UNIQUENESS OF A ZARA GRAPH ON 126 POINTS AND NON-EXISTENCE OF A COMPLETELY REGULAR TWO-GRAPH ON 288 POINTS

by

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Dedicated to J.J. Seidel on the occasion of his retirement.

Abstract. There is a unique graph on 126 points satisfying the following three conditions:

(i) every maximal clique has six points;

(ii) for every maximal clique C and every point p not in C, there are exactly two neighbours of p in C;

(iii) no point is adjacent to all others.

Using this we show that there exists no completely regular two-graph on 288 points, cf. [4], and no (287,7,3)-Zara graph, cf. [1].

1. INTRODUCTION

A *Zara graph* with clique size K and nexus e is a graph satisfying:

(i) every maximal clique has size K;

(ii) every maximal clique has nexus e (i.e., any point not in the clique is adjacent to exactly e points in the clique).

For a list of examples, due to Zara, we refer to [1] and [6]. In this note we prove that there is only one Zara graph on 126 points with clique size 6 and nexus 2, which also has the property that no point is adjacent to all others. This graph, Z^*, is defined as follows:

Let W be a 6-dimensional vector space over GF(3), together with the bilinear form $\langle x|y\rangle = x_1y_1 + \dots + x_6y_6$. Points of Z^* are the one-dimensional subspaces of W generated by a point x of norm 1, i.e., $\langle x|x\rangle = 1$. Two such subspaces are adjacent if they are orthogonal: $\langle x\rangle \sim \langle y\rangle$ iff $\langle x|y\rangle = 0$.
In the following section Z will denote any Zara graph on 126 points with $K = 6$ and $e = 2$.

2. BASIC PROPERTIES OF ZARA GRAPHS

A *singular subset* of a Zara graph is a set of points which is the intersection of a collection of maximal cliques. Let $\mathcal{S}$ denote the collection of singular subsets. From [1] we quote the main theorem for Zara graphs (a graph is called *coconnected* if its completement is connected):

THEOREM 1. Let G be a coconnected Zara graph. There exists a rank function $\rho : \mathcal{S} \to \mathbb{N}$ such that

(i) $\rho(\emptyset) = 0$

(ii) If $\rho(x) = i$ and C is a maximal clique containing x while $p \in C\backslash x$, then $\exists y \in \mathcal{S}$ with $\rho(y) = i+1$ and $x \cup \{p\} \subset y \subset C$.

(iii) $\exists r : \rho(c) = r$ for all maximal cliques C.

(iv) $\exists R_0, R_1, \dots, R_r : \rho(x) = i \Rightarrow x$ is in R_i maximal cliques.

(v) $\exists K_0, K_1, \dots, K_r : \rho(x) = i \Rightarrow |x| = K_i$.

(vi) The graph defined on the rank 1 sets by $x \sim y$ iff $\xi \sim \eta$ for all $\xi \in x$ and $\eta \in y$ is strongly regular. □

The number r is called the *rank* of the Zara graph. A coconnected rank 2 Zara graph with $e = 1$ is essentially a *generalized quadrangle*. In this case singular subsets are the empty set (rank 0) the points (rank 1) and the maximal cliques (rank 2). This graph is also denoted by $GQ(K-1,R_1-1)$. As an example we mention $GQ(4,2)$. This is a graph on 45 points, maximal cliques have size 5, and each point is in three maximal cliques. This graph is unique [5] and has the following description:

Let W be a 4-dimensional vector space over GF(4) with hermitian form $\langle x|y\rangle = x_1\bar{y}_1 + \dots x_4\bar{y}_4$, where $\bar{y}_i = y_i^2$. Points are the one-dimensional subspaces $\langle x\rangle$ with $\langle x|x\rangle = 0$ and $\langle x\rangle \sim \langle y\rangle$ if $\langle x|y\rangle = 0$ (and $\langle x\rangle \neq \langle y\rangle$). Another description of this graph is the following: Let W' be a 5-dimensional vector space over GF(3) with bilinear form $\langle x|y\rangle = x_1y_1 + \dots + x_5y_5$. Points are the one-dimensional subspaces $\langle x\rangle$ with $\langle x|x\rangle = 1$ and $\langle x\rangle \sim \langle y\rangle$ if $\langle x|y\rangle = 0$.

From the main theorem on Zara graphs one can prove:

THEOREM 2. Z is a strongly regular graph, with $(v,k,\lambda,\mu) = (126,45,12,18)$. Each point is in 27 maximal cliques, each pair of adjacent points in 3. The induced graph on the neighbours of a given point is (isomorphic to) $GQ(4,2)$. □

3. A FEW REMARKS ON GQ(4,2), Z^* AND FISCHER SPACES

The following facts can be checked directly from the description of $GQ(4,2)$ and Z^* and the definition of Z. If x and y are points at distance two in the graph G then $\mu_G(x,y)$ (or just $\mu(x,y)$) denotes the induced graph on the set of common neighbours of x and y in G.

Fact 1. If $x \not\sim y$ in Z then $\mu(x,y)$ is a subgraph of $GQ(4,2)$ on 18 points, regular with valency 3. If $x \not\sim y$ in Z^* then $\mu(x,y) \simeq 3 \times K_{3,3}$.

Fact 2. $GQ(4,2)$ contains 40 subgraphs isomorphic to $3 \times K_{3,3}$. Through each 2-claw (i.e. $K_{1,2}$) in $GQ(4,2)$ there is a unique $3 \times K_{3,3}$ subgraph, even a unique $K_{3,3}$.

Let $x \in Z^*$. Let $\Gamma(x)$ denote the induced graph on the neighbours of x, $\Delta(x)$ the induced graph on the non-neighbours, different from x. $\Gamma(x) \simeq GQ(4,2)$ and each point $y \in \Delta(x)$ determines the subgraph $K_y \simeq 3 \times K_{3,3}$ in $\Gamma(x)$, where $K_y = \mu(x,y)$

Fact 3. To each subgraph $K' \simeq 3 \times K_{3,3}$, of $\Gamma(x)$ there correspond exactly two points $y, y' \in \Delta(x)$, such that $K_y = K_{y'} = K'$. Note that $y \not\sim y'$.

This property can be used to show that Z^* is a *Fischer space*.

DEFINITION. A *Fischer space* is a linear space (E,L) such that

(i) All lines have size 2 or 3;

(ii) For any point x, the map $\sigma_x : E \to E$, fixing x and all lines through x, and interchanging the two points distinct from x on the lines of size 3 through x, is an automorphism.

THEOREM 3. There is a unique Fischer space on 126 points with 45 two-lines on each point.

The proof of this fact can be found in [2] p. 14. □

4. THE UNIQUENESS PROOF, PART I

Using a few lemmas, it will be shown that Z carries the structure of a Fischer space. By Theorem 3 then $Z \simeq Z^*$.

Notation: For a subset S of Z, we denote by $S^\perp$ the induced subgraph on the set of points adjacent to all of S.

LEMMA 1. Let $\{a,b,c\}$ be a two-claw in Z: $a \sim b$, $a \sim c$, $b \not\sim c$. Then $\{a,b,c\}^\perp \simeq \bar{K}_3$ and there is a unique point $d \sim a$ such that $\{a,b,c,d\}^\perp = \{a,b,c\}^\perp$. Moreover, $d \not\sim b$, $d \not\sim c$.

Proof. Apply fact 2 to $\Gamma(a) \simeq GQ(4,2)$. □

LEMMA 2. Let $a \not\sim b$ in Z. Then $\mu(a,b) \simeq 3 \times K_{3,3}$.

This is the *main lemma*; the proof will be the subject of the next section. □

LEMMA 3. Let $a \not\sim b$ in Z. There is a unique point $c \in Z$ such that $\{a,b\}^\perp = \{a,b,c\}^\perp$. Moreover, $c \not\sim a$, $c \not\sim b$.

Proof. Consider a 2-claw $\{x,y,z\}$ in $\mu(a,b)$. By Lemma 1 there is a point c in $\{x,y,z\}^\perp$ and $c \not\sim a$, $c \not\sim b$. By Lemma 2 $\mu(a,b) \simeq 3K_{3,3}$ and by fact 2 this subgraph of $\Gamma(a)$ is unique, hence $\mu(a,b) = \mu(a,c)$. □

THEOREM 4. Z carries the structure of a Fischer space with 126 points and 45 two-lines on each point.

Proof. Let the two-lines correspond to the edges of Z, the 3-lines to the triples {a,b,c} as in Lemma 3. This turns Z into a linear space with 45 two-lines on each point. It remains to be shown that σ_x is an automorphism for all $x \in Z$. Since $\sigma_x^2 = 1$ it suffices to show that $y \sim z$ implies $\sigma_x(y) \sim \sigma_x(z)$. The only non-trivial case is when $y,z \in \Delta(x)$. Let $Y = \Gamma(y) \cap \Delta(x)$, $Y' = \Gamma(\sigma_x(y)) \cap \Delta(x)$. Then $Y \cap Y' = \emptyset$ and $|Y| = |Y'| = 27$.
Since $|\{y,u,x\}^{\perp}| = 6$ for all $u \in Y$ (there are three maximal cliques passing through y, and x has two neighbours on each of them), and since $\mu(y,x) = \mu(\sigma_x(y),x)$, we also have $|\{y',u,x\}^{\perp}| = 6$ for $u \in Y$ and similarly $|\{y,u',x\}^{\perp}| = 6$ for $u' \in Y'$.
Counting edges between $\mu(x,y)$ and $\Delta(\infty)$ it follows that the average of $|\{y,u,x\}^{\perp}|$, with $u \in U = \Delta(\infty)\backslash(Y \cup Y' \cup \{y\} \cup \{\sigma_x(y)\})$ is 9. Consider an edge in $\mu(x,y) = \mu(x,y')$. There are three maximal cliques passing through that edge, containing x,y,y' respectively. Hence {y,u,x'} is a coclique for $u \in U$, whence $|\{y,u,x\}^{\perp}| \leq 9$. Combining this yields $|\{y,u,z\}^{\perp}| = 9$ for all $u \in U$.
Next, consider a point z in Y. Since $\mu(x,z) = \mu(x,\sigma_x(z))$, we must have $\sigma_x(z) \in Y$ or $\sigma_x(z) \in Y'$. If $\sigma_x(z) \in Y$, then $y \sim z$ and $y \sim \sigma_x(z)$ but $y \sim x$, contradiction. Hence, $z \in Y'$, i.e., $\sigma_x(y) \sim \sigma_x(z)$. □

This finishes the uniqueness. It remains to prove Lemma 2.

5. THE UNIQUENESS PROOF, PART II: PROOF OF THE MAIN LEMMA

Main lemma. Let $\infty \not\sim \infty'$ in Z. Then $\mu(\infty,\infty') \simeq 3K_{3,3}$.
The proof will be split into a number of lemmas.

LEMMA 4. Let $S = \{a,b,c,d\}$ be a square in Z, i.e., $a \sim b \sim c \sim d \sim a$ and $a \not\sim c$, $b \not\sim d$. Then $|S^{\perp}| \in \{0,1,3\}$.

Proof. Clearly $S^{\perp}$ has at most three points, so it suffices to show that two points is impossible.

Let $\infty,\infty' \in S^{\perp}$. By Lemma 1 there is a point a' such that $\{d,a,b\}^{\perp} = \{a',\infty,\infty'\}$. Similarly there are points b',c',d'. If two of the points a',b',c',d' coincide, then we have found a third point adjacent to all of S. Hence, assume they are all different. There are three maximal cliques containing ab. One contains ∞, another ∞', whence the third one contains a' and b'. Hence $a' \sim b' \sim c' \sim d' \sim a'$. Considering again the clique $\{a,b,a',b'\}$, notice that $c' \not\sim a$, $c' \sim b$ and $c' \sim b'$. It follows that $c' \not\sim a'$ and similarly $b' \not\sim d'$. The situation is summarized in figure 1 where $A = \{a,a'\}$.

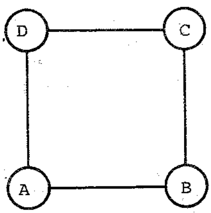

figure 1

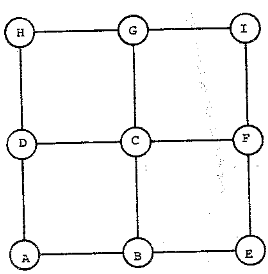

figure 2

Using the Zara graph property it follows that the picture can be completed to figure 2:

where $E = \{e,e'\}$ etc.: Indeed, the clique $\{a,a',b,b'\}$ can be completed with points e,e'. Similarly DC can be completed and $\{e,e'\} \cap \{f,f'\} = \emptyset$. Having found E,F,G,H, complete the clique $\{e,e',f,f'\}$ using $\{i,i'\}$. Since i and i' have no neighbours in A,B,C,D, they must be adjacent to G and H.

Now ∞ and ∞' have one neighbour in each of A,B,C,D. It follows that both are adjacent to i,i'. However, there are three maximal cliques through I, two of them already visible, whence ∞ and ∞' must be in the third clique. This is a contradiction since $\infty \not\sim \infty'$. The conclusion is that $a' = b' = c' = d'$ is the third point in $S^{\perp}$. □

LEMMA 5. If $\infty \not\sim \infty'$ in Z and $\mu(\infty,\infty')$ contains a square, then $\mu(\infty,\infty') \simeq 3K_{3,3}$, and there is a unique point ∞'' such that $\{\infty,\infty',\infty''\}^{\perp} = \mu(\infty,\infty')$.

Proof. Let $S = \{a,b,c,d\}$ be a square in $\mu(\infty,\infty')$. From the previous lemma it follows that there is a third point, e, adjacent to the square $\{\infty,a,\infty',c\}$. Similarly there is a point f adjacent to $\{\infty,b,\infty',d\}$, and $\{a,b,c,d,e,f\}$ is a $K_{3,3}$ in $\mu(\infty,\infty')$. Now $\mu(\infty,\infty')$ is a subgraph of $\Gamma(\infty) \simeq GQ(4,2)$ with 18 points and valency 3, containing a $K_{3,3}$. This is enough to guarantee that $\mu(\infty,\infty') \simeq$ $\simeq 3K_{3,3}$. Let ∞'' be the third point adjacent to S. Since S is in a unique $K_{3,3}$ in $\Gamma(\infty)$ it follows that $\mu(\infty,\infty'') = \mu(\infty,\infty')$. □

LEMMA 6. Let $a,b,c \in Z$ with $|\{a,b,c\}^{\perp}| = 18$. Then $\{a,b,c\}^{\perp} \simeq 3K_{3,3}$.

Proof. First note that $\{a,b,c\}$ is a coclique. Let $M = \{a,b,c\}^{\perp}$ and $A = \Gamma(a)\backslash M$; B and C are defined similarly. Finally $R = Z\backslash(A \cup B \cup C \cup M \cup \{a\} \cup \{b\} \cup \{c\})$.

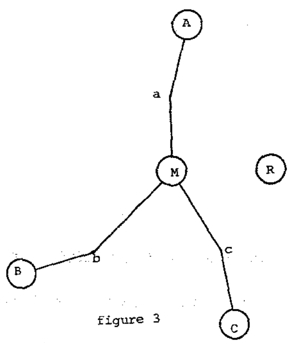

figure 3

$|A| = |B| = |C| = 27$

$|R| = 24, \quad |M| = 18$.

Two adjacent points in M have twelve common neighbours, three in A,B and C and none in R. It follows that the neighbours of a point $r \in R$ in M form a coclique. A point $m \in M$ has three neighbours in M, nine in

A,B and C (since Z is strongly regular with $\lambda = 12$). Hence m has twelve neighbours in R. Since the neighbours of $r \in R$ in M form a coclique, r has at most nine neighbours in M. But $9 \times 24 = 12 \times 18$, so it is exactly nine. If M is connected, there are at most two nine cocliques in M, whence at least twelve points of R are adjacent to the same 9-coclique. If there is an edge between two of the twelve we have a contradiction, if not also Hence M is disconnected. In this case however, one easily sees that M contains a square and hence $M \simeq 3K_{3,3}$. □

From now on we will identify $\Gamma(\infty) \simeq GQ(4,2)$ with the set of isotropic points in PG(3,4) w.r.t. a unitary form.

For $a \in \Delta(\infty)$ let $M_a = \mu(a,\infty)$. The graph M_a has 18 vertices and is regular of valency 3. By Lemma 5, if M_a contains a square, then $M_a \simeq 3K_{3,3}$. A computer search for all 18-point subgraphs of valency 3 and girth ≥ 5 of GQ(4,2) reveals that such a graph is necessarily (connected and) bipartite, i.e., a union of two ovoids. Now GQ(4,2) contains precisely two kinds of ovoids, plane ovoids and tripod ovoids (cf. [3]).

Let $x \in PG(3,4)\backslash U$, where U is the set of isotropic points.

A *plane ovoid* is a set of the form $x^{\perp} \cap U$.

A *tripod ovoid* (on x) is a set of the form

$$\bigcup_{i=1}^{3} xz_i \cap U ,$$

where $\{x,z_1,z_2,z_3\}$ is an orthonormal basis. On each non-isotropic point there are four tripod ovoids. Since two plane ovoids always meet, we find that each set M_a is one of the following (where T_x denotes some tripod ovoid on x):

I. $(x^{\perp} \cup T_x) \cap U$ ($M_a \simeq 3K_{3,3}$ in this case).

II. $(x^{\perp} \cup T_z) \cap U$ where $z \in x^{\perp}$ and $x \notin T_z$.

III. $(T_z \cup T'_z) \cap U$, the union of two tripod ovoids on the same point.

(Note that $(T_x \cup T_z) \cap U$ for $z \in x^{\perp}$ and xz in T_z but not in T_x does contain squares, in fact $K_{2,3}$'s.)

If $a \sim b$, $a,b \in \Delta(\infty)$, then ∞ has two neighbours on each of the three 6-cliques on the edge ab, so that $M_a \cap M_b \simeq 3K_2$.
By studying the intersections between sets of the three types, I,II,III we shall see that necessarily all sets M_a are of type I. Let us prepare this study by looking at the intersections of two ovoids in GQ(4,2).

A. $$|x^{\perp} \cap y^{\perp} \cap U| = \begin{cases} 9 \text{ if } x = y\ ; \\ 3 \text{ if } x \perp y\ ; \\ 1 \text{ otherwise.} \end{cases}$$

B. $$|x^{\perp} \cap T_z \cap U| = \begin{cases} 0 \text{ if } x = z \text{ or } (z \in x^{\perp} \text{ and } x \notin T_z)\ ; \\ 6 \text{ if } z \in x^{\perp} \text{ and } x \in T_z\ ; \\ 2 \text{ otherwise.} \end{cases}$$

C. $$|T_x \cap T_z \cap U| = \begin{cases} 9 \text{ if } T_x = T_z\ ; \\ 3 \text{ if } z \in x^{\perp} \text{ and } xz \text{ occurs in both or none of } T_x, T_z\ ; \\ 0 \text{ if } (x = z \text{ and } T_x \neq T_z) \text{ or } (z \in x^{\perp} \text{ and } xz \text{ in one of } T_x, T_z); \\ 4 \text{ if } z \neq x \text{ and } z \notin x^{\perp} \text{ and } (xz)^{\perp} \text{ meets } T_x \cap T_z; \\ 1 \text{ otherwise.} \end{cases}$$

Next let us determine which intersections of the sets of types I,II,III are of the form $3K_2$.

a) $(x^\perp \cup T_x) \cap (y^\perp \cup T_y) \cap U \simeq 3K_2$ iff $y \in x^\perp$, $x \notin T_y$ and $y \notin T_x$.

b) $(x^\perp \cup T_x) \cap (y^\perp \cup T_z) \cap U \simeq 3K_2$ iff either $(x \in \{y,z\}^\perp$ and $y \notin T_x)$ or $(x \notin y^\perp \cup z^\perp$ and $T_x \ni w$ where $w \in \{y,z\}^\perp)$.

c) $(x^\perp \cup T_x) \cap (T_z \cup T_z') \cap U \not\simeq 3K_2$.

d) $(x^\perp \cup T_y) \cap (z^\perp \cup T_w) \cap U \simeq 3K_2$ iff either $(x = w$ and $y = z)$ or $(x = w$ and $y \in z^\perp)$ or $(w \in x^\perp$ and $y = z)$.

e) $(x^\perp \cup T_y) \cap (T_z \cup T_z') \cap U \not\simeq 3K_2$.

f) $(T_x \cup T_x') \cap (T_z \cup T_z') \cap U \not\simeq 3K_2$.

It follows immediately that no set M_a can be of type III, since no type is available for M_b when $b \sim a$. Each edge of $\Delta(\infty)$ is in three 6-cliques and these have two points each in $\Gamma(\infty)$, so that we find 4-cliques in $\Delta(\infty)$.

If some 4-clique $\{a, b,c,d\}$ has $M_a = (x^\perp \cup T_y) \cap U$ and $M_b = (y^\perp \cup T_z) \cap U$ (with $z \in x^\perp$), then M_c and M_d cannot both be of type I (for let $\{x,y,z,w\}$ be an orthonormal basis; if $M_c = (v^\perp \cap T_v) \cap U$ where $v \neq w$ then $v \in w^\perp$ and $w \in T_v$; now M_d cannot be $(w^\perp \cup T_w) \cap U$ so $M_d = (u^\perp \cup T_u) \cap U$ where $u \in \{v,w\}^\perp$ and $w \in T_u$, $v \notin T_u$, impossible by the definition of a tripod); so w.l.o.g. $M_c = (z^\perp \cup T_x) \cap U$.

Consequently the three 4-cliques on the edge ab each contain a point c with $M_c = (z^\perp \cup T_x) \cap U$, and by Lemma 6 these three sets are distinct, so we see that the three possibilities for T_x $(z \notin T_x)$ all occur. Now fixing a and c and repeating the argument we find three points b with $M_b = (y^{\perp} \cup T_z) \cap U$ and similarly three points a with $M_a = (x^{\perp} \cup T_y) \cap U$ and thus a subgraph $\simeq K_{3,3,3}$ in $\Delta(\infty)$. But that is impossible:

LEMMA 7. Let K be a subgraph of Γ with $K \simeq K_{3,3,3}$. Then any point x outside K is adjacent to precisely three points of K.

Proof. Standard counting arguments. □

Next: no 4-clique $\{a,b,c,d\}$ has M_a of type II and M_b, M_c, M_d all of type I: Let $M_a = (x^{\perp} \cup T_y) \cap U$ and $\{x,y,z,w\}$ be an orthonormal basis; let the three sets M_b, M_c and M_d be $(v_i^{\perp} \cup T_{v_i}) \cap U$ $(i = 1,2,3)$, then the points v_1,v_2,v_3 are pairwise orthogonal and each is in $\{w,z\} \cup w^{\perp} \cup z^{\perp}$; if $v_1 = w$, $v_2 = z$ then $v_3 \in \{x,y\}$, impossible; if $v_1 = w$, $v_2,v_3 \in w^{\perp}\backslash\{z\}$ then T_{v_2} must contain w and must not contain w, impossible; if $v_1,v_2,v_3 \in (w^{\perp} \cup z^{\perp})\backslash\{w,z\}$ then we may suppose $v_2,v_3 \in w^{\perp}\backslash\{z\}$ and the same contradiction arises.

It follows that if a 4-clique $\{a,b,c,d\}$ has M_a of type II, then there is precisely one other set of type II among M_b, M_c, M_d - if $M_a = (x^{\perp} \cup T_y) \cap U$ then $M_b = (y^{\perp} \cup T_x) \cap U$, but a is on 27 four-cliques and for each of the three possible b the edge ab is on only 3 four-cliques, a contradiction. This shows that sets of type II do not occur at all: the main lemma is proved. □

6. APPLICATIONS

THEOREM 4. There does not exist a rank 4 Zara graph G on 287 points with clique size 7 and nexus 3.

Proof. Using the main theorem for Zara graphs it is not difficult to show that G is a strongly regular graph with $(v,k,\lambda,\mu) = (287,126,45,63)$. Moreover, for each point $\infty \in G$, $\Gamma(\infty) \simeq Z^*$. To finish the proof we need two lemmas.

LEMMA 8. Let $a \not\sim b$ in G . Then $\mu(a,b)$ is a graph on 63 points, and for each $c \in \mu(a,b)$ we have $\Gamma_{\mu(a,b)}(c) \simeq 3K_{3,3}$.

Proof. Consider $\Gamma_G(c) \simeq Z^*$. In $\Gamma_G(c)$, $\mu(a,b) \simeq 3K_{3,3}$, but this just means that $\Gamma_{\mu(a,b)}(c) \simeq 3K_{3,3}$. □

LEMMA 9. Z^* does not contain a subgraph T on 63 points which is locally $3K_{3,3}$.

Proof. Let $\infty \in Z^*$ and suppose $\infty \in T$. Let $K \simeq 3K_{3,3}$ be the subgraph of $\Gamma_Z(\infty)$ also in T. Let figure 4 be one of the components of K, and consider $\Gamma_T(a)$. We see the points ∞,u,v,w.

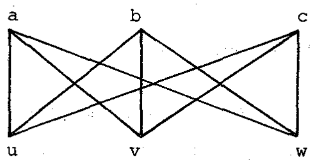

figure 4

Since $\Gamma_T(a) \simeq 3K_{3,3}$ there are points ∞' and ∞'' in T also adjacent to a,u,v,w. But we know these points, they are unique in Z. Hence: ∞,∞',∞'' have precisely the same neighbours in T. As a consequence the points of T can be divided into 21 groups of 3. Let T' be the graph defined on the 21 triples by $t_1 \sim t_2$ if $\tau_1 \sim \tau_2$ for all $\tau_1 \in t_1$, $\tau_2 \in t_2$. Then T' is a strongly regular graph on 21 points with $k = 6$, $\lambda = 1$ and $\mu = 1$ (this is a direct consequence of the structure of Z^*). Now such a graph does not exist, since it violates almost all known existence conditions for strongly regular graphs. This proves the lemma and the theorem. □

THEOREM 5. There does not exist a (non-trivial) completely regular two-graph on 288 points.

Proof. (For definitions and results about completely regular two-graphs see [4]).

A completely regular two-graph on 288 points, gives rise to at least one rank 4 Zara graph on 287 points with clique size 7 and nexus 3. But such a graph does not exist by the previous theorem. □

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