The association scheme on the points off a quadric

F. Vanhove
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Abstract
The parameters of the association scheme on the points off a quadric are computed. This corrects a mistake in the literature.

In [BCN, Theorem 12.1.1], the existence of a certain association scheme is claimed, and details are given for \( n = 3 \). Here we correct the statements given there for odd \( n \geq 5 \).

Let \( q \) be a power of 2, and \( n \geq 3 \). Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{F}_q \) provided with a nondegenerate quadratic form \( Q \). Let \( B \) be the associated symmetric bilinear form, given by \( B(x, y) = Q(x + y) - Q(x) - Q(y) \). If \( n \) is odd, there will be a nucleus \( N = V^\perp \).

We construct an association scheme with point set \( X \), where \( X \) is the set of projective points not on the quadric defined by \( Q \) and (for odd \( n \)) distinct from \( N \). For \( n = 3 \) and for even \( n \), the relations will be \( R_0, R_1, R_2, R_3 \) where

\[
R_0 = \{(x, x) \mid x \in X\}, \text{ the identity relation;}
R_1 = \{(x, y) \mid x + y \text{ is a hyperbolic line (secant)}\}; \\
R_2 = \{(x, y) \mid x + y \text{ is an elliptic line (exterior line)}\}; \\
R_3 = \{(x, y) \mid x + y \text{ is a tangent}\}.
\]

For odd \( n, n \geq 5 \), it is necessary to distinguish \( R_{3a} \) and \( R_{3n} \), defined by

\[
R_{3a} = \{(x, y) \mid x + y \text{ is a tangent not on } N\}; \\
R_{3n} = \{(x, y) \mid x + y \text{ is a tangent on } N\}.
\]

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Note that every line on $N$ is a tangent, and that for $n = 3$ there are no other tangents, so that $R_{3n}$ is empty. For $q = 2$ a hyperbolic line contains only one nonisotropic point, and a tangent on $N$ contains only one nonisotropic point distinct from $N$, so that $R_1$ and $R_{3n}$ are empty.

We show that if $n = 3$ or $n$ is even, then $(X, \{R_0, R_1, R_2, R_3\})$ is an association scheme. Also that if $n$ is odd, $n \geq 5$, then $(X, \{R_0, R_1, R_2, R_{3a}, R_{3n}\})$ is an association scheme. We give the parameters $p^i_{jk}$ and the eigenmatrix $P$ in both cases.

1 Quadric size

The number $M$ of isotropic projective points on a nonisotropic quadric in $V$, where $V$ has vector space dimension $n$ equals

$$M = \begin{cases} \frac{q^{2m} - 1}{q - 1} & \text{if } n = 2m + 1 \\ \frac{q^m - \epsilon}{(q^m - 1 + \epsilon)} & \text{if } n = 2m. \end{cases}$$

Equivalently,

$$M = \frac{q^{n-1} - 1}{q - 1} + \epsilon q^{n/2 - 1}$$

with $\epsilon = \pm 1$ if $n$ is even, and $\epsilon = 0$ if $n$ is odd.

2 $n = 3$

Suppose first that $n = 3$. The parameters $(p^i_{jk})$ were given in [BCN], p. 375. Let us call them $(a^i_{jk})$ here in the special case $n = 3$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (a^i_{1j}) = \begin{pmatrix} 0 & \frac{1}{4}q(q - 2) & 0 & 0 \\ 1 & \frac{1}{4}(q - 2)^2 & \frac{1}{4}q(q - 2) & \frac{1}{4}q - 2 \\ 0 & \frac{1}{4}(q - 2)^2 & \frac{1}{4}q(q - 2) & \frac{1}{4}q - 1 \\ 0 & \frac{1}{4}q(q - 4) & \frac{1}{4}q^2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & \frac{1}{4}q^2 & 0 \\ 0 & \frac{1}{4}q(q - 2) & \frac{1}{4}q^2 & \frac{1}{4}q \\ 1 & \frac{1}{4}q(q - 2) & \frac{1}{4}q^2 & \frac{1}{4}q - 1 \\ 0 & \frac{1}{4}q^2 & \frac{1}{4}q^2 & 0 \end{pmatrix}, \quad (a^i_{2j}) = \begin{pmatrix} 0 & 0 & 0 & q - 2 \\ 0 & \frac{1}{4}q - 2 & \frac{1}{4}q & 0 \\ 0 & \frac{1}{4}q - 1 & \frac{1}{4}q - 1 & 0 \\ 1 & 0 & 0 & q - 3 \end{pmatrix}.$$
We see that $R_3$ is an equivalence relation (and the equivalence classes are the tangent lines, that is, the lines on $N$). We also see that $R_2$ has only three distinct eigenvalues, and hence defines a strongly regular graph.

Now suppose that $\dim V = 3$ but the quadratic form $Q$ on $V$ is degenerate in such a way that $N := V^\perp$ is a (single) isotropic point. Then the space is a cone over a hyperbolic or elliptic line. We have $v = |X| = q^2 - \epsilon q$ and the valencies are $k_0 = 1$, $k_3 = q - 1$ and $k_1 = q^2 - 2q$, $k_2 = 0$ if $\epsilon = 1$, $k_1 = 0$, $k_2 = q^2$ if $\epsilon = -1$. Call the corresponding parameters $(h^i_{jk})$ and $(e^i_{jk})$, respectively. Then

\[
(h^i_{ij}) = \begin{pmatrix} 0 & q^2 - 2q & 0 & 0 \\ 1 & q^2 - 3q & 0 & q - 1 \\ * & * & * & * \\ 0 & q^2 - 2q & 0 & 0 \end{pmatrix}, \quad (h^i_{j3}) = \begin{pmatrix} 0 & 0 & 0 & q - 1 \\ 0 & q - 1 & 0 & 0 \\ * & * & * & * \\ 1 & 0 & 0 & q - 2 \end{pmatrix},
\]

\[
(e^i_{ij}) = \begin{pmatrix} 0 & 0 & q^2 & 0 \\ * & * & * & * \\ 1 & 0 & q^2 - q & q - 1 \\ 0 & 0 & q^2 & 0 \end{pmatrix}, \quad (e^i_{j3}) = \begin{pmatrix} 0 & 0 & 0 & q - 1 \\ 0 & 0 & q - 1 & 0 \\ * & * & * & * \\ 1 & 0 & 0 & q - 2 \end{pmatrix}.
\]

(with undefined * since relation $R_2$ (resp. $R_1$) does not occur).

Finally, suppose that $\dim V = 3$ and the quadratic form $Q$ on $V$ is a double line (that is, $B$ vanishes identically, $Q$ is the square of a linear form). Now $k_0 = 1$, $k_1 = k_2 = 0$, $k_3 = q^2 - 1$. Call the corresponding parameters $(z^i_{jk})$. Then

\[
(z^i_{j3}) = \begin{pmatrix} 0 & 0 & q^2 - 1 \\ * & * & * \\ * & * & * \\ 1 & 0 & 0 \end{pmatrix}.
\]

### 3 n even

Now let $n$ be even, say $n = 2m$, where $m \geq 2$. Let the form have type $\epsilon$, with $\epsilon = 1$ for a hyperbolic and $\epsilon = -1$ for an elliptic quadric.

The number of points of the scheme equals $v = |X| = q^{2m-1} - \epsilon q^{m-1}$.

For the valencies $k_i$ of the relations $R_i$ we find

\[
k_0 = 1
\]

\[
k_1 = (q - 2)q^{m-1}(q^{m-1} + \epsilon)/2
\]

\[
k_2 = q^m(q^{m-1} - \epsilon)/2
\]

\[
k_3 = q^{2m-2} - 1
\]

If $n = 2$, $m = 1$, then only one type of line occurs (since all of $V$ is just a line), and $P = \begin{pmatrix} 1 & q - 2 \\ 1 & -1 \end{pmatrix}$ if $\epsilon = 1$, and $P = \begin{pmatrix} 1 & q \\ 1 & -1 \end{pmatrix}$ if $\epsilon = -1$.

Let $n \geq 4, m \geq 2$. If $(x, y) \in R_h$ for a certain $h \in \{1, 2, 3\}$ then for each plane on the line $x + y$ we find the same relation, and a contribution as just computed for
the case \( n = 3 \). In the plane we did not count the nucleus, but here that nucleus contributes 1 to \( p_{33}^h \) for \( h \neq 3 \). If \( h = 3 \) then \( x \) or \( y \) might itself be the nucleus of a nondegenerate plane on \( x + y \). The details follow.

Let \( L \) be a hyperbolic line, and consider the \( (q^{n-2} - 1)/(q - 1) \) planes on \( L \). A degenerate plane must be the span \( L + z \) of \( L \) and an isotropic point \( z \) in \( L^\perp \). Now \( L^\perp \) has the same type \( \varepsilon \) as \( V \) and dimension \( n - 2 \), so has \( a := (q^{2m-3} - 1)/(q - 1) + \varepsilon q^{m-2} \) isotropic points. Hence \( L \) is on a degenerate planes \( L + z \), and on \( (q^{n-2} - 1)/(q - 1) - a = q^{n-3} - \varepsilon q^{m-2} \) nondegenerate planes. All parameters \( p_{jk}^1 \) follow by summing such parameters of these two types of planes: If \( (x, y) \in R_1 \), then \( L = x + y \) is a hyperbolic line that contributes \( q - 3 \) to \( p_{11}^1 \), and nothing to \( p_{jk}^1 \) for \( \{j, k\} \not\subseteq \{0, 1\} \). A degenerate plane on \( L \) is a cone over a hyperbolic line, and contributes \( h_{1j}^1 \). Thus

\[
p_{11}^1 = q - 3 + (q^{n-3} - \varepsilon q^{m-2})(a_{11}^1 - q + 3) + a(h_{11}^1 - q + 3)
\]

and

\[
p_{33}^1 = (q^{n-3} - \varepsilon q^{m-2})(a_{33}^1 + 1) + ah_{33}^1
\]

and

\[
p_{jk}^1 = (q^{n-3} - \varepsilon q^{m-2})a_{jk}^1 + ah_{jk}^1
\]

for nonzero \( j, k \) not both 1 or both 3.

Let \( L \) be an elliptic line, and consider planes on \( L \). This time \( L^\perp \) has the opposite type, so has \( b := (q^{2m-3} - 1)/(q - 1) - \varepsilon q^{m-2} \) isotropic points, and \( L \) is on \( (q^{n-2} - 1)/(q - 1) - b = q^{n-3} + \varepsilon q^{m-2} \) nondegenerate planes. We find

\[
p_{22}^2 = q - 1 + (q^{n-3} + \varepsilon q^{m-2})(a_{22}^2 - q + 1) + b(e_{22}^2 - q + 1)
\]

and

\[
p_{33}^2 = (q^{n-3} + \varepsilon q^{m-2})(a_{33}^2 + 1) + be_{33}^2
\]

and

\[
p_{jk}^2 = (q^{n-3} + \varepsilon q^{m-2})a_{jk}^2 + be_{jk}^2
\]

for nonzero \( j, k \) not both 2 or both 3.

Let \( L \) be a tangent, with isotropic point \( z \). Then \( L^\perp \) is an \((n-2)\)-space containing \( L \). The line \( L \) is on \( q^{n-3} \) nondegenerate planes (where \( Q \) is a conic, \( L \) a tangent to the conic, and the nucleus of the plane is a nonisotropic point of \( L \)), namely those not contained in \( z^\perp \). The line \( L \) is on \( (q^{n-4} - 1)/(q - 1) \) planes contained in \( L^\perp \) (on which the symplectic form vanishes identically, and the quadratic form is a double line). The line \( L \) is on \( q^{n-4} \) degenerate planes with radical \( z \) (contained in \( z^\perp \) but not in \( L^\perp \)). The space \( z^\perp/z \) is a nondegenerate \((n-2)\)-space of the same type \( \varepsilon \) in which \( L \) is a nonisotropic point. The quadric in that space has size \( (q^{n-3} - 1)/(q - 1) + \varepsilon q^{m-2} \), and through the point \( L \) there are \( (q^{n-4} - 1)/(q - 1) \) tangents, and \( (q^{n-4} + \varepsilon q^{m-2})/2 \) hyperbolic lines, and \( (q^{n-4} - \varepsilon q^{m-2})/2 \) elliptic lines. Consequently, of the \( q^{n-4} \) degenerate planes \( \pi \) on \( L \) with radical \( z \), for \( (q^{n-4} + \varepsilon q^{m-2})/2 \) the quotient \( \pi/z \) is hyperbolic, and for \( (q^{n-4} - \varepsilon q^{m-2})/2 \) elliptic. Each of the \( q \) nonisotropic points of \( L \) is nucleus of \( q^{n-4} \) nondegenerate planes. For the computation of \( p_{3k}^3 \) starting with two points \( x, y \) where \( L = x + y \)
is a tangent, the $q^{n-4}$ nondegenerate planes in which $x$ is nucleus each contribute
\( \frac{1}{2} q(q-2) \) for $k = 1$ and $\frac{1}{2} q^2$ for $k = 2$. There are $q^{n-4}(q-2)$ such planes where
none of $x, y$ is nucleus. Altogether, we find
\[
p_{jk}^3 = q^{n-4}(q-2) a_{jk}^3 + \frac{1}{2} (q^{n-4} + \epsilon q^{m-2}) h_{jk}^3 + \frac{1}{2} (q^{n-4} - \epsilon q^{m-2}) e_{jk}^3
\]
for $j, k \neq 0, 3$, and
\[
p_{31}^3 = \frac{1}{2} q^{n-3}(q-2),
\]
\[
p_{32}^3 = \frac{1}{2} q^{n-2},
\]
\[
p_{33}^3 = q - 2 + \frac{q^{n-4} - 1}{q-1} (z_{33}^3 - q + 2).
\]

Since we could compute all $p_{jk}^i$, this proves that we have an association scheme.
Let us substitute the values of $a_{jk}^i, h_{jk}^i, e_{jk}^i$ and $z_{jk}^i$ and compute the eigenmatrix $P$
of the scheme. In order to save space, we abbreviate $r := q - 2$.

For $(p_{ij}^1)_{ij}$ one finds
\[
\begin{pmatrix}
0 & \frac{1}{2} q^{m-1}(q^{m-1} + \epsilon) r \\
1 & \frac{1}{2} q^{m-2}(q^{m-1} + \epsilon) r \\
0 & \frac{1}{2} q^{m-1}(q^{m-1} + \epsilon) r \\
0 & \frac{1}{2} q^{m-2} + \epsilon q^{m-3} r
\end{pmatrix}
\]
with eigenvalues $\frac{1}{2} q^{m-1}(q^{m-1} + \epsilon)(q - 2), \frac{1}{2} q^{m-2}(q + 1)(q - 2), -\epsilon q^{m-1}, 0$.

For $(p_{ij}^2)_{ij}$ one finds
\[
\begin{pmatrix}
0 & \frac{1}{2} q^{m-1}(q^{m-1} - \epsilon) \\
0 & \frac{1}{2} q^{m-1}(q^{m-1} - \epsilon) \\
1 & \frac{1}{2} q^{m-1}(q^{m-1} + \epsilon) r \\
0 & \frac{1}{2} q^{m-2} + \epsilon q^{m-3} r
\end{pmatrix}
\]
with eigenvalues $\frac{1}{2} q^{m}(q^{m-1} - \epsilon), \epsilon q^{m-1}, -\frac{1}{2} \epsilon q^{m-1}(q - 1), 0$.

For $(p_{ij}^3)_{ij}$ one finds
\[
\begin{pmatrix}
0 & \frac{1}{2} (q^{m-1} - \epsilon)(q^{m-2} + 2 \epsilon) \\
0 & \frac{1}{2} q^{m-1}(q^{m-1} - \epsilon) \\
1 & \frac{1}{2} q^{m-2}(q^{m-1} + \epsilon) r \\
0 & \frac{1}{2} q^{m-2} + \epsilon q^{m-3} r
\end{pmatrix}
\]
with eigenvalues $q^{m-2} - 1, q^{m-1} - 1, -q^{m-1} - 1, \epsilon q^{m-2} - 1$.

The $P$-matrix is
\[
P = \begin{pmatrix}
1 & \frac{1}{2} q^{m-1}(q^{m-1} + \epsilon)(q - 2) & \frac{1}{2} q^{m}(q^{m-1} - \epsilon) & q^{m-2} - 1 \\
1 & 2 \epsilon q^{m-2}(q + 1)(q - 2) & -\frac{1}{2} \epsilon q^{m-1}(q - 1) & \epsilon q^{m-2} - 1 \\
1 & 0 & -\epsilon q^{m-1} & \epsilon q^{m-1} - 1 \\
1 & -\epsilon q^{m-1} & 0 & \epsilon q^{m-1} - 1
\end{pmatrix}.
\]

The multiplicities (in the order of the rows of $P$) are $1$, $q^{2}(q^{n-2} - 1)/(q^2 - 1)$,
$\frac{1}{2} q(q^{m-1} - \epsilon)(q^{m} - \epsilon)/(q + 1)$, $\frac{1}{2} (q - 2)(q^{m-1} + \epsilon)(q^{m} - \epsilon)/(q - 1)$.
4 $n$ odd

Now let $n$ be odd, say $n = 2m + 1$, where $m \geq 2$. Let $Q$ be a nondegenerate quadric, and let $N$ be its nucleus. We compute the $p^j_{jk}$ as before, this time splitting relation $R_3$ (being joined by a tangent) into the two relations $R_{3a}$ and $R_{3n}$, depending on whether the tangent does not or does pass through $N$.

The number of points of the scheme equals $v = |X| = q^{n-1} - 1$.

For the valencies $k_i$ of the relations $R_i$ we find

$$k_0 = 1$$
$$k_1 = \frac{1}{2} q^{n-2} (q - 2)$$
$$k_2 = \frac{1}{2} q^{n-1}$$
$$k_{3a} = q^{n-2} - q$$
$$k_{3n} = q - 2$$

The number of planes on a line $L$ is $(q^{n-2} - 1)/(q - 1)$. If $L$ is hyperbolic or elliptic, then a degenerate plane must be the span $L + z$ of $L$ and an isotropic point $z$ in $L^\perp$. Now $L^\perp$ is a nondegenerate $(n - 2)$-space, and has $(q^{n-3} - 1)/(q - 1)$ isotropic points, so there are $q^{n-3}$ nondegenerate planes, and $(q^{n-3} - 1)/(q - 1)$ degenerate planes on $L$. We find for $i = 1, 2$ that

$$p^i_{jk} = q^{n-3} (a^i_{jk} - c) + \frac{q^{n-3} - 1}{q - 1} (x^i_{jk} - c) + c$$

with $x = h$ for $i = 1$ and $x = e$ for $i = 2$, and $c = q - 3$ if $i = j = k = 1$, $c = q - 1$ if $i = j = k = 2$ and $c = 0$ otherwise.

If $L$ is a tangent on $N$, with isotropic point $z$, then the $q^{n-3}$ nondegenerate planes on $L$ are the planes not in $z^\perp$. The remaining $(q^{n-3} - 1)/(q - 1)$ planes on $L$ are contained in $L^\perp$, and the form induces a double line on these. Hence

$$p^i_{jk} = q^{n-3} a^i_{jk}$$

for $i = 3n$ when not $\{j, k\} \subseteq \{0, 3a, 3n\}$.

If $L$ is a tangent not on $N$, with isotropic point $z$, then the $q^{n-3}$ nondegenerate planes on $L$ are the planes not in $z^\perp$. Each nonisotropic point of $L$ is the nucleus of $q^{n-4}$ of these planes. There are $(q^{n-4} - 1)/(q - 1)$ planes on $L$ contained in $L^\perp$, where the form induces a double line. The remaining planes are degenerate, cones over a hyperbolic or elliptic line, $\frac{1}{2} q^{n-4}$ of each.

Relation $R_{3n}$ is an equivalence relation with equivalence classes of size $q - 1$. If $L$ does not pass through $N$, then it is on a unique plane $L + N$ on $N$, and the points that have relation $R_{4n}$ with $x$ or $y$ live in that plane. We find $p^1_{1,3n} = \frac{1}{2} q - 2$, $p^2_{2,3n} = \frac{1}{2} q$, $p^2_{1,3n} = p^2_{2,3n} = \frac{1}{2} q - 1$.

For $(p^i_{ij})$ one finds

$$
\begin{pmatrix}
0 & \frac{1}{2} q^{n-2} (q - 2) & 0 & 0 & 0 \\
1 & \frac{1}{2} q^{n-3} (q - 2)^2 & \frac{1}{2} q^{n-2} (q - 2) & \frac{1}{2} (q^{n-3} - 1) (q - 2) & \frac{1}{2} q - 2 \\
0 & \frac{1}{2} q^{n-3} (q - 2)^2 & \frac{1}{2} q^{n-2} (q - 2) & \frac{1}{2} (q^{n-3} - 1) (q - 2) & \frac{1}{2} q - 1 \\
0 & \frac{1}{2} q^{n-3} (q - 2)^2 & \frac{1}{2} q^{n-2} (q - 2) & \frac{1}{2} q^{n-3} (q - 2) & 0 \\
0 & \frac{1}{2} q^{n-2} (q - 4) & \frac{1}{2} q^{n-1} & 0 & 0
\end{pmatrix}
$$
with eigenvalues $\frac{1}{2}q^{2m-1}(q - 2), \pm \frac{1}{2}q^{m-1}(q - 2), \pm \frac{1}{2}q^m$.

For $(p^i_{2j})$ one finds

$$
\begin{pmatrix}
0 & 0 & \frac{1}{2}q^{n-1} & 0 & 0 \\
0 & \frac{1}{2}q^{n-2}(q - 2) & \frac{1}{2}q^{n-1} & \frac{1}{2}q(q^{n-3} - 1) & \frac{1}{2}q \\
1 & \frac{1}{2}q^{n-2}(q - 2) & \frac{1}{2}q^{n-1} & \frac{1}{2}q(q^{n-3} - 1) & \frac{1}{2}q - 1 \\
0 & \frac{1}{2}q^{n-2}(q - 2) & \frac{1}{2}q^{n-1} & \frac{1}{2}q^2 & 0 \\
0 & \frac{1}{2}q^{n-1} & \frac{1}{2}q^{n-1} & 0 & 0
\end{pmatrix}
$$

with eigenvalues $\frac{1}{2}q^{2m}, \pm \frac{1}{2}q^m$ (each twice).

For $(p^i_{3a,j})$ one finds

$$
\begin{pmatrix}
0 & 0 & 0 & q(q^{n-3} - 1) & 0 \\
0 & \frac{1}{2}(q^{n-3} - 1)(q - 2) & \frac{1}{2}q(q^{n-3} - 1) & q^{n-3} - 1 & 0 \\
0 & \frac{1}{2}(q^{n-3} - 1)(q - 2) & \frac{1}{2}q(q^{n-3} - 1) & q^{n-3} - 1 & 0 \\
1 & \frac{1}{2}q^{n-3}(q - 2) & \frac{1}{2}q^{n-2} & q^{n-3} - 2q + 1 & q - 2 \\
0 & 0 & 0 & q(q^{n-3} - 1) & 0
\end{pmatrix}
$$

with eigenvalues $q(q^{2m-2} - 1), (q^{m-1} - 1)(q - 1), -(q^{m-1} + 1)(q - 1), 0$ (twice).

For $(p^i_{3n,j})$ one finds

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & q - 2 \\
0 & \frac{1}{2}q - 2 & \frac{1}{2}q & 0 & 0 \\
0 & \frac{1}{2}q - 1 & \frac{1}{2}q - 1 & 0 & 0 \\
0 & 0 & 0 & q - 2 & 0 \\
1 & 0 & 0 & 0 & q - 3
\end{pmatrix}
$$

with eigenvalues $q - 2$ (three times) and $-1$ (twice).

Since we could compute all $p^i_{jk}$, this is indeed an association scheme.

The $P$-matrix is

$$
P = \begin{pmatrix}
1 & \frac{1}{2}q^{2m-1}(q - 2) & \frac{1}{2}q^{2m} & q(q^{2m-2} - 1) & q - 2 \\
1 & \frac{1}{2}q^{m-1}(q - 2) & \frac{1}{2}q^m & -(q^{m-1} + 1)(q - 1) & q - 2 \\
1 & -\frac{1}{2}q^{m-1}(q - 2) & -\frac{1}{2}q^m & (q^{m-1} - 1)(q - 1) & q - 2 \\
1 & \frac{1}{2}q^m & -\frac{1}{2}q^m & 0 & -1 \\
1 & -\frac{1}{2}q^m & \frac{1}{2}q^m & 0 & -1
\end{pmatrix}
$$

The multiplicities (in the order of the rows of $P$) are $1, \frac{1}{2}q(q^m + 1)(q^{m-1} - 1)/(q - 1), \frac{1}{2}q(q^m - 1)(q^{m-1} + 1)/(q - 1), \frac{1}{2}(q - 2)(q^{2m} - 1)/(q - 1)$ (twice).

5 Conclusion

F. Vanhove computed all $p^i_{jk}$ and communicated both $P$ matrices by email. This note was written by A. E. Brouwer, and confirms his results.
References
