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GRAPHS WITH BALANCED STAR-HYPERGRAPH

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Graphs with balanced star-hypergraph

by

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ABSTRACT

A characterization is given of simple graphs $G = (X,\Gamma)$ such that $H_G = (\Gamma(x) \cup \{x\} | x \in X)$ is a balanced hypergraph.

KEY WORDS & PHRASES: *balanced hypergraph.*

BERGE [1,p.278] asked for a characterization of simple graphs $G = (X,\Gamma)$ such that the hypergraph $H_G = (\Gamma(x) \cup \{x\} | x \in X)$ is balanced. (A hypergraph $(E_i | i \in I)$ is called *balanced* if for each circuit $(x_1,E_1,x_2,E_2,\ldots,x_n,E_n,x_1)$, where n is odd, there exists an E_i $(1 \le i \le n)$ such that $|E_i \cap \{x_1,\ldots,x_n\}| \ge 3$). The following theorem gives such a characterization.

<u>THEOREM</u>. *Let* $G = (X,\Gamma)$ *be a simple graph and let* $H_G = (\Gamma(x) \cup \{x\} | x \in X)$. *Then the following conditions are equivalent:*

(1) H_G *is balanced;*

(2) *every subgraph of* G *induced by a circuit of length at least* 4 *has a point with valency at least* 3 *and every subgraph of* G *induced by a circuit of length* 4k + 2 *has at most* 2k *points with valency* 2

<u>PROOF</u>. Define for each $x \in X$: $\Gamma_*(x) = \Gamma(x) \cup \{x\}$.

(1) ⇒ (2). Let $C = (x_1,\ldots,x_{4k+i},x_1)$ be a circuit in G, where $k \ge 1$ and $0 \le i \le 3$.

If i = 2, C can be interpreted as a representation of an odd circuit in H_G, namely the circuit

$C' = (x_1,\Gamma_*(x_2),x_3,\Gamma_*(x_4),\ldots,\Gamma_*(x_{4k+2}),x_1)$. Since H_G is balanced some x_j must be adjacent to at least three other vertices x_i, i.e. the valency in C' of x_j is at least 3.

If i = 1 then in order to get an odd circuit in H_G we have to repeat one point:

$C' = (x_1,\Gamma_*(x_1),x_2,\Gamma_*(x_3),\ldots,\Gamma_*(x_{4k+1}),x_1)$.

If i = 0 we have to repeat two non-adjacent points, and if i = 3 we have to repeat three non-adjacent points in order to get an odd circuit in H_G. E.g. in the last case:

$C' = (x_1,\Gamma_*(x_1),x_2,\Gamma_*(x_3),x_3,\Gamma_*(x_4),x_5,\Gamma_*(x_5),x_6,\ldots,\Gamma_*(x_{4k+3}),x_1)$.

In all cases it follows that for some j the set $\Gamma_*(x_j)$ contains at least three vertices of C', since H_G is balanced. It is not possible that $\Gamma_*(x_j) = \{x_{j-1},x_j,x_{j+1}\}$ (otherwise x_{j-1},x_j,x_{j+1} would be vertices of C', but only non-adjacent vertices of C were repeated), hence the valency of x_j in C is at least 3. This shows that every circuit in G with at least 4 vertices contains a point with valency at least 3. Next assume we have

a circuit C in G with $4k + 2$ vertices of which at least $2k + 1$ have in C valency 2:
$C = (x_1,x_2,\dots,x_{4k+2},x)$. If x_i and x_{i+1} both have valency 2 then a minimal circuit C_0, which is contained in C and which contains x_i and x_{i+1}, also contains x_{i-1} and x_{i+2}. Hence C_0 has at least 4 vertices. Therefore C_0 must contain a point x_j with valency at least 3, i.e. C_0 contains a diagonal. But this contradicts the minimality of C_0. Hence at most one of two adjacent points has valency 2 in C and we can index C in such a way that the points of valency 2 are just the points $x_2,x_4,\dots,x_{4k+2}$ with even subscript. Taking $C' = (x_1,\Gamma_*(x_2),x_3,\Gamma_*(x_4),\dots,\Gamma_*(x_{4k+2}),x_1)$ we find that H_G is not balanced.

(2) ⇒ (1). Assume H_G is not balanced and that we have a circuit $C = (x_1,E_1,\dots,x_{2n+1},E_{2n+1},x_1)$ such that for each i:
$|E_i \cap \{x_1,\dots,x_{2n+1}\}| = 2$. Let $E_i = \Gamma_*(y_i)$ with $y_i \in X$. Note that we may suppose that all x_i are different. If all y_i differ from each x_j then we have a circuit $(x_1,y_1,\dots,x_{2n+1},y_{2n+1},x_1)$ in G with length $4n + 2$ and $2n + 1$ vertices have valency 2, contrary to the hypothesis. Therefore assume y_1 equals some x_j. But: $E_1 \cap \{x_1,\dots,x_{2n+1}\} \supset \{x_1,x_2,x_j\}$, hence x_j must be either x_1 or x_2 and we may suppose $y_1 = x_1$. Now let $G_0 = \{x_1,\dots,x_{2n+1},y_1,\dots,y_{2n+1}\}$ and let C_0 be a minimal circuit in G_0 containing $x_1 = y_1$. By hypothesis C_0 is a triangle: $C_0 = (x_1,z_1,z_2,x_1)$.
If $z_1 \in \{x_2,\dots,x_{2n+1}\}$ then: $E_1 \cap \{x_1,\dots,x_{2n+1}\} \supset \{x_1,x_2,z_1\}$, so $z_1 = x_2$.
Therefore $\{z_1,z_2\} \not\subset \{x_2,\dots,x_{2n+1}\}$. Hence we may assume:
$z_2 \notin \{x_1,\dots,x_{2n+1}\}$, i.e. $z_2 = y_k$ for some $k \neq 1$.
But now: $E_k \cap \{x_1,\dots,x_{2n+1}\} \supset \{x_k,x_{k+1},x_1\}$, so $k = 2n + 1$ and $z_2 = y_{2n+1}$.
Therefore $\{z_1,z_2\} \not\subset \{z_1,z_2\} \not\subset \{y_1,\dots,y_{2n+1}\}$. Thus $C_0 = (x_1,x_2,y_{2n+1})$, and this means that: $E_{2n+1} \cap \{x_1,\dots,x_{2n+1}\} \supset \{x_1,x_2,x_{2n+1}\}$. But this contradicts our assumption. □

REFERENCE

[1] C. BERGE & D. RAY-CHAUDHURI (eds.), *Hypergraph Seminar*, Lecture Notes in Mathematics 411, Springer Verlag, Berlin 1974.