by*

$$\mathcal{N}_{V_{10}} = \mathcal{V}(j_2, j_4, j_6, A_6, B_6, j_8, j_9, j_{10}, A_{12}, j_{14}, A_{14}).$$

Proof. Since $k \in V_4$ we can apply Lemma 3.3(ii) and conclude that if $j_4 = j_6 = 0$ then k is a nullform. Without loss of generality we consider three cases: $k = 0$, $k = x^4$ and $k = x^3 y$.

Case 1: $k = 0$. Denote by $I = (j_2, k)$ the ideal generated by j_2 and the coefficients of k. Define

$$\begin{aligned}
A_4 &= (q,q)_8, \qquad & A_{10} &= (m_q, k_q)_4, \\
A_8 &= (k_q, k_q)_4, & B_{12} &= ((k_q, k_q)_2, k_q)_4,
\end{aligned}$$

Since $q \in V_8$, in order to show that q is a nullform it suffices by Lemma 3.3(iv) to show that each of A_4, B_6, A_8, A_{10}, B_{12} and j_{14} vanishes.

Easy Gröbner basis computations show that $A_4, A_8, A_{10} \in I$ and $B_{12} \in (I, B_6)$. It follows that if $k = 0$ and $j_2 = B_6 = j_{14} = 0$ then q is a nullform. Now Lemma 3.5 implies that f is a nullform.

Case 2: $k = x^4$. Then we have:

$$\begin{aligned}
A_{12} &\sim a_{10}^2, \\
j_{10} &\sim -a_9^2 + a_8 a_{10}, \\
j_8 &\sim 3a_8^2 - 4a_7 a_9 + a_6 a_{10}, \\
A_6 &\sim -10a_7^2 + 15a_6 a_8 - 6a_5 a_9 + a_4 a_{10}.
\end{aligned}$$

If $A_{12} = j_{10} = j_8 = A_6 = 0$ then it follows that $a_{10} = \ldots = a_7 = 0$. If we substitute this in k we obtain

$$\begin{aligned}
k = &70a_6^2 y^4 + 56a_5 a_6 x y^3 + (168a_5^2 - 252a_4 a_6)x^2 y^2 + \\
&(56a_4 a_5 - 112a_3 a_6)x^3 y + (70a_4^2 - 112a_3 a_5 + 56a_2 a_6)x^4,
\end{aligned}$$

and as we supposed $k = x^4$ we get also $a_6 = a_5 = 0$, which implies that f is a nullform.

Case 3: $k = x^3 y$. Then we have:

$$\begin{aligned}
j_9 &\sim a_9, \\
A_{14} &\sim a_7 a_9 - a_8^2, \\
j_{10} &\sim -5a_7^2 + 2a_6 a_8 + 3a_5 a_9, \\
A_6 &\sim -10a_6^2 + 15a_5 a_7 - 6a_4 a_8 + a_3 a_9.
\end{aligned}$$

If $j_9 = A_{14} = j_{10} = A_6 = 0$ then $a_9 = \ldots = a_6 = 0$. We substitute this in k and j_2:

$$\begin{aligned}
k = &2a_2 a_{10} y^4 + 4a_1 a_{10} x y^3 + (168a_5^2 + 2a_0 a_{10})x^2 y^2 + \\
&56a_4 a_5 x^3 y + (70a_4^2 - 112a_3 a_5)x^4, \\
j_2 = &-252a_5^2 + 2a_0 a_{10}
\end{aligned}$$

From $168a_5^2 + 2a_0 a_{10} = -252a_5^2 + 2a_0 a_{10} = 0$ we find $a_5 = 0$, which contradicts $k = x^3 y$. $\square$

So far, we defined the nullcone using 11 invariants, but we need a definition using 8 invariants. As a first step, replace the two invariants of degree 14 by a single one.

Now for $f = x^2 y(2a_1 x^7 + 9a_8 y^7)$ all invariants from Proposition 3.7 vanish, except A_{14}. And for $f = y^3(120a_3 x^7 + a_{10} y^7)$ all invariants from Proposition 3.7 vanish, except j_{14}. That means that the single invariant of degree 14 cannot be either j_{14} or A_{14}. However, as it turns out we can use $j_{14} + A_{14}$.

3.2 Finding the system of parameters

Proposition 3.7 gives an explicit set of invariants (and in particular an explicit set of degrees of invariants) that define the nullcone. Having that, only a finite amount of work is left.

The final part of the construction of the system of parameters was done by computer. All computations were carried out in the ring R generated by the 106 invariants found in Section 2. Or, more precisely, in the quotient $Q = R/j_2R$, reduced mod p, where this time $p = 197$ (the different p has no significance), and again a_4, a_7 and a_9 were taken to be zero. It was checked that the graded parts of the resulting ring have the expected dimension (for degree up to 54), so that no collapse occurred as a consequence of the reduction mod p or the substitution of variables.

The ideal generated in this ring by all invariants of degrees 4, 6, 8, 9, 10, 14 has full dimension 542 for its graded part of degree 24. We know that $\dim I_{24} = 1429$ and $\dim I_{22} = 887$ and multiplication by j_2 is an injection, so $\dim I_{24}/j_2I_{22} = 542$. It follows that the ideal generated by these invariants, together with j_2, contains all of I_{24}, so that no invariants of degree 12 are needed to define the nullcone (since their squares are in I_{24}, and they themselves are in the radical).

With only $j_{14} + A_{14}$ instead of all invariants of degree 14 in the set of generators of the ideal, one finds full dimension 1148 for the graded part of degree 28, so this single invariant of degree 14 suffices.

With only j_{10} instead of all invariants of degree 10, one finds full dimension 221 in degree 20, so this single invariant of degree 10 suffices.

With only j_9 instead of all invariants of degree 9, one finds full dimension 890 in degree 27, so this single invariant of degree 9 suffices.

With only j_8 instead of all invariants of degree 8, one finds full dimension 2279 in degree 32, so this single invariant of degree 8 suffices.

That only leaves the invariants of degree 6. After some work it turned out that with only A_6 and C_6 one finds full dimension 37892 in degree 54, so these suffice, and we have constructed the homogeneous system of parameters promised in Theorem 3.1.

Note that one knows what to expect if all is well: the coefficients of the polynomial $a(t)$ from Section 2 give for each degree the codimension of the set of invariants in the ideal generated by the hsop in the space of all invariants of that degree. Since 54 is the smallest multiple of 6 where $a(t)$ has zero coefficient, that explains why the computation had to extend to there.

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