Group Theory Assignments
Course number 2F790, TU/e, Eindhoven, November 2004
Arjeh M. Cohen

Rules
• One person per assignment.
• Let me know your preferences before November 20, 2004.
  I will then fix determine your assignment by November 22, 2004.
• The assignment should be submitted to me before December 5, 2004.
• You are allowed (and encouraged) to consult me.
• Visit the library and the internet!
• Use any CA package (especially GAP and Magma can be very useful).
• If there are no satisfactory assignments in the list below, contact me with a
  wish for another assignments, preferably with a indication of your interests.

Assignments
(1) Deal (succinctly) with the representation theory of the symmetric groups
\( S_n \) \( (n \in \mathbb{N}) \). You know that conjugacy classes correspond to partitions of \( n \).
There is a way to let irreducible representations correspond to partitions
of \( n \) as well, and to compute the corresponding characters \( (n = 25 \) is close
to today’s computational limit). Use this theory to determine the charac-
ter table of \( \text{Sym}_6 \). See, for instance, [M.A.A. van Leeuwen, A.M. Cohen,
and B. Lisser, \textit{LiE Manual}, manual for the software package LiE for Lie
group theoretical computations, CWI/CAN, 1992] or [G.D. James & A.
Kerber: The Representation Theory of the Symmetric Group. Encyclope-
(2) The finite group \( \text{SL}(2, \mathbb{F}_9) \) of all \( 2 \times 2 \) matrices of determinant 1 with coefficients in the finite field of order 9, has order 720.
(a) Determine its character table.
(b) The group has a center of order two. With which familiar group is the
  quotient isomorphic? Give a proof of this result.
(c) In which dimension does a faithful representation occur? Can you
  construct it?
(d) If \( \alpha \) is an automorphism of \( \text{SL}(2, \mathbb{F}_9) \) and \( \rho \) a representation of this
  group, then \( \rho \circ \alpha \) is again a representation of \( \text{SL}(2, \mathbb{F}_9) \). Prove this.
(e) Apply the previous part to the automorphism that changes each entry
  of a matrix of \( \text{SL}(2, \mathbb{F}_9) \) to its third power. It induces a permutation
  of characters occurring in the character table. Indicate which permu-
tation.
(3) The finite group \( G = \text{SL}(2, \mathbb{F}_5) \) of all \( 2 \times 2 \) matrices of determinant 1 with coefficients in the finite field of order 5, has order 120.
(a) Determine its character table.
(b) The group has a center of order two. With which familiar group is the
  quotient isomorphic? Give a proof of this result.
(c) Conclude from the previous items that \( G \) has a faithful irreducible
  representation of degree 2. Denote by \( \rho \) its character.
(d) For \( \chi \) an irreducible character of \( G \), write
  \[ P_\chi(t) = \sum_{i=0}^{\infty} (\chi, \rho^i) t^i. \]
  This is in fact a rational expression in \( t \). Try and find this expression! (Hint:
view $G$ as a subgroup of $SU(2)$ and consider the restrictions to $G$ of the irreducible characters of the latter group described in the notes.)

(4) Let $G$ be a finite group, with a permutation representation $\pi : G \to S_n$ of degree $n$.

(a) Prove that $(1, \pi)$ is the number of orbits of $G$ on $X = \{1, \ldots, n\}$.

(b) Suppose that $\pi$ is transitive (that is, $\pi(G)$ has only one orbits on $X$). Show that $(\pi, \pi)$ is the number of orbits of $G$ on $X \times X$ (via the diagonal action).

(c) An orbit of $G$ on $X \times X$ is called an orbital. So, an orbital is a relation on $X$. Suppose $E$ is a symmetric orbital. Prove that $\pi(G)$ is a group of automorphisms of the graph whose vertex set is $X$ and in which $x, y \in X$ are adjacent vertices iff $(x, y) \in E$.

(d) Show that every graph $(X, E)$ whose automorphism group is transitive on $X$ as well as on $E$ can be constructed as in the previous item.

(e) Give a two parameter series of examples by letting $X$ be the collection of sets of size $k$ from a set of size $d$, and $G = S_d$. The best examples are obtained by letting adjacency be ‘having intersection of size $k - 1$’. Show that, with this choice of adjacency, the orbitals are precisely the relations ‘having distance $i$ in the graph’.

(f) Analyse the characters involved in the permutation character in the examples of the previous item (as far as you can, possibly explicitly for small $d$).

(5) Treat the induced representations. Let $G$ be a group with subgroup $H$, and suppose that a representation $\rho : H \to \text{GL}(V)$ is given.

(a) Describe the character of the induced representation $\rho^G$ of $G$.

(b) Prove that, if $\chi$ denotes the character of $\rho$, and $\chi^G$ is the character of $\rho^G$, while $\psi$ is an arbitrary character of $G$, we have: $(\psi, \chi^G) = (\psi|_H, \chi)$, where $|_H$ denotes the restriction (of a function) to $H$.

(c) Verify that all irreducible representations of the dihedral groups are induced from linear (that is, 1-dimensional) representations.

(d) Check which irreducible representations of $\text{Alt}_6$ are induced from proper subgroups.

(e) Can you also determine which irreducible representations of $\text{Alt}_6$ occur in induced representations from proper subgroups?

(6) Consider the character table below ... that is, I claim it is a character table of a finite group. Find all groups $G$ up to isomorphism, of which this table is a character table. All properties of the group that you can derive from the table are welcome. Record all your arguments.
Hint: a character table records the information

\[ a_{ijk} = \#\{z \in \text{ConjClass}_i \mid \exists y \in \text{ConjClass}_j \exists z \in \text{ConjClass}_k \quad z = xy\}. \]

(Consult literature.) Besides, the character table shows that the group has a faithful representation of degree 4. Character \( \chi_3 \) shows that there is a normal subgroup of order 8. Determine its structure, and start your ‘reconstruction’ of the faithful 4-dimensional representation with this subgroup.