BMW algebras
of simply laced type

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BMW algebra of type $A_n$

- BMW = Birman & Wenzl, Murakami
- Definition for type $A_n$ carries over to any graph $M$
BMW algebra of type $M$ over $\mathbb{Q}(l, m)$

**generators**
\[ \{ g_i \mid i \in M \} \]

**relations**
- (braid1) \[ g_i g_j = g_j g_i \]
  when \( i \not\sim j \),
- (braid2) \[ g_i g_j g_i = g_j g_i g_j \]
  when \( i \sim j \),
- (skein) \[ me_i = l(g_i^2 + mg_i - 1) \]
  for all \( i \),
- (self-intersection1) \[ g_i e_i = l^{-1}e_i \]
  for all \( i \),
- (self-intersection2) \[ e_i g_j e_i = le_i \]
  when \( i \sim j \).
Properties

- $J$ coclique in $M$ of size $p$

  $$e_J := x^{-p} \prod_{j \in J} e_i$$

  idempotent, where $x = 1 - (l - l^{-1})/m$ in $\mathbb{Q}(l, m)$

- $I_p :=$ ideal generated by all $e_J$ for $|J| = p$

- Quotient by $I_1$ gives Hecke algebra of type $C$,

  where $C$ is type of $C_{W(M)}(\alpha_1)$

- The left module spanned by $e_1$ modulo $I_2$ is the Lawrence-Krammer representation

  with coefficients in Hecke algebra of type $C$
Theorem*

$BMW(M)$ is finite-dimensional

$\iff$

connected components of $M$ in $\{A_n, D_n, E_6, E_7, E_8 \mid n \in \mathbb{N}\}$.

\[
\begin{align*}
\dim BMW(A_n) &= (n + 1)!! \\
\dim BMW(D_n) &= (2^n + 1)n!! - (2^{n-1} + 1)n! \\
\dim BMW(E_6) &= 1,440,585 \\
\dim BMW(E_7) &= 439,670,025 \\
\dim BMW(E_8) &= 53,328,069,225 \\
\end{align*}
\]

$n!! = \text{the number of matchings of } 2n \text{ points } = 1 \cdot 3 \cdots (2n - 1)$
Contents

• A
• B
• D
• E
Part A

- braid group $A$
- Kauffman algebra
- braid group representations
Braid group $A(A_{n-1})$ on $n$ strands

The generators $g_1$ and $g_2$.  

The noncommuting braid relation

\[ g_1 g_2 g_1 = g_2 g_1 g_2 \]
The braid group algebra

- An infinite-dimensional algebra
- Finite-dimensional quotients?
- The symmetric group algebra
  by ignoring crossing details
- Finite-dimensional quotients containing
  the braid group?
Extending the braid group algebra with idempotents

The generators $e_1$ and $e_2$. 
The idempotent and the circle

\[ x^{-1}e_i \text{ is an idempotent.} \]
The self-intersection

\[ l - 1 \]

Straightening at a cost of \( l^{-1} \).
Relation ensued

\[ e_1 g_2 e_1 = le_1 \]
The left $A$-module generated by

$$e_1e_3e_5\cdots e_{2t-1}$$

- each tangle has $\geq t$ bottom strands
- more bottom strands may arise
- quotient out tangles with more than $t$ bottom strands
- modules are infinite-dimensional
The skein relation

\[ g_i + m = g_i^{-1} + me_i \]
The skein relation

\[ g_i + m = g_i^{-1} + m e_i \]
equivalently:

\[ e_i = l m^{-1}(g_i^2 + m g_i - 1) \]
Mod out the skein relation

- \( x = 1 - (l - l^{-1})/m \) in \( \mathbb{Q}(l, m) \)
- gives: \textbf{Kauffman algebra} on \( n \) strands
- dimension(Kauffman algebra) at least \( n!! \)
- the \textbf{Brauer algebra} is obtained by putting \( l = 1 \), so \( m = 0 \)

\[ n!! = \text{the number of matchings of } 2n \text{ points} = 1 \cdot 3 \cdot \ldots (2n - 1) \]
Summary of Kauffman algebra over $\mathbb{Q}(l, m)$

generators $g_1, \ldots, g_n$ and $e_1, \ldots, e_n$

relations found

(braid1) $g_i g_j = g_j g_i$ when $i \not\sim j$,

(braid2) $g_i g_j g_i = g_j g_i g_j$ when $i \sim j$,

(skein) $m e_i = l(g_i^2 + mg_i - 1)$ for all $i$,

(self-intersection1) $g_i e_i = l^{-1} e_i$ for all $i$,

(self-intersection2) $e_i g_j e_i = l e_i$ when $i \sim j$. 
Theorem \((M = A_{n-1})\)

- Morphism \(BMW \rightarrow\) Kauffman algebra is an isomorphism.
- \(I_{p}/I_{p+1}\) is a matrix algebra of size over \(\text{Hecke}(A_{n-1-2p})\).
- \(BMW\) is a sum of matrix algebras and has dimension

\[
\sum_{p=0}^{\lfloor n/2 \rfloor} \left( \frac{n!}{2^p p!(n-2p)!} \right)^2 \cdot (n-2p)! = n!!
\]


\(n!! =\) the number of matchings of \(2n\) points \(= 1 \cdot 3 \cdot \cdots (2n - 1)\)
BMW structure in terms of roots

- $I_p/I_{p+1}$ has a basis consisting of triples $B, w, B'$ with
  - $B, B'$ sets of $p$ orthogonal positive roots
  - $w \in W(C)$ with $C = A_{n-1-2p}$
- top/bottom strand $\{i, j\} \leftrightarrow \text{root } \epsilon_i - \epsilon_j$
- for given $p$, all $B$ are in a single $W(A_{n-1})$ orbit $\mathcal{B}$
- triples $\leftrightarrow$ pictures in Brauer algebra
**BMW structure for other** $M$

| $M$     | $|B|$            | $|B|$                          | $Y$       | $C$      | $N_W(B)$                              |
|---------|-----------------|-------------------------------|-----------|----------|---------------------------------------|
| $A_n$   | $t$             | $(n+1)!$                       | $A_{n-2t}$| $A_{n-2t}$| $2^t \text{ Sym}_t \text{ Sym}_{n+1-2t}$|
| $D_n$   | $t$             | $t!(n-2t)!$                    | $A_1^t D_{n-2t}$ | $A_1 D_{n-2t}$ | $2^{2t} \text{ Sym}_t W(D_{n-2t})$ |
| $D_n$   | $2t$            | $n!$                          | $D_{n-2t}$ | $A_{n-2t-1}$ | $2^{2t} W(B_t) W(D_{n-2t})$            |
| $E_6$   | 1               | 36                            | $\emptyset$ | $\emptyset$ | $2 \text{ Sym}_6$                      |
| $E_6$   | 2               | 270                           | $A_3$      | $A_2$    | $2^{2+1} \text{ Sym}_4$               |
| $E_6$   | 4               | 135                           | $\emptyset$ | $\emptyset$ | $2^4 \text{ Sym}_4$                   |
| $E_7$   | 1               | 63                            | $A_5$      | $A_5$    | $2W(D_6)$                             |
| $E_7$   | 2               | 945                           | $A_3$      | $A_2$    | $2^{2+1+1} W(D_4)$                    |
| $E_7$   | 3               | 315                           | $D_4$      | $A_2$    | $2^3 \text{ Sym}_3 W(D_4)$            |
| $E_7$   | 4               | 945                           | $A_1^3$    | $A_1$    | $2^{4+3} \text{ Sym}_4$              |
| $E_7$   | 7               | 135                           | $\emptyset$ | $\emptyset$ | $2^7 L(3,2)$                         |
| $E_8$   | 1               | 120                           | $E_7$      | $E_7$    | $2W(E_7)$                             |
| $E_8$   | 2               | 3780                          | $D_6$      | $A_5$    | $2^{2+1} W(D_6)$                      |
| $E_8$   | 4               | 9450                          | $D_4$      | $A_2$    | $2^4 \text{ Sym}_3 W(D_4)$            |
| $E_8$   | 8               | 2025                          | $\emptyset$ | $\emptyset$ | $2^8+3 L(3,2)$                        |

$Y$ is root system orthogonal to $B$; $C$ maximal subsystem of $Y$ on nodes of $M$
Approach to proof

- Upper bound on dimension from root system analysis
- Lower bound for $D_n$ from orbifold tangles
- For $E_6$, $E_7$, $E_8$ from better understanding triples
Part B

Affine BMW and type $M = B_n$

- Replace knot space by solid torus
- Allcock: strands in presence of a pole
- Analog of Morton-Wassermann by Goodman & Hauschild
Extra generator crossing the pole
Part D

- Replace solid torus by orbifold
- Allcock: strands in presence of a pole of order 2
- Analog of Morton-Wassermann?
Orbifold motivation

\[(g_4^{-1}g_3g_4)g_2(g_4^{-1}g_3g_4) = g_2(g_4^{-1}g_3g_4)g_2\]
\[(g_4g_3g_4)g_3 = g_3(g_4g_3g_4)\]

works iff \(g_4^2 = 1\)
The half twist for a pole of order 2
Once more the pole of order 2
Pole of order 2, cont’d

No Reidemeister rules for the pole. For $D_2 = A_1A_1$ Goodman & Hauschild have two more relations.
Generators for $M = D_2 = A_1A_1$
Braid relation with pole

\[ \text{\includegraphics{braid_relation_with_pole.png}} \]
Self-intersections around the pole

\[ \begin{align*}
\text{\large{\includegraphics[width=0.3\textwidth]{image1}}} & = \text{\large{\includegraphics[width=0.3\textwidth]{image2}}} \\
\text{\large{\includegraphics[width=0.3\textwidth]{image3}}} & = \text{\large{\includegraphics[width=0.3\textwidth]{image4}}} \\
\text{\large{\includegraphics[width=0.3\textwidth]{image5}}} & = \text{\large{\includegraphics[width=0.3\textwidth]{image6}}}
\end{align*} \]
Self-intersections around the pole, cont’d
Corresponding idempotents for $M = D_2 = A_1A_1$
$e_1 e_2 = e_2 e_1$
Double closed loop relations

\[
\begin{align*}
\quad & = \quad x
\end{align*}
\]
Double closed loop relations, cont’d

\[ \text{Diagram} \]

\[ = x \]
Theorem \((M = D_n)\)

- Invariant bottom patterns of triples (pictures) are \(t\) horizontal strands, and the presence of a pole crossing.

- For fixed \(t\) the dimension of the representation
  - with a pole crossing is \(\frac{n!}{2^t!(n-2t)!}\) over Hecke\((A_{n-2t-1})\)
    (if \(n = 2t\) there are 2)
  - without a pole crossing is \(\frac{(n+1)!}{2^t!(n-2t+1)!}\) over Hecke\((A_1 D_{n-2t})\)

- root correspondence:
  - bottom strand from \(i\) to \(j\) without pole crossing \(\leftrightarrow \epsilon_i - \epsilon_j\)
  - bottom strand from \(i\) to \(j\) with pole crossing \(\leftrightarrow \epsilon_i + \epsilon_j\)
Part E

Transition to roots

- Representation bases: $W(M)$-orbits of sets of mutually orthogonal positive roots
- Only admissible orbits occur:
  no reflection moves exactly 3 roots of any set in the orbit
Theorem* ($M$ simply laced)

1. If $B$ is admissible, then there is a subdiagram $C_B$ of $M$ and an irreducible representation of $BMW$ over $\text{Hecke}(C_B)$ with basis $B$

2. $\dim(BMW) = \sum_{B \text{ admissible}} |B|^2 |W(C_B)|$
### Admissible $W$-orbits

| $M$  | $|B|$  | $|B|$ | $Y$     | $C$     | $N_W(B)$               |
|------|-------|-------|---------|---------|------------------------|
| $A_n$ | $t$   | $\frac{(n+1)!}{2^t t! (n-2t+1)!}$ | $A_{n-2t}$ | $A_{n-2t}$ | $2^t \text{Sym}_t \text{Sym}_{n+1-2t}$ |
| $D_n$ | $t$   | $\frac{n!}{t! (n-2t)!}$ | $A_1^t D_{n-2t}$ | $A_1 D_{n-2t}$ | $2^{2t} \text{Sym}_t W(D_{n-2t})$ |
|       | $2t$  | $\frac{n!}{2^t t! (n-2t)!}$ | $D_{n-2t}$ | $A_{n-2t-1}$ | $2^{2t} W(B_t) W(D_{n-2t})$ |
| $E_6$ | 1     | 36    | $A_5$   | $A_5$   | $2 \text{Sym}_6$       |
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|       | 2     | 945   | $A_1 D_4$ | $A_1 D_4$ | $2^{2+1+1} W(D_4)$     |
|       | 3     | 315   | $D_4$   | $A_2$   | $2^3 \text{Sym}_3 W(D_4)$ |
|       | 4     | 945   | $A_1^3$ | $A_1$   | $2^4+3 \text{Sym}_4$    |
|       | 7     | 135   | $\emptyset$ | $\emptyset$ | $2^7 L(3, 2)$       |
| $E_8$ | 1     | 120   | $E_7$   | $E_7$   | $2 W(E_7)$             |
|       | 2     | 3780  | $D_6$   | $A_5$   | $2^{2+1} W(D_6)$       |
|       | 4     | 9450  | $D_4$   | $A_2$   | $2^4 \text{Sym}_3 W(D_4)$ |
|       | 8     | 2025  | $\emptyset$ | $\emptyset$ | $2^8+3 L(3, 2)$      |
Theorem*

\(BMW(M)\) is finite-dimensional \iff connected components of \(M\) in \(\{A_n, D_n, E_6, E_7, E_8 \mid n \in \mathbb{N}\}\).

\[
\begin{align*}
\dim BMW(A_n) &= (n + 1)!! \\
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\end{align*}
\]

\(n!! = \text{the number of matchings of } 2n \text{ points} = 1 \cdot 3 \cdots (2n-1)\)
Thanks
Correspondence roots and horizontal strands

For $A_{n-1}$

- bottom strand $e_i \leftrightarrow \epsilon_i - \epsilon_{i+1}$
- bottom strand from $i$ to $j \leftrightarrow \epsilon_i - \epsilon_j$

For $D_n$

- bottom strand from $i$ to $j$ without pole crossing $\leftrightarrow \epsilon_i - \epsilon_j$
- bottom strand from $i$ to $j$ with pole crossing $\leftrightarrow \epsilon_i + \epsilon_j$