# What is wave speed?

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## ABSTRACT

The concept of "wave speed" is at the heart of most descriptions of wave motion, whether qualitative or quantitative. However, it is argued in this paper that the term wave speed is interpreted differently by analysts using different mathematical methods of representing wave motion. It is shown that, whereas the implications of the inconsistency are often insignificant, that outcome is not guaranteed. Furthermore, even if the different usages never had any consequences for practical predictions of wave behaviour, the ambiguous use of terminology inevitably creates communication barriers between disciplines. The discussion begins with a description of simple waves as observed by lay persons and their interpretations have a strong influence on the assessment of the alternative mathematical approaches. Overall, in the matter of wave speed, it is shown that lay persons' perspectives coincide more closely with frequency-domain methods than with MOC methods.

## NOMENCLATURE

| | |
|---|---|
| $a$ | phase speed in the frequency domain |
| $c$ | speed of sound in stationary fluid |
| $c_0$ | constant speed of sound |
| $C_{RHS}$ | RHS terms in the continuity equation |
| $D$ | pipe diameter |
| $f$ | frequency |
| i | imaginary unit |
| $K$ | bulk modulus |
| $L$ | pipe length |
| LHS | left hand side |
| $M_{RHS}$ | RHS terms in the continuity equation |
| MOC | method of characteristics |
| $p$ | pressure |
| $\overline{p}$ | amplitude of pressure wave |
| RHS | right hand side |
| $T$ | wave period |
| $t$ | time co-ordinate |
| TMM | transfer matrix method |
| $U$ | mean velocity |
| $\bar{U}$ | amplitude of velocity wave |
| $x$ | distance co-ordinate |

*Greek symbols*

| | |
|---|---|
| $\delta$ | infinitesimal small quantity |
| $\Delta$ | finite interval or difference |
| $\varphi$ | typical term on RHS of MOC equation |
| $\kappa$ | wave number |
| $\lambda$ | wavelength |
| $\rho$ | fluid density |
| $\tau$ | shear stress |
| $\omega$ | angular frequency |

*Subscripts*

| | |
|---|---|
| A | actual |
| av | average value |
| L | left |
| R | right |
| w | wall |
| 0 | constant, initial, linearised value |

## 1 INTRODUCTION

The propagation of pressure waves is an important aspect of many practical phenomena involving unsteady flows. Indeed, a majority of the papers presented at the conference series on Pressure Surges refer to pressure waves. Often, the most important characteristics of such waves are their amplitudes and speed of travel. In some cases, account must also be taken of wavelength and frequency.

External observers might reasonably assume that analysts use a common terminology when describing key phenomena related to pressure surge. In reality, however, this practice is less widespread than these observers would expect. This paper focusses on one particular property that is widely used ambiguously by analysts, namely "wave speed". It is not suggested that the ambiguities are a source of widespread error in prediction methodologies or in the interpretation of physical measurements. However, the authors know from direct personal experience that they can be a source of confusion that acts as a powerful communication barrier when, for example, a person experienced in time-domain interpretations attempts to understand frequency-domain methodologies (or vice-versa).

Waves have a gregarious character; they are rarely seen alone and, more often than not, they congregate in disparate groups with a wide range of amplitudes and frequencies. Nevertheless, for obvious reasons, much of the terminology in widespread use is related to individual waves – and this is one cause of the confusion that can arise. In fact, although this paper focusses on "wave speed", an even more obvious confusion arises in the interpretation of the word "wave" itself. Does this term necessarily imply the existence of a peak and a trough (as in a sine wave for instance) or can it include an isolate monotonic change (as in classical water-hammer following rapid valve closure)? The answers to questions such as this might depend upon the context in which the questions are raised. Indeed, it might even be difficult to reach a consensus on what the answer should be. Consider, for example, discussions between groups of researchers specialising in time-domain interpretations of wave mechanics and groups specialising in frequency-domain approaches. These two groups are attempting to describe the world from very different points of view, as can be illustrated by considering the validity of the following broad statements:

***Statement-1***: *In frequency-domain analyses, a step is typically interpreted as an infinite series of sine waves.*

***Statement-2***: *In time-domain analyses, a sine wave is typically interpreted as an infinite series of steps.*

These statements are not offered as expressions of universal truths. Nevertheless, the underlying message is surely true, namely that the mind-sets of the two broad groups are very different indeed. It is not surprising that this has consequences for the mathematical methodologies that are developed to describe physical behaviour in general – and wave propagation in particular. Differences also exist between definitions used *within* the individual groups of course, but these are less striking than generic differences between the broad groups.

The most important purpose of this paper is simply to draw attention to generic differences between the most natural interpretations of "wave speed" in time-domain and frequency-domain analyses. A secondary purpose is to explain the reasons for the differences. Less attention is paid to illustrating the *consequences* of the differences.

This is partly because the consequences for practical applications are unlikely to be important and partly because it is more useful to reduce the risk of confusion than to develop methods of allowing for it.

### 1.1 Scope of paper

The remainder of the paper begins with a discussion of wave speed from a physical standpoint. This is necessarily qualitative, but it provides a convenient background for the more detailed discussion that follows. In Section 3, the phenomenon is explored from a time-domain perspective, using the Method of Characteristics (MOC) to provide a specific focus, and Section 4 repeats the exercise from a frequency-domain perspective. In both of these Sections, the discussion utilises the introductory physical description. Section 5 highlights some of the differences between the time-domain and frequency-domain outcomes from an academic standpoint and it also includes more practical complications that are expressly excluded from the main body of the paper. For example, no attention is paid in the following development to the influence of pipe properties on wave propagation. In reality, waves propagate simultaneously in pipe walls and in the contained fluid – and they do so at different speeds and in an interactive manner. Allowances are made in most practical analyses for complications such as this, but their inclusion in the main body of this paper would be an unnecessary and unhelpful distraction from the main purpose of the discussion.

## 2 PHYSICAL INTERPRETATION OF WAVE SPEED

The first waves with which many of us became aware are free-surface waves (Fig 1), typically enjoyed at the seaside or a lakeside. Playing in such waves is one of the joys of many happy childhoods. As youngsters, we simply delight in the beauty of the phenomenon, without feeling any need to "understand" it. Indeed, many persons never need to move beyond this highly pleasurable phase. Suppose, however, that a stimulating school teacher decides to use waves as the basis of a challenging project of exploration for children. Further suppose that, at some stage, the project requires the children to measure the speed with which the waves travel.

As a starter project, the teacher might take the class to a local pond on a calm day and create waves by tossing a small stone into the water to create circular ripples spreading radially outwards. In addition to providing an enjoyable learning environment, this is quite a good way of helping children to recognise that although the waves spread out in their pretty circles, the water is staying put (except for small local movements). It also shows that the waves do not lose energy rapidly; a single stone causes ripples that persist for a considerable time and spread out for a considerable distance.

To measure the wave speed, the children will need to devise a means of estimating the distance travelled by a ripple in a fixed time or, perhaps more likely, the time taken for a ripple to travel a fixed distance. Some ingenuity will be needed to achieve this at the pond itself so perhaps a wise teacher will also make use of video footage that can be studied relatively easily in the classroom. An introduction to practical difficulties such as providing recording suitable length scales and time scales would be an additional educational benefit for the children.

The next phase of the project might involve a trip to the seaside, either for the whole class or just for the teacher. This will introduce a number of complications related to wave motion, one of which is that the speed of travel of waves will vary as they approach the beach and travel up it. To measure how the speed varies, the children could track the motion of wave peaks. It would even be possible to provide some height markers to enable rough estimates to be made of the relationship between speed and depth.

Of course, the measurements at the beach will not be anywhere near as accurate as can be obtained in a controlled laboratory environment. Nevertheless, the methodology described above is intuitively correct. It fits nicely with an intelligent lay person's understanding of the phenomenon. However, it contains an important assumption, namely that the speed of propagation of the wave peak can be interpreted as the true "wave speed". Is this also what you (the reader) mean by wave speed? Is it also what is meant by wave speed in your preferred type of theoretical analysis of waves? These questions are addressed in Sections 3 and 4 from the perspectives of time-domain and frequency domain analysts respectively.

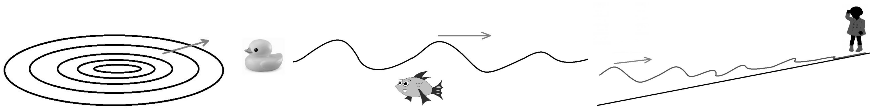

**Fig 1 (a) Ripples in a pond, (b) waves in the sea, (c) waves on a beach.**

## 3 TIME-DOMAIN ANALYSIS

The Method of Characteristics is now used to illustrate the interpretation of wave speed in an analytical time-domain analysis. It is not claimed to be representative of all time-domain approaches, but it enables certain features to be pinpointed easily. Also, it is widely used in water-hammer simulations by researchers and by practising engineers.

The continuity and momentum equations for unsteady liquid flow in a rigid-walled pipe of uniform cross-section can be expressed as

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + \rho\frac{\partial U}{\partial x} = C_{RHS} \tag{3.1}$$

$$\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = M_{RHS} \tag{3.2}$$

in which $p$, $\rho$ & $U$ denote the pressure, mean velocity & density and $x,t$ are space and time co-ordinates. The terms $C_{\mathrm{RHS}}$ and $M_{\mathrm{RHS}}$ depend upon the nature of the particular case under study. They enable allowance to be made for, say, mass flow sources, skin friction and body forces. For instance, if the only such complication is skin friction and this is modelled using quasi-steady relationships, $C_{\mathrm{RHS}} = 0$ and

$$M_{RHS} = \frac{4\tau_{\mathrm{w}}}{D} \tag{3.3}$$

where $\tau_{\mathrm{w}}$ is the wall shear stress and $D$ is the pipe diameter. By introducing a speed of sound $c$ satisfying

$$\frac{dp}{d\rho} = \frac{K}{\rho} = c^2 \tag{3.4}$$

in which $K$ denotes the bulk modulus of the liquid, the continuity and momentum equations can be combined and expressed in the characteristic compatibility form

$$\frac{dp}{dt} \pm \rho c \frac{dU}{dt} = c^2 C_{RHS} \pm c M_{RHS} \;\; \coloneqq \;\; \varphi, \text{say} \qquad (3.5a,b)$$

These equations are applicable only in the characteristic directions

$$\frac{dx}{dt} = U \pm c \qquad (3.6a,b)$$

### 3.1 $\Delta U = \delta$, $RHS = 0$

Consider first the special case of flows for which the right hand side terms in the continuity and momentum equations may be neglected and all fluctuations $\Delta U$ are small amplitude (i.e. sufficiently small to justify neglecting consequential variations in coefficients of differential terms in the above equations). These approximations might be acceptable in studies of the propagation of audible sound, for instance. They are a useful starting point for a critical assessment of the interpretation of wave speed.

For this case, the right hand sides of the MOC compatibility equations 3.5 are zero and the coefficient $\rho c$ on the left hand side may be regarded as constant and equal to its initial value $\rho_0 c_0$. The equations can then be integrated exactly without recourse to finite difference approximations. Assuming that the initial velocity is uniform, the gradients of the characteristics are equal to $U_0 \pm c_0$ at all locations $\{x,t\}$. Relative to axes moving with the background uniform flow ($U_0$), every part of any particular wave/disturbance will travel at the initial speed of sound $c_0$ and its amplitude will not change. It follows that no ambiguity will arise if the speed of propagation of the crest of a wave is used as a measure of the overall wave speed. Relative to the flow, both upstream-moving and downstream-moving waves will propagate uniformly at the speed $c_0$. Furthermore, neither the amplitude nor the shape of pressure disturbances will vary as they propagate.

This behaviour is independent of whether the pressure is increasing or decreasing and it is also independent of whether the waves are propagating upstream or downstream (relative to the chosen axes). Furthermore, waves travelling in opposite directions along the pipe will pass one another without distortion and without change of speed. During the crossing period, the waves will sum linearly and, thereafter, no lasting effects of the crossing period will exist. Of course, the assumptions on which this Section is based represent only an asymptotic state in the realm of physically possible conditions. Nevertheless, they provide a good approximation to some real phenomena, especially in the field of acoustics.

### 3.2 $\Delta U > \delta$, $RHS = 0$

Now consider a slightly more general case, still assuming that the right hand side terms in the equations of continuity and momentum may be neglected, but allowing for finite amplitude disturbances. This complicates the outcome in at least three distinct ways. These are now illustrated for the special case of a positive solitary wave (Fig 2).

First, the gradients of the characteristics (i.e. $U \pm c$) will vary with position along any particular wave – because both the fluid velocity $U$ and the speed of sound $c$ will vary along it (NB: $c$ depends on $K$ and $\rho$, both of which vary with pressure). In practice, high-pressure parts of the wave will travel faster than low-pressure parts and, as a consequence, the shape of the wave will evolve continually as it propagates downstream (eventually becoming a shock front). Also, the fastest-moving part of the wave is at the point of greatest pressure, namely at its crest. Therefore, the lay-person's intuitive interpretation of wave speed presented above now applies only to the crest itself, not to the whole wave.


**Fig 2 Inertial propagation of a pressure pulse.**

Second, in addition to causing distortion and the associated variation of wave speed along a wave, finite-amplitude effects influence the behaviour of waves crossing one another in opposite directions. During the crossing period, the local wave speeds ($U \pm c$) at any location at any instant will be determined by the instantaneous velocity and the speed of sound, both of which will differ from the values applicable to either wave alone. In general, therefore, wave speeds during crossing will differ from those before and after crossing and this will influence overall average speeds of propagation of the waves along the pipe.

Third, a further feature of finite amplitude waves arises from the fact that the product $\rho c$ varies along them. Because of this, the ratio $\Delta p/\Delta U$ implied by Eqs 3.5 must also vary. It follows that the shape of the overall wave expressed as $p\{x\}$ at any instant will differ from the shape expressed as $U\{x\}$. For example, if the pressure distribution at some instant is exactly sinusoidal, then the corresponding velocity distribution cannot also be exactly sinusoidal. Likewise, if the instantaneous pressure distribution is sinusoidal, then the pressure variation $p\{t\}$ at a fixed location crossed by the wave cannot be sinusoidal.

In many applications of practical importance, all of these finite amplitude effects will be small. Nevertheless, each can be important in special applications. For the purposes of this paper, it is instructive to note that all of them can be reproduced with reasonable accuracy in appropriately designed MOC analyses. This requires suitable choices of grid structures and care in the selection/assessment of grid sizes, but it is possible in principle.

### 3.3 $\Delta U = \delta$, $RHS \neq 0$

Next consider cases where one or both of $C_{RHS}$ and $M_{RHS}$ is non-zero and, to avoid unnecessary complication, initially restrict consideration to small amplitude waves. This enables the gradients of the characteristic lines to be approximated by $U_0 \pm c_0$ and so the waves are not influenced by the inertial distortion discussed in the preceding Section. Furthermore, the coefficient of $\mathrm{d}U/\mathrm{d}t$ on the left hand side of the MOC equations 3.5 may be treated as constant and equal to $\rho_0 c_0$. Notwithstanding these simplifications, however, the existence of non-zero values of $C_{RHS}$ and/or $M_{RHS}$ changes the MOC equations and, as a consequence, it changes the predicted shapes of waves as they propagate along a pipe (compare Fig 3 with Fig 2). The dominant difference is dispersive (i.e. a re-distribution of energy) although the process will usually also involve some dissipation (conversion of mechanical energy to internal energy).

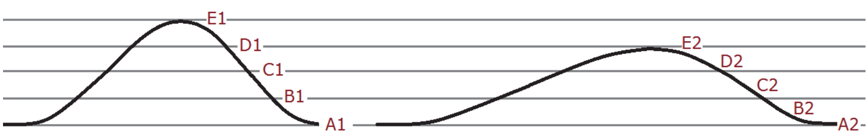


**Fig 3 Inertial and dispersive propagation of a pressure pulse.**

In MOC analyses, such terms are usually integrated numerically, typically as the product of an instantaneous value and the integration time step. For example, denoting a typical RHS term as $\varphi$, and using the notation shown in Fig 4, the integration along a typical characteristic LA might be approximated as

$$\int_L^A \varphi dt \approx \varphi_{\mathrm{av}} \Delta t \tag{3.7}$$

in which $\Delta t = (t_A - t_L)$.

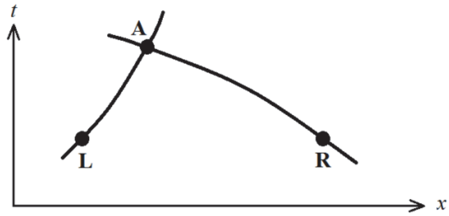


**Fig 4 Paths of characteristics in {*x*,*t*} space.**

The consequences of the RHS terms on the predicted pressures and velocities depend upon the nature of the parameter typified by $\varphi$ and upon the chosen method of approximating its average value $\varphi_{\mathrm{av}}$. In special cases, $\varphi$ might be a constant, but it is more usually expressed as a function of the coordinates $\{x,t\}$ or of the flow parameters $\{U,p\}$ or of both of these. In the cases where it is a function of the flow-variables, obvious ways of approximating $\varphi_{\mathrm{av}}$ include, for example

$$\varphi_{\mathrm{av}} \approx \varphi_{\mathrm{L}} \tag{3.8a}$$

$$\varphi_{\mathrm{av}} \approx \varphi_{\mathrm{A}} \tag{3.8b}$$

$$\varphi_{\mathrm{av}} \approx ½(\varphi_{\mathrm{L}} + \varphi_{\mathrm{A}}) \tag{3.8c}$$

The particular choices are inherently arbitrary and will be influenced by user preferences as well as by issues related to accuracy, stability, linearity and ease of programming. Commonly, the end results will be almost identical provided that sufficiently small grid sizes are chosen. Hence, when CPU time is of little consequence, the simple approach typified by Eq 3.8a coupled with a fine grid size might be the most suitable. There would be little point in optimising computational usage at the expense of excessive human time devoted to developing and checking nominally more accurate approximations. In contrast, when CPU time is relatively scarce (e.g. when hundreds or thousands of simulations are needed in optimisation analyses), the use of higher order approximations for $\varphi_{\mathrm{av}}$ will be advantageous because it improves the accuracy obtained using coarser numerical grids.

Irrespective of the nature of the chosen approximation, the RHS terms will cause the solution at any particular grid point A to differ from that which would have been obtained with RHS = 0 for the same given values at the points L and R (except in trivial cases where $\varphi_{\mathrm{L}} = \varphi_{\mathrm{R}} = \varphi_{\mathrm{A}} = 0$). This behaviour will exist at all points along the wave and so, even with identical initial conditions, the subsequent RHS = 0 and RHS $\neq$ 0 conditions will differ. Furthermore, it is possible (indeed common) for the difference to be such that the locus of peak pressure (i.e. the nominal wave crest) will not coincide with a characteristic line. This is an example of dispersive behaviour. For present purposes, the consequence of greatest significance is that the paths of MOC characteristics will rarely, if ever, correspond exactly with the most natural physical interpretation of wave speed.

To avoid unnecessary complexity, the above description is only qualitative. For completeness, however, note that a particular instance is discussed in more detail in a twin paper by Vardy & Tijsseling (2015). In that paper, it is shown that different approximations to RHS terms will cause different effective wave speeds.

### 3.4 $\Delta U > \delta$, $RHS \neq 0$

The final case for consideration is a combination of the two complications considered above, namely finite amplitudes and RHS ≠ 0. This general (and common) case includes all of the complications discussed in the Sections 3.2 & 3.3 and it will also include some interactions between the two. Nevertheless, from a conceptual standpoint, it does not create any new uncertainties in the search for a meaning of "wave speed". Accordingly, this case is not discussed explicitly herein.

### 3.5 Comparison of MOC wave speed and physical wave speed

Having concluded that MOC analyses will predict changes in wave shapes during propagation along a pipe, it is appropriate to consider when these will be truly representative of the changes that would be observed by experimenters. This matter can be addressed in a step-by-step manner. First, however, it is useful to simplify the discussion by making two stipulations, namely:

*(i)* the measured data are exactly correct;
*(ii)* the chosen analytical equations are a true representation of the physics.

In reality, neither of these stipulations is achievable. Nevertheless, they are justified herein because errors due to these causes are irrelevant to the central purposes of the present paper.

Subject to these stipulations, we may conclude:

*(i)* For small amplitude waves and ideal conditions represented by RHS = 0, the wave speed is uniform and constant. Relative to a uniform background flow, the wave speed is equal to the speed of sound $c_0$. In these ideal asymptotic conditions, waves will propagate without change of shape. This behaviour can be reproduced correctly in MOC analyses. It never exists exactly in physics, but close approximations are possible.

*(ii)* For finite amplitude waves and ideal conditions represented by RHS = 0, the wave speed varies in space and time (because both $U$ and $c$ vary). At any position $\{x,t\}$, the local wave speed is $U \pm c$, namely the gradients of the characteristics. This condition can be reproduced correctly in MOC analyses that are able to track the characteristics correctly. [NB: Numerical errors in tracking MOC characteristic lines have not been discussed explicitly above]. Again, the condition never exists exactly in physics, but close approximations are possible.

*(iii)* For all cases with RHS ≠ 0, MOC predicts changes of wave shape that do not arise from inertial behaviour. In almost all practical cases, it is not possible to determine exactly how accurately the RHS terms have been evaluated and how errors will accumulate. It is especially difficult to estimate the accuracy when some or all of the RHS terms are nonlinear. In principle, however, the uncertainty can be reduced almost indefinitely by the careful assessment of solutions obtained using successively smaller grid sizes. Accordingly, with sufficiently fine grid sizes, MOC should closely reproduce the physical behaviour.

*(iv)* Although not discussed above, it should be noted that the presentation has been restricted to one-dimensional wave propagation in regions of pipe between discrete boundaries. Further important complications arise when waves interact with boundaries, partly because of local three-dimensional effects and often also because the boundaries themselves respond to waves (e.g. Rudinger 1957). These complications are outwith the scope of this paper.

In summary, using time-domain methodology:

*(i)* wave speed can be defined usefully for purely inertial waves, but it varies with position along waves;

*(ii)* wave speed can be defined usefully, but not exactly, for waves that are influenced by non-inertial phenomena.

### 3.6 Implications for the meaning of wave speed

In numerous books and papers describing applications of MOC to wave propagation phenomena, it is stated that the paths of characteristic lines coincide with the paths of wave propagation. That is, the gradients of the characteristic lines are interpreted as equal to the local wave speed. However, it has been shown in Section 3.3 that, in nearly all real flows, the wave crest will not follow a characteristic line exactly and hence that this widespread interpretation is not strictly valid. Furthermore, it has been shown in Section 3.2 that even in cases where MOC could nominally follow a crest exactly, there is an ambiguity. That is, both physically and in the analysis, all parts of the pressure wave except the crest travel more slowly than the crest. Therefore, even the lay-person's intuitive interpretation is of limited value for definitive theoretical purposes.

It might be tempting for some readers to dismiss all of the above as the stuff of pedants. Indeed, for the purposes of water-hammer simulations, it is common to accept even greater approximations than those discussed above – specifically to approximate $U \pm c$ by $c_0$ irrespective of actual values of $U$ and $c$. Nothing herein should be interpreted as implying that this is bad practice. Indeed, the simplification has many advantages, especially as it can be difficult to estimate real values of the speed of sound in a pipe with high accuracy. Nevertheless, there are also applications when the non-ideal behaviour has a crucial influence on outcomes even with low-amplitude waves. One such case is the influence of ballast track on wave propagation in railway tunnels (e.g. Vardy & Brown 2000). Except in short tunnels, dispersive influences of the ballast prevent the potential radiation of sonic booms from the tunnel portals even though such booms are possible in the case of slab track tunnels. MOC analyses that approximate $U \pm c$ by $c_0$ are unable to predict either of these behaviours.

Given this background, if MOC water-hammer specialists are to communicate successfully with other MOC specialists, it will be desirable for all to use a common terminology. A seemingly inevitable conclusion from the above discussion is that great care will be needed if the term "wave speed" should be introduced in detailed discussions. It is a splendid descriptive term for use in qualitative discussion, but its detailed interpretation is likely to depend upon the background of the user so its use in a quantitative context carries a significant risk of ambiguity and misunderstanding.

## 4 FREQUENCY-DOMAIN ANALYSIS

Besides watching waves at the lake and beach, the children introduced in Section 2 can enjoy hearing and even exciting waves when playing with an acoustic guitar. Again, they might simply delight in the beauty of music, without feeling any need to "understand" it. Suppose that the school teacher decides to use a guitar for another challenging project and a skipping-rope to visualise things. Playing with the guitar strings will familiarise the children with (over)tones, octaves and harmonics. Playing with the rope will reveal sinusoidal-like shapes, the notion of reflections, and the visualisation of travelling and standing waves. It might be possible to stimulate the children (or at least some of them) to recognise that organ pipes are resonating air tubes. Likewise, they could be enthused by a (transparent) pipe filled with a column of water (Wylie 1999). By considering sound in general – and music in particular – it should be easy to help children to realise that vast numbers of waves can exist simultaneously and that the result can be hugely more satisfying than individual waves. Producing a constant monotone by blowing across the open end of an air-filled tube might give a wonderful feeling of achievement, but it is nevertheless less inspiring than listening to an orchestra.

All of these examples, and also the ripples on the surface of the pond, are examples of periodic wave phenomena. This type of wave is so common that many lay persons might instinctively feel that all "true" waves are periodic. For present purpose, though, a more important thought is that, since so many vibrations are periodic, it should make sense to develop mathematical means of describing these overall outcomes instead of focussing on individual points along the wave, as is done in MOC analyses. Indeed, this thought has had such a profound influence on scientific thinking that it underpins a large proportion of the whole field of wave mechanics, far beyond the borders of fluid mechanics as well as within them.

## 4.1 $\Delta U = \delta$, $RHS = 0$

In frequency-domain analyses, it is common to focus attention on small amplitude waves and to pay special attention to cases where the RHS terms in Eqs 3.1 and 3.2 have relatively simple mathematical forms. In the particular case of $C_{\mathrm{RHS}} = 0$ and $M_{\mathrm{RHS}} = 0$, the equations can be combined to give a well-known wave equation

$$\frac{\partial^2 p}{\partial t^2} - (U_0 \pm c_0)^2 \frac{\partial^2 p}{\partial x^2} = 0 \tag{4.1}$$

In this Section the constant velocity $U_0$ has been eliminated from this equation by choosing axes moving with the flow. Equations such as this have been studied in great depth by mathematicians and powerful tools have been developed to enable solutions to be expressed in standard forms. For this particular equation, the solutions are usually presented as a sum of harmonic functions.

A harmonic, small-amplitude, pressure wave propagating along a pipe in a plane-wave manner can be represented by:

$$p(x,t) = \overline{p} \sin 2\pi \left( \frac{t}{T} \mp \frac{x}{\lambda} \right) \tag{4.2}$$

where $\overline{p}$, $T$ and $\lambda$ are the amplitude, period and length of the wave and $p$ should strictly be written as $p - p_0$. In principle, the pipe and the periodic wave can be imagined to extend to infinity in both directions along the pipe. Indeed, for this expression to be a complete description of the conditions, it is necessary that there are no boundaries within the domain under consideration and also that there are no other waves. Assuming small amplitudes, ideal conditions corresponding to those considered in Section 3.1, and also assuming zero initial flow relative to the chosen axes, the wave will travel along the pipe at the speed of sound. In one wave period $T$, the whole wave will travel a distance equal to its wavelength $\lambda$ and so each crest, say, will occupy the position that was occupied by the adjacent crest at the beginning of the period. Accordingly, we may conclude that $c_0 = \lambda/T$.

In practical situations, it is common for the speed $c_0$ to be known a priori (because it is function of material properties, not of the flow). As a consequence, for any particular wavelength, the wave period can be deduced directly using $T = \lambda/c_0$. Alternatively, for any particular wave period, the wavelength can be deduced directly using $\lambda = Tc_0$. In practice, it is more common to focus on the wave frequency $f = 1/T$ or on its angular frequency $\omega = 2\pi f$ than on its period.

In general, it is possible for any number of harmonic waves to exist simultaneously in an infinite pipe, each with its own frequency and amplitude and each extending along the whole pipe. For the stated limitations of small amplitude and ideal conditions, all of the waves will travel independently, but each at the identical speed $c_0$. The overall result will be equivalent to a single, composite, periodic wave with a shape that differs from a simple sine wave, perhaps greatly so (e.g. Fig 5). If measurements are made in the pipe, it will be impossible to recognise the existence of the individual waves themselves. Instead, the measurements will be interpreted as a single wave of compound shape propagating along the pipe at a steady speed $c_0$.

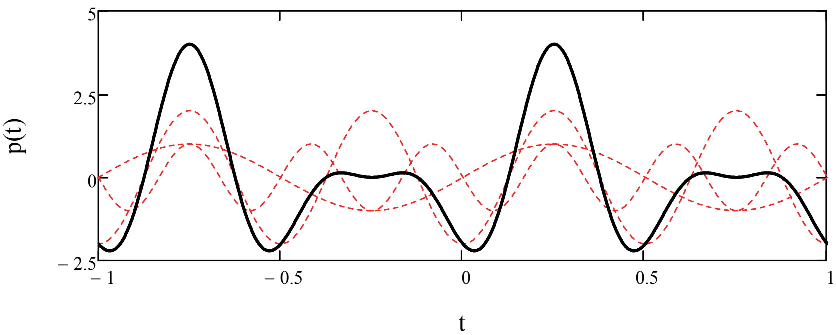


**Fig 5 $p(t)$ (continuous line) composed of $\sin(2\pi t)$, $2\sin(4\pi t - \pi/2)$ and $-\sin(6\pi t)$ (broken lines).**

In practical situations, observers see only the composite wave, not its individual components. Mathematically, however, it is useful to de-construct the wave into simple components that can be handled easily. The best known way of doing this is by use of Fourier series, which interpret the composite wave as a sum of harmonic waves, each with its own amplitude and frequency. In principle, the single compound wave may also be interpreted as a combination of multiple non-harmonic waves and, indeed, this is standard practice in the analysis of less simple flows than those considered in this Section.

In the above descriptions, it is implicitly assumed that each of the individual waves is travelling in the same direction. In that case, if an observer were to travel along the pipe at exactly the speed of sound $c_0$, he/she would be moving with the wave and would not notice any change in conditions. If, however, one or more of the waves is travelling in the opposite direction to any of the others, the resulting pressure patterns will be more difficult to discern because they will vary continuously. No matter which wave the wave-follower tracks, he/she will always see continual changes caused by the wave(s) moving in the opposite direction. This is, of course, the behaviour of which great advantage is taken in MOC.

It is possible to derive a transfer matrix that enables harmonic oscillations at any location ($x_1$) to be related explicitly to those at any other location ($x_2$). A suitable form presented by Tijsseling *et al* (2010) is

$$\begin{pmatrix} \bar{U}_2 \\ \bar{p}_2 \end{pmatrix} = \begin{pmatrix} \cos(\kappa\,\Delta x) & -\mathrm{i}\sin(\kappa\,\Delta x)\,/\,(\rho_0 c_0) \\ -\mathrm{i}\sin(\kappa\,\Delta x)\;(\rho_0 c_0) & \cos(\kappa\,\Delta x) \end{pmatrix} \begin{pmatrix} \bar{U}_1 \\ \bar{p}_1 \end{pmatrix} \tag{4.3}$$

in which $\Delta x = x_2 - x_1$ is the distance between the two locations, $\kappa = \omega/c_0$ is the wave number, $\mathrm{i}^2 = -1$ and the overbars denote the local amplitudes of the velocity and pressure oscillations. If the conditions at any particular point are known at some instant, then the conditions at all other points can be deduced from this transfer matrix. In a similar manner, the local ratio of amplitudes of pressure and velocity – i.e. the hydraulic impedance – can be derived. Transfer-matrix methods (TMM) are widely used in frequency-domain representations of waves (e.g. see textbooks by Constantinesco 1922, Wylie & Streeter 1993 and Chaudhry 2014). They are mentioned here because they illustrate a very important property of the frequency-domain approach, namely that the solutions are inherently global. In stark contrast with MOC, if the solution is known at *any* ($x$), it is also known at all other ($x$).

One special case merits attention, namely when the set of waves travelling in one direction is identical to the set travelling in the opposite direction. This is the situation that gives rise to the standing wave behaviour that can be created so readily by the children and their skipping rope. It is not necessary for the set of waves to be sinusoidal, but the superposition of one sine wave in each direction is the most commonly cited example, partly because it leads to a particularly simple outcome with stationary nodes and a simple harmonic oscillation at all other points. At nodes in the standing wave, a crest of the wave moving in one direction coincides with a trough of the wave moving in the other direction. Likewise, midway between these nodes, opposite moving crests arrive simultaneously and, half a wave period later, opposite-moving troughs arrive simultaneously. Mathematically, this behaviour can be represented by:

$$p(x,t) = 2\bar{p}\cos\left(2\pi\frac{x}{\lambda}\right)\sin\left(2\pi\frac{t}{T}\right) \tag{4.4}$$

### 4.2 $\Delta U > \delta$, $RHS = 0$

Now consider finite amplitude disturbances, but still assuming that the right hand side terms in the equations of continuity and momentum may be neglected. In this case, it is no longer possible to characterise a compound wave as an infinite series of sine waves because such summations inevitably lead to a non-changing overall waveform, whereas we saw in Section 3.2 that the actual waveform will evolve, eventually leading to the formation of a shock. Linear transform techniques cannot reproduce this behaviour. To quote Huang *et al* (1999, p. 446): *"We must break with the earlier paradigm of wave analysis, and emphasize again that Fourier analysis is not a good method for studying waves. The reasons are many: water waves are nonlinear; therefore, we should not expect to use a linear expansion and be able to represent it. With the Fourier expansion, the harmonics have only a mathematical significance, but no physical meaning. Furthermore, as the wave evolution is local, Fourier expansion simply cannot represent* *this non-stationary process. The only way the Fourier method can represent a local frequency change is through harmonics. But such a representation is no longer local."*

Advanced methods exist for the analysis of finite amplitude waves in the frequency domain. However, they are outwith the scope of this paper, which is designed to highlight contradictions inherent in the term "wave speed" in a manner that is readily accessible to a large proportion of potential readers, notably the conference delegates.

### 4.3 $\Delta U = \delta$, $RHS \neq 0$

Now return to the case considered in Section 3.3, namely small amplitude waves in the presence of source terms such as friction, pipe tapering, gravity and distributed leakage. For the small amplitude case, the Eqs 3.1 and 3.2 can be linearised by regarding all coefficients of derivative terms as constants, giving, with the help of Eq 3.4,

$$\frac{\partial p}{\partial t} + K_0 \frac{\partial U}{\partial x} = c_0^2 C_{RHS} \left\{ p, U, \frac{\partial p}{\partial t}, \frac{\partial p}{\partial x}, \frac{\partial U}{\partial t}, \frac{\partial U}{\partial x} \right\} \tag{4.5}$$

$$\frac{\partial p}{\partial x} + \rho_0 \frac{\partial U}{\partial t} = M_{RHS} \left\{ p, U, \frac{\partial p}{\partial t}, \frac{\partial p}{\partial x}, \frac{\partial U}{\partial t}, \frac{\partial U}{\partial x} \right\} \tag{4.6}$$

in which it is further assumed that the RHS terms are linear(ised) functions of $p$, $U$ and their first derivatives and, with no loss of generality, the chosen axes are moving with the flow (so that $U_0 = 0$).

Once again, it is possible to express solutions to the water-hammer equations in a harmonic form – that is, the component corresponding to any particular frequency again has the form of a sine wave that is travelling along the pipe at a steady speed. However, there is a major difference from the case presented in Section 4.1, namely that, in the presence of effects such as friction, this steady speed is smaller than the speed of sound $c_0$. From a physical point of view, the inertial/elastic disturbances (represented by the LHS terms in Eqs 4.5 & 4.6) still propagate at the speed $c_0$, but the constraining influences (represented by the RHS terms) have the effect of slowing the process. As a consequence, the relationship between the wavelength and the wave period no longer satisfies $\lambda/T = c_0$. Instead, it satisfies $\lambda/T = a < c_0$, where $a$ denotes the actual speed of propagation of the (single-frequency) wave. It is known as the phase speed (or phase velocity) of the wave. This is the propagation speed that would be measured by our school children using their method of tracking wave crests. That is, for this single-frequency wave, the phase velocity is what the lay person would regard as the wave speed.

From the preceding description, it is clear that the phase speed depends upon the relative importance of the inertial (LHS) terms and the non-inertial (RHS) terms in Eqs 4.5 & 4.6. This relationship depends upon the frequency of the wave and so the phase speed also depends upon the frequency. As an example, Fig 6 shows the dependence of phase speed on frequency already given by Kries (1892) for a particular case with $C_{RHS} = 0$ and $M_{RHS}$ representing linear friction. In fact, this is part of the solution of the well-known telegrapher's (transmission line) equation. It can be seen that the influence of friction is much greater at low frequencies than at higher ones. This is physically reasonable because, for any particular amplitude of the velocity oscillation, the amplitudes of the inertial terms in Eqs 4.5 & 4.6 will increase with frequency (NB: higher frequencies imply greater accelerations) whereas the amplitudes of the friction terms will not. For completeness, it is emphasized that Fig 6 is based on linear, quasi-steady friction.

Qualitatively similar behaviour might be expected with nonlinear, quasi-steady friction and qualitatively different behaviour is likely with frequency-dependent friction.

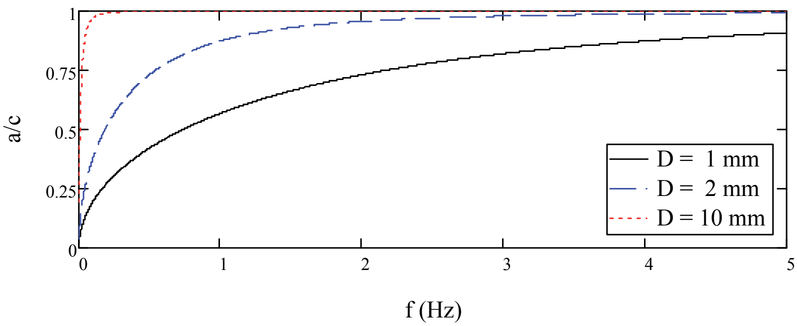


**Fig. 6 Frequency-dependent phase speed due to quasi-steady friction in one-dimensional laminar pipe flow according to Kries (1892) ($M_{RHS} \sim U$); formula (5) in Tijsseling & Anderson (2007) with kinematic viscosity $10^{-6}$ m$^2$/s.**

If two or more individual components exist simultaneously, the composite outcome will not appear as a single compound wave travelling at a fixed speed. Instead, each component will travel at its own speed and the faster ones will overtake the slower ones. The combined result will never be steady, but will always exhibit wave motion.

Consider a case when just two components exist and their frequencies are different. The maximum combined amplitude will occur when a crest of the faster moving wave coincides with a crest of the slower moving component. However, since the wavelengths will be different, the nearby crests will not coincide at the same instant. The distance between successive locations at which crests coincide will depend upon the relative wavelengths of the two components as well as upon their absolute values. This can cause a beat-like pattern in the composite result. Similarly, the time interval between any given crest coinciding with successive crests of the other component will depend upon the relative phase speeds. It is possible to imagine cases where an observer's attention would instinctively focus on the propagation of the beat rather than on the individual wave crests. In this case, the inferred wave speed would have a very different meaning. However, it is not likely that this would cause significant confusion because it would be obvious that individual crests are not travelling at the same speed as the beat. In studies of surface wave motion, attention is commonly drawn to this sort of behaviour. That is, individual waves can be seen propagating twice as fast as the group to which they "belong".

### 4.4 $\Delta U > \delta$, $RHS \neq 0$

In common with the discussion on time-domain methods, the general case of finite amplitude waves with RHS $\neq$ 0 is not discussed. An important reason for this is that, as explained in Section 4.2, the study of finite amplitude waves in the frequency domain requires advanced mathematical treatment. However, it is also true that the discussion would be unlikely to add anything of high significance for the purposes of this paper – other than what might be revealed by an expanded Section 4.2.

### 4.5 Comparison of phase velocity and physical wave speed

It has been shown in Section 4.3 that, for a small-amplitude, single-frequency wave, the phase speed is identical to the layperson's intuitive interpretation of wave speed. Such waves could be generated in laboratory experiments by small-amplitude excitation of a long, fluid-filled pipe at a constant frequency. If the excitation were at one end of the pipe, the amplitude of the resulting oscillations would reduce with increasing distance, but the waves would nevertheless propagate away from the source of excitation. Provided that suitable precautions are taken at the remote end of the pipe to minimise the amplitudes of any reflected waves, the principal wave motion would be clear and its speed would be equally reasonably be described as a phase speed or as a wave speed.

### 4.6 Implications for the meaning of wave speed

Assuming that the phase speed of an individual, single-frequency, small-amplitude disturbance is indeed considered to be the appropriate interpretation of the term "wave speed", it follows that the latter term, like the former, is meaningful only for single-frequency waves. No useful purpose would be served by attempting to broaden the definition to include waves with multiple-frequency components. Just as a mathematician would regard each component as having its own phase speed, so a layman would interpret the physical behaviour as comprising multiple waves propagating at different speeds.

## 5 CORRELATION OF TIME-DOMAIN AND FREQUENCY-DOMAIN ANALYSES

From the discussion presented in Sections 3 and 4, it is clear that analysts using the two different methods have different objectives. In the MOC time-domain approach, the focus is on *local* behaviour whereas in the frequency-domain approach, it is on *overall* behaviour. Furthermore, each method is inherently incapable of addressing the other viewpoint explicitly.

At any particular point $(x_1,t_1)$ the gradients of the characteristic lines considered in MOC analyses are the eigenvalues of the matrix of coefficients of differential terms in the wave equations and so are wholly independent of any non-differential RHS terms in the continuity and momentum equations. In the most common implementations, this means that the characteristics paths are curved only by the inertial components of the wave. The numerical solution process includes the RHS terms and so the calculated wave amplitude varies in the characteristic direction, as indeed it should. With sufficiently small integration steps and sufficiently small rounding errors, the method is inherently capable of reproducing overall outcomes with good accuracy. Nevertheless, in principle, it does so by analysing each bit of the overall wave in detail at every solution instant. Nothing inherent in the methodology itself is directly related to the overall form of individual solutions at a multiplicity of $(x,t)$ points. The closest that MOC comes internally to reflecting a lay person's interpretation of wave speed is its evaluation of eigenvalues – and, in principle, even this is done independently at every individual $(x,t)$. Perhaps the calculation of a global mean of all of the individual characteristic directions would provide a means of approximating to an overall wave speed in cases with RHS = 0, but even this approach would fail to miss the instantaneous influence of non-zero RHS terms (although it would capture subsequent consequences). For completeness, note that the overall outcome will also be influenced by the method of selecting the individual $(x,t)$ points – e.g. by whether a fixed grid or a natural grid is used.

In contrast, the frequency-domain approach implicitly seeks to describe the overall result at all $(x,\omega)$ simultaneously. Transfer matrix methods that effectively discretise the solution can be used for convenience, but the solution is inherently available over the whole domain. It is not necessary to isolate local behaviour because, for each frequency component, the whole wave propagates in unchanged form. That is, each part of it travels at the same speed. The overall behaviour of the composite wave is regarded as the sum of individual contributions of multiple single-frequency components. This makes the method well-suited to representing overall behaviour of small-amplitude waves, including allowing for the linear(ised) RHS terms. However, the requirement for each part of the overall wave to travel at the same speed is in conflict with the physical behaviour of large-amplitude waves. As a consequence, standard versions of the method are unable to represent phenomena such as wavefront steepening that can be of great importance in special circumstances, but are rarely of interest in typical water-hammer applications.

In a nutshell, for single-frequency waves:

*(i)* If the phenomenon under study would be described by lay persons as the steady wave propagation:
- the speed of propagation can reasonably be considered to be a "wave speed"
- the phase speed in frequency-domain analyses should be equal to this wave speed
- the gradients of MOC characteristic lines will not reflect dispersive influences on this wave speed

*(ii)* If the phenomenon under study would be described by lay persons as a progression of evolving waveforms:
- the inferred wave speed will depend upon the observers' points of reference on the waveform
- linear frequency-domain methods will be inapplicable
- MOC gradients will reflect the evolving behaviour, but will still not reflect dispersive influences

## 5.1 Other considerations

Nothing in this paper is considered by the authors to reflect an advancement of knowledge. Instead, the paper is intended to promote greater understanding, especially between proponents of alternative ways of interpreting wave motion – especially time-domain and frequency-domain methods. We expect that most readers, like ourselves, will need to think quite deeply before being confident that they fully understand the differences – and their implications.

To enable the issues discussed above to be explored in some depth, a large number of other contradictions between the two approaches have been avoided. More important, a large number of other ambiguities and uncertainties in the interpretation of "wave speed" have been omitted. Very short pointers to some of these are listed here.

***Influence of pipe properties on "effective" wave speed***

Because pipes are not rigid, it is common to define "effective" wave speeds that include allowance for pipe flexibility as well as fluid compressibility. As a first approximation, much of the paper remains applicable provided that the bulk modulus $K$ is re-interpreted as an effective bulk modulus. This is not possible, however, when simultaneous account needs to be taken of waves propagating primarily in the pipe walls.

***Inference of "wave speed" from measured time intervals between successive reflections in pipes***

It is quite common to see "wave speed" inferred from time intervals between successive peaks in water-hammer signals. At best, this is a measure of phase speed, not a direct measure of the speed of sound. Even if dispersive effects are assumed totally absent, an exact estimate of speed of sound would require allowance to be made for different propagation speeds in the two directions (i.e. $U+c$ & $U-c$). In fact, the existence of this difference is utilised in some methods of measuring fluid velocity.

***Influence of boundary movement on apparent wave speeds***
Time delays can be introduced when waves reflect at boundaries, especially at open ends. Phase speeds or sound speeds deduced without allowing for this effect can be significant in pipes of relatively small $L/D$.

***Identifying the "exact" leading edge of a step wave front (as in water-hammer)***
Wave speeds are sometimes deduced from the time taken for waves to travel between known points along a pipe. In such cases, high accuracy is possible only if the times of arrival of the two locations can be determined with sufficient accuracy. Cross-correlation techniques can reduce the uncertainty.

***Unsteady friction and nonlinear pipe properties, etc.***
The reliability of cross-correlation techniques reduces when the shape of a wave changes strongly during propagation. Phenomena such as unsteady friction have a greater influence on regions of rapid acceleration than on regions of gentle acceleration and hence they change wave shapes. Similar (often much stronger) distortion is caused by nonlinear pipe properties, whether elastic or not.

## 6 CONCLUSIONS

The authors' message is that the meaning of "wave speed" is not quite as obvious as would appear from the widespread use of the term in publications on water-hammer. In these, the term is typically used to represent the gradients of characteristic lines. However, it has been shown above that such an interpretation is strictly valid only in the absence of RHS terms in the underlying equations. Even in that case, this use of the term is ambiguous in analyses that account for large-amplitude wave behaviour.

When the RHS terms are not zero, even small-amplitude waves do not propagate at the speeds implied by the gradients of characteristic lines. If, on average, the RHS terms are small – as in many practical water-hammer applications – the differences will be small and no serious harm will result from ignoring them. However, when they are potentially large – as in water-hammer in laminar flow or low-Reynolds number turbulent flow – the use of the term wave speed to represent characteristic gradients will conflict with every-day interpretations of laypersons. Even more important for scientific development, it will create a powerful communication barrier between MOC specialists (often engineers) and frequency-domain specialists (often mathematicians). This barrier is real and it is strong, but it derives its strength primarily because, for most people, it is invisible. The authors know this only too well from personal experience in discussions together. Habits of a lifetime will now have to change.

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