Universally noncommutative loops

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Abstract

We call a loop *universally noncommutative* if it does not have a loop isotope in which two non-identity elements commute. Finite universally noncommutative loops are equivalent to latin squares that avoid the configuration:

$$\begin{pmatrix} \cdot & \alpha & \beta \\ \alpha & \cdot & \gamma \\ \beta & \gamma & \cdot \end{pmatrix} \tag{1}$$

By computer enumeration we find that there are only two species of universally noncommutative loops of order $\leqslant 11$. Both have order 8.

We assume familiarity with the basic terminology of latin squares and loops, such as can be found in [2].

The aim of this note is to answer a question asked by Nick Cavenagh at the British Combinatorial Conference in 2007 and by Aart Blokhuis at Oisterwijk in 2008. Both researchers asked whether there are any latin squares that avoid the configuration (1). This means that no 3×3 submatrix should be equivalent to (1) up to permutation of the rows and columns. Here α, β, γ stand for any three distinct symbols and $\cdot$ denotes a cell whose contents are arbitrary. The property of avoiding (1) is invariant across a species (also known as main class).

There are several well studied problems that hinge on avoiding configurations in latin squares. For example, the problem of avoiding *intercalates* (2×2 latin subsquares) was solved in the 1970s (see [4] for details). There has also been work on avoiding short cycles in latin squares (see, for example, [1, 5, 6]). Problems of avoiding configurations are also well known in the study of steiner triple systems (STS). A notable example is the solution [3] of Erdős' anti-pasch conjecture, which took considerable effort and indicates that such problems can be hard. The problem we consider here is analogous to avoiding the "window" or "grid" configuration in a STS.

Any latin square $L = [L_{ij}]$ can be interpreted as the Cayley table for a loop by choosing one row r and one column c of the square and defining a binary operation $\star$ by $L_{ic} \star L_{rj} = L_{ij}$. Changing our choice of r and/or c produces *loop isotopes* of the loop we first defined. It is easy to see that L contains a copy of (1) if and only if some loop isotope

*Research supported by ARC grants DP0662946 and DP1093320

contains two non-identity elements that commute. When r and c are chosen to be the first row and column of (1) we find that $\alpha \star \beta = \gamma = \beta \star \alpha$. Note that in this situation α and β are not the identity element, which would appear in the cell marked $\cdot$ in the top left hand corner of (1).

Independent computer enumerations by the two authors show that there are only two species of universally noncommutative loops of order $\leqslant 11$. Both have order 8. Equivalently, up to order 11 there are only two species of latin squares avoiding (1). Representatives of these two species are

$$A = \begin{pmatrix} 1&2&3&4&5&6&7&8\\ 2&1&4&3&6&5&8&7\\ 3&6&1&8&2&7&4&5\\ 6&3&8&1&7&2&5&4\\ 7&4&5&2&8&3&6&1\\ 4&7&2&5&3&8&1&6\\ 5&8&7&6&1&4&3&2\\ 8&5&6&7&4&1&2&3 \end{pmatrix} \qquad B = \begin{pmatrix} 1&2&3&4&5&6&7&8\\ 2&1&4&3&6&5&8&7\\ 3&6&8&1&2&7&5&4\\ 6&3&1&8&7&2&4&5\\ 7&4&2&5&3&8&6&1\\ 4&7&5&2&8&3&1&6\\ 5&8&7&6&4&1&2&3\\ 8&5&6&7&1&4&3&2 \end{pmatrix}$$

It is striking that these two squares possess similar structure. Both are composed of 16 disjoint intercalates (although A has 32 additional intercalates). To convert between A and B it suffices to replace each of the six shaded intercalates by the other possible intercalate on the same symbols.

Both A and B have autotopism groups acting transitively on their 64 (row, column, symbol) triples. The autotopism groups are of respective orders 64 and 192. Both A and B have paratopism groups of order 384. So A is isotopic to all 6 of its conjugates, whereas B is isotopic to its transpose via the symbol permutation $(46)(57)$.

One last feature is worth noting: the loops corresponding to A and B are so called G-loops (group-like loops), meaning that they are isomorphic to all of their loop isotopes. With that knowledge, it is easy to check that they are universally noncommutative.

We leave open the existence of universally noncommutative loops of order $\geqslant 12$.

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