Work and life of Piotr Szymański

K Urbanowicz
Department of Mechanical Engineering and Mechatronics
West Pomeranian University of Technology, Poland

A S Tijsseling
Department of Mathematics and Computer Science
Eindhoven University of Technology, The Netherlands

ABSTRACT

The work and life of Piotr Szymański is reviewed. He is the author of a classical paper on accelerating pipe flow, but is relatively unknown to the scientific community of today. The paper by Szymański is one of the first related to unsteady friction, clearly explaining the underlying physics and with rigorous mathematics. Exactly half a century after his death, the authors of this current paper would like to pay a modest tribute to the man and his achievements.

INTRODUCTION

In 1932 in the prestigious *Journal de Mathématiques Pures et Appliquées* a paper appeared with the title "*Quelques solutions exactes des équations de l'hydrodynamique du fluide visqueux dans le cas d'un tube cylindrique*". The author was Piotr Szymański from Varsovie. The paper became famous, but its author always remained anonymous. The translated title is "*Some exact solutions of the equations of hydrodynamics of viscous fluid in the case of a cylindrical tube*". The paper gives the analytical solution of laminar pipe flow accelerating from rest and as such forms a basis for the nowadays popular research topic "*unsteady friction*". It is our intention to inform the reader about the man behind the paper. Szymański had an interesting life in a turbulent time in a dangerous part of the world. He was deported twice, escaped from concentration camp, became ambassador of Poland, and ended up as full professor in Warsaw. The paper itself was in French and a summary by Szymański (translated into English by the authors) is given in Appendix B herein. The paper will be placed in its historical context, both regarding predecessors and successors. Related work by Szymański will be mentioned and his full biography is listed in Appendix A. In fact, his main contribution to science can be dated two years earlier than 1932: in 1930 he presented a condensed version of his1932 paper at an esteemed conference in Stockholm.

Fig 1 Professor Piotr Szymański.

UNSTEADY FRICTION

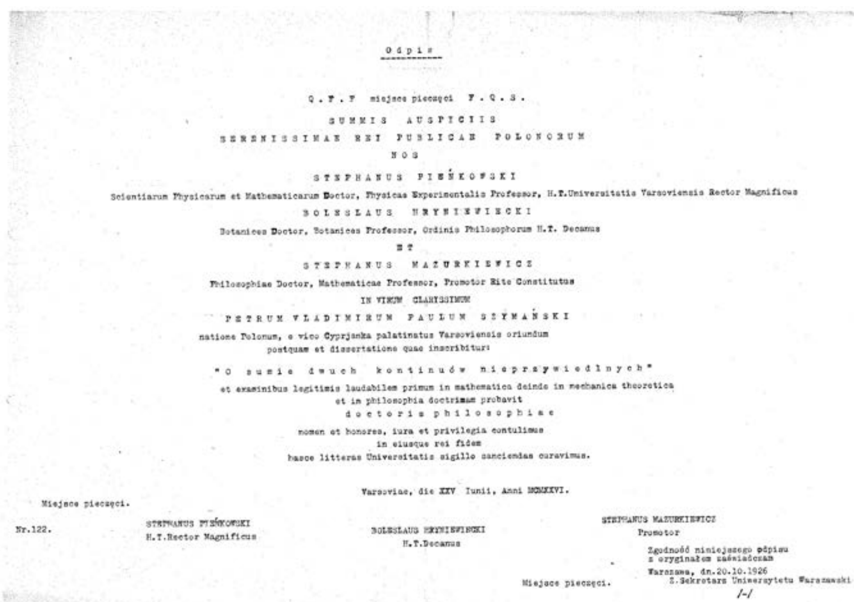
Unsteady friction (UF) in unsteady pipe flow refers to situations where a quasi-steady law is not sufficient to describe skin friction at the wall. The unsteady velocity profile is not parabolic as function of the radial coordinate *r*, but can take different shapes depending on the pressure gradient as function of time *t*. All early work concentrated on the calculation of velocity profiles in laminar flow. [Schönfeld (3) appears to be the first to consider turbulence in modelling UF.] The mathematical problem is one-dimensional in space, with – compared to waterhammer – the radial coordinate *r* instead of the axial coordinate *x*. For steady oscillatory flow driven by a sinusoidal pressure-gradient, analytical solutions were derived independently in the 1920s by Crandall (4), Grace (5) and Sexl (6), and by Gerbes (7), Fassò (8-9) and Womersley (10) in the 1950s. Important – but less well known – works preceding these investigations were by Witzig (11), Gromeka (12-13) and Boussinesq (14-15).

Szymański (1-2) (in French) considered starting of a flow due to a suddenly applied pressure gradient. He used Fourier-Bessel (Hankel) transforms to derive the analytical solution presented herein in the section "His Work". Gromeka (12) (in Russian) and Gerbes (13) (in German, and using Laplace transforms) derived the same solution. Gerbes also dealt with stopping of a flow and found that – in contrast to starting flow – the unsteady velocity profile keeps a parabola-like shape during deceleration to rest. Brereton (16) considered – besides pipe flow – unsteady flow between two parallel plates, and Brereton and Jiang (17) systematically calculated all relevant functional dependencies. Vardy and Brown (18-19) presented analytical solutions derived for the extra complication of time-dependent viscosity.

Laminar flow is of less significance for civil engineering practice, where nearly all flows are turbulent, but nowadays the subject becomes increasingly important because of micro- and nano-tubes conveying liquids. Via d'Souza and Oldenburger (20) and Zielke (21), the papers mentioned above also stood at the basis of UF in turbulent flow and waterhammer (22-23).

HIS LIFE

Piotr Szymański (Fig. 1) was born on 29 June 1900 in the village of Cypryjanka in the administrative district of Lipno (Poland). His father (Stefan) worked as a technician in the Chełmica sugar factory. During World War I his father was evacuated to the city of Berdyansk on the northern coast of the Sea of Azov (Ukraine). In that city Piotr Szymański spent his formative years (from 1915-1921). In 1918 he finished high school and started his studies in the Department of Mathematics and Physics at the University of Kharkov (Ukraine). Because it was (civil) war time, he could not study in a normal way and was forced to learn independently. In 1921 he went back to Poland and started mathematical studies in the Department of Philosophy at the University of Warsaw. He specialised in the theory of sets and topology. In 1926 he finished and defended his Ph.D. Thesis with the title "About the sum of two irreducible continua" under promotor Professor S. Mazurkiewicz (Fig. 2). In 1928, on the initiative of Professor C.M. Witoszyński, he gained a one-year scholarship in Paris to specialise in hydrodynamics and aerodynamics. In Paris he wrote his famous work on the hydrodynamics of viscous pipe flow, which was subsequently published in the *Journal de Mathématiques Pures et Appliquées* (1). This contribution to science has excited specialists ever since. The theoretical solutions presented in the article were experimentally validated shortly later in the University of Lille (the person who validated the solutions had a private letter communication with Piotr Szymański). After Paris, Szymański returned to Poland and took on positions at Warsaw Polytechnic.


Odpis

Q. F. F miejsce pieczęci F. Q. S.

SUMMIS AUSPICIIS

SERENISSIMAE REI PUBLICAE POLONORUM

NOS

STEPHANUS PIEŃKOWSKI

Scientiarum Physicarum et Mathematicarum Doctor, Physicae Experimentalis Professor, H.T.Universitatis Varsoviensis Rector Magnificus

BOLESLAUS HRYNIEWIECKI

Botanices Doctor, Botanices Professor, Ordinis Philosophorum H.T. Decanus

ET

STEPHANUS MAZURKIEWICZ

Philosophiae Doctor, Mathematicae Professor, Promotor Rite Constitutus

IN VIRUM CLARISSIMUM

PETRUM VLADIMIRUM PAULUM SZYMAŃSKI

natione Polonum, e vico Cyprjanka palatinatus Varsoviensis oriundum
postquam et dissertatione quae inscribitur:

"O sumie dwuch kontinuów nieprzywiedlnych"

et examinibus legitimis laudabilem primum in mathematica deinde in mechanica theoretica
et in philosophia doctrinam probavit

doctoris philosophiae

nomen et honores, iura et privilegia contulimus
in eiusque rei fidem
hasce litteras Universitatis sigillo sanciendas curavimus.

Varsoviae, die XXV Iunii, Anni MCMXXVI.

Miejsce pieczęci.

Nr.122.

STEPHANUS PIEŃKOWSKI
H.T.Rector Magnificus

BOLESLAUS HRYNIEWIECKI
H.T.Decanus

STEPHANUS MAZURKIEWICZ
Promotor

Zgodność niniejszego odpisu
z oryginałem zaświadczam
Warszawa, dn.20.10.1926
Z.Sekretarz Uniwersytetu Warszawski
/-/

Miejsce pieczęci.

Fig 2 Copy of Szymański's Ph.D. Diploma (1926).

During the German-Russian-Slovak invasion of Poland in September 1939, which marked the beginning of World War II, Szymański was deported to Romania and interned in a concentration camp in Tulcea. In October 1939, when transported to another camp, he was lucky enough to escape by running away from the train. He ended up in Cluj (Romania) (1939-1940) where he gave a series of lectures at Cluj University on the "theory of sets" (there was large interest in hearing them). From Cluj he moved to Craiova (Romania) where he became teacher at and director of the local Polish high school. He stayed in Romania during the entire war and wrote many articles published in Romanian journals (see App. A).

During his stay abroad he was socially and politically active. He became involved in the Association of Polish Patriots and was elected president of the Polish Democratic Union in Romania. This fact, and because he was highly educated and fluent in Russian, French and Romanian, and also understood English and German, led to an appointment within the Ministry of Foreign Affairs after his return to Poland in July 1945. An additional explanation is that he joined the Polish Communist Workers Party. He became diplomat in Denmark, Hungary and Romania and made it up to Polish Ambassador in Romania. His interests for science and didactics made him in 1949 return to Academia and Applied Mathematics. He ended his long career as Full Professor at Warsaw Polytechnic. His impressive Curriculum Vitae is summarised in Table 1.

He married to Halina Smolarkiewicz and had one son: Jan Kazimierz (born on 8 October 1928) who became an interior architect in Warsaw. He had a brother Jerzy who too was employed at Warsaw Polytechnic and a sister Helena Wojtulewska who worked as a clerk in cooperatives at Białystok. He died on 13 January 1965 in Warsaw probably due to problems with his heart.

Table 1 *Curriculum Vitae.*

From	**To**	**Institute**	**Position**	**City**
IX 1924	VIII 1925	Teachers Seminar	Teacher	Warsaw-Ursynow
1925	1928	Aerodynamic Institute – Warsaw Polytechnic	Scientific Worker	Warsaw
IX 1927	VIII 1928	Jan Zamoyski High School	Teacher	Warsaw
VIII 1928	V 1929	Advanced studies	Post-Doc	Paris
VIII 1929	VIII 1934	Aerodynamic Institute Warsaw Polytechnic	Research Fellow	Warsaw
X 1929	VIII 1939	Department of Mathematics Warsaw Polytechnic	Senior Assistant	Warsaw
X 1934	VIII 1939	School of Aviation Cadet Officers	Lecturer	Bydgoszcz Warsaw
I 1940	III 1945	Craiova High School	Teacher and Director	Craiova (Romania)
VI 1945	X 1945	Ministry of Foreign Affairs	Advisor	Warsaw
XI 1945	V 1946	Polish Embassy Copenhagen	Chargé d'Affaires	Copenhagen
VII 1946	VII 1947	Polish Embassy Budapest	Chargé d'Affaires	Budapest
VIII 1947	II 1949	Polish Embassy Bucharest	Ambassador	Bucharest
II 1949	V 1949	Ministry of Foreign Affairs	Advisor	Warsaw
V 1949	VIII 1950	Course for Railroad Engineers	Lecturer	Warsaw
V 1949	VII 1951	Ministry of (University and) Education	Assistant Manager	Warsaw
VII 1951	VI 1962	Warsaw Polytechnic – Dept. of Geodesy and Cartography	Assistant and Full Professor	Warsaw

HIS WORK

Szymański's much celebrated paper (1) [8] appeared in the *Journal de Mathématiques Pures et Appliquées* in1932, but was preceded by a fine and to-the-point summary (2) [6] presented at the *Troisième Congrès International de Mécanique Appliquée* held in 1930 in Stockholm. Renowned participants were Th. von Karman and E. Jouguet, the latter known from his early text on waterhammer (25). Szymański's other mathematical treatments of fluid mechanics were on Stokes flow [7], on tapered tubes [9] and on boundary layers [15].

His main contribution is the analytical solution describing the instantaneous acceleration of an initially stationary fluid. The Newtonian fluid is incompressible and contained in a cylindrical pipe of circular cross-section. The governing partial differential equation for the velocity profile $v(r,t)$ is linear:

$$\frac{\partial v}{\partial t} = C_1 + \tilde{\nu}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right), \quad v(r,0)=0\,,\ v(\frac{D}{2},t)=0\,,\ \left|v(0,t)\right| < \infty\,, \qquad (1)$$

where r is the radial coordinate, t is time, D is the pipe diameter, and $\tilde{\nu}$ is the kinematic viscosity. The constant $C_1 = \Delta P/(\rho L) > 0$ represents a suddenly applied acceleration with ΔP the corresponding pressure step, ρ is the mass density and L is the pipe length. The solution is in terms of Bessel functions (J) of the first kind of orders 0 and 1:

$$v(r,t) = 2V_\infty \left\{ \left[1-\left(\frac{2r}{D}\right)^2\right] - 8\sum_{n=1}^{\infty} \frac{\mathrm{J}_0\left(\lambda_n \frac{2r}{D}\right)}{\lambda_n^3\,\mathrm{J}_1\left(\lambda_n\right)}\,\mathrm{e}^{-\frac{1}{8}\lambda_n^2 C_3 t} \right\} \qquad (2)$$

with final (cross-sectional) average velocity $V_\infty = C_1/C_3 > 0$ and $C_3 = 32\tilde{\nu}/D^2 > 0$. The first 21 values of λ_n (n^{th} zero of J_0) are:

λ_1 = 2.40482555769577	λ_8 = 24.3524715307493	λ_{15} = 46.3411883716618
λ_2 = 5.52007811028631	λ_9 = 27.4934791320403	λ_{16} = 49.4826098973978
λ_3 = 8.65372791291101	λ_{10} = 30.6346064684320	λ_{17} = 52.6240518411150
λ_4 = 11.7915344390143	λ_{11} = 33.7758202135736	λ_{18} = 55.7655107550200
λ_5 = 14.9309177084878	λ_{12} = 36.9170983536640	λ_{19} = 58.9069839260809
λ_6 = 18.0710639679109	λ_{13} = 40.0584257646282	λ_{20} = 62.0484691902272
λ_7 = 21.2116366298793	λ_{14} = 43.1997917131767	λ_{21} = 65.1899648002069

For these 21 values, approximately: $\lambda_n = -0.635 + 3.131\,n$.

Plotting the truncated Eq. (2) for different instants of time leads to Szymański's elucidating drawing: Figure 3. The difference with the related issue of entrance flow (from reservoir to pipe) is that entrance flow is steady and involves radial velocities, though Atabek (26) found analytical solutions for an unsteady entrance region.

The cross-sectional average of the flow velocity, or discharge divided by cross-sectional area, is found directly from Eq. (2) as

$$V(t) = V_\infty \left(1 - 32 \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{8}\lambda_n^2 C_3 t}}{\lambda_n^4} \right). \quad (3)$$

From Eqs. (2) [with $C_1(t)$] and (3) a relation between $\frac{\partial v}{\partial r}$ at the pipe wall (at $r = D/2$) and $V(t)$ can be obtained, such that the wall shear stress can be included (as convolution-type UF) in one-dimensional (waterhammer) models governing V and P.

The derivation of Eq. (2) is well described in (1) and (2), where (1) contains a range of mathematical proofs on convergence and asymptotic behaviour of the analytical solutions. In the 1930s that was most useful, since there were no computers to carry out quick checks (or produce a drawing like Fig. 3). For periodic and arbitrary pressure-gradients driving the flow, Szymański found general solutions in terms of a series and an integral, respectively. Furthermore, in (2) he presented (with the aid of Green's functions) integral expressions describing unsteady flow in conduits of arbitrary cross-section.

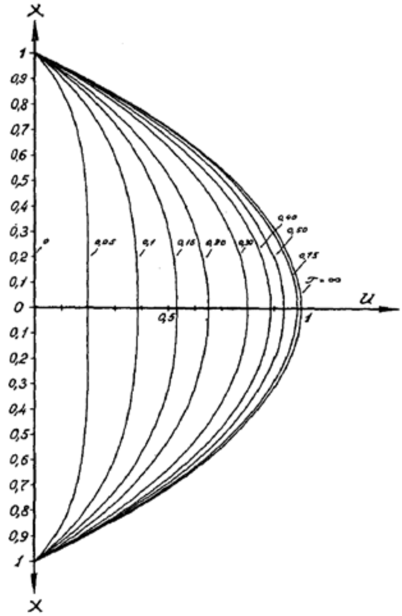


Fig 3 Szymański's dimensionless velocity profiles published in 1930 (2): development from no-flow at time zero to Poiseuille flow at time infinity.

CONCLUSION

Szymański was a remarkable man with an interesting life in an eventful era. Mathematics and politics were his areas of focus. In both he kept the highest standards and reached the highest positions. Nevertheless, hydraulicians will remember him for his seminal paper (1).

ACKNOWLEDGEMENTS

Figure 1: Karta (www.karta.org.pl).

Figure 2, Table 1: Warsaw Polytechnic, Department of Human Resources.

Museum of Warsaw Polytechnic (http://www.muzeumpw.com.pl/).

Piotr Kijkowski (Warsaw, Poland), former colleague of Szymański, provided much authentic information.

The section "His life" is written by the first author and based on two telephone conversations with Piotr Kijkowski (in November 2012 and June 2015) and on a self-written biography kept in Szymański's personal folder at the Human Resources Department of Warsaw Polytechnic.

Patrick Vaugrante (Paris, France) advised on translations from French.

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APPENDIX B: English translation of Szymański's 1930 paper:

ON THE NON-PERMANENT FLOW OF VISCOUS FLUID IN A PIPE

by P. SZYMAŃSKI, Warsaw.

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Introduction.

This note is devoted to the study of the flow of incompressible viscous fluid in a cylindrical pipe. The objective that I pose myself is to obtain the exact solutions of the equations of STOKES, constituting generalisations of the well-known Poiseuille solution. The latter represents permanent laminar flow of viscous fluid in a pipe of circular cross-section. Well then, I intend to study how the laminar flow regime establishes from rest. Therefore I look for integrals of the equations of Stokes that depend on time. I assume that the fluid velocity is constantly parallel to the pipe axis and that it is zero everywhere at the initial time.

I first study the case of a pipe of circular cross-section and I find the solution of the problem in the form of a series. The solution depends essentially on the manner in which the difference of the pressures at the ends of the tube varies with time. Of course, one assumes this function as given.

I show that if the said function has a certain limit at infinity, the solution for infinite time tends uniformly to the Poiseuille solution.

As for the general case of arbitrary pipe cross-section, I prove the uniqueness of solutions and I express these with a function similar to that of GREEN.[1]

I. Flow through a pipe of circular cross-section.

The flow is symmetrical with respect to the pipe axis; we use the cylindrical coordinates (r, ϑ, z). We have the OZ axis along the axis of the pipe such that the ends of the pipe correspond to $z = 0$ and $z = l$. Assuming zero velocity components v_r and v_ϑ, one obtains by virtue of the continuity equation: $\frac{\partial v_z}{\partial z} = 0$. The equations of motion are written, therefore, as follows:

$$-\frac{1}{\sigma}\frac{\partial p}{\partial r} = 0 \tag{1}$$

$$-\frac{1}{\sigma}\frac{\partial p}{\partial z} + \frac{\mu}{\sigma}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) = \frac{\partial v_z}{\partial t} \tag{2}$$

$$\frac{\partial v_z}{\partial z} = 0 \tag{3}$$

where p denotes the pressure, σ − the density of the fluid and μ − the coefficient of viscosity.

(The above equations are written for the case where the external forces are zero. If a potential for the external forces exists, the equations have a similar shape.)

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[1] I limit myself here to indicating and phrasing the theorems. An article about this topic and containing all the details of proofs will be published soon.

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The equations (1) and (3) show that p is a function of z and t, while v_z depends only on r and t, where, by virtue of equation (2), $\frac{\partial p}{\partial z}$ is a function of t only. This is a given function equal to $\frac{p_l(t) - p_0(t)}{l}$ where $p_l(t)$ and $p_0(t)$ indicate the pressures at the terminations of the pipe.

The problem reduces to finding one function v_z of the two variables r and t satisfying the equation:

$$\frac{\mu}{\sigma}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) - \frac{\partial v_z}{\partial t} = \frac{1}{\sigma}\cdot\frac{p_1(t) - p_0(t)}{l} \tag{I}$$

and conditions at boundaries:

$$v_z = 0 \quad \text{for} \quad r = r_0 \text{ and } t \geq t_0\,, \tag{α}$$

$$v_z = 0 \quad \text{for} \quad t = t_0 \text{ and } 0 \leq r \leq r_0\,, \tag{β}$$

$$\frac{\partial v_z}{\partial r} = 0 \quad \text{for} \quad r = 0 \text{ and } t \geq t_0\,. \tag{γ}$$

where r_0 is the radius of the pipe cross-section and t_0 − the initial time. (One may obviously assume $t_0 = 0$.)

The pressure is determined by the formula:

$$p(t) = p_o(t) - \frac{p_o(t) - p_l(t)}{l} \cdot z$$

Let us introduce new variables that do not depend on the particular choice of units of measurement by taking:

$$x = \frac{r}{r_0}, \qquad v_z = \frac{\mu}{\sigma l} u,$$

$$\tau = \frac{t\mu}{\sigma r_0^2}, \qquad T = \frac{\sigma r_0^2}{\mu^2}(p_l - p_0).$$

The equation is then written as:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial \tau} = T(\tau) \qquad \text{(II)}$$

and the function u satisfies the conditions:

$$u = 0 \ \text{ for } \ x = 1 \text{ and } \tau \geq 0, \qquad (\alpha)$$

$$u = 0 \ \text{ for } \ \tau = 0 \text{ and } 0 \leq x \leq 1, \qquad (\beta)$$

$$\frac{\partial u}{\partial x} = 0 \ \text{ for } \ x = 0 \text{ and } \tau \geq 0. \qquad (\gamma)$$

In addition, the function u must be continuous and also its derivatives $\frac{\partial u}{\partial \tau}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ for $0 \leq x \leq 1$ and $\tau \geq 0$. We call such a function *regular*.

I solve the problem by the following method. For each given function $T(\tau)$, I find a particular solution $u_0(x, \tau)$ of the equation (II) satisfying the conditions (α) and (γ). Then I find the solution w of the equation without second member (right-hand side):

$$\frac{\partial^2 w}{\partial x^2} + \frac{1}{x}\frac{\partial w}{\partial x} - \frac{\partial w}{\partial \tau} = 0 \qquad \text{(III)}$$

satisfying the conditions (α) and (γ) and reducing to $u_0(x, 0)$ for $\tau = 0$ and $0 \leq x \leq 1$. The function $u = u_0 - w$ will then be the solution of the problem.

I develop the function w in a series:

$$w = \sum_{n=1}^{n=\infty} c_n J(a_n x) e^{-a_n^2 \tau}$$

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where $J(\xi)$ denotes the Bessel function of the first kind and of zero order. This series formally satisfies equation (III) and the conditions (α) and (γ), if the numbers a_n are the roots of the equation: $J(a) = 0$. (It is sufficient to consider positive roots only.)

The condition (β) can be fulfilled by a suitable choice of the coefficients c_n, if $u_0(x, 0)$ is developable in a series of FOURIER-BESSEL:

$$u_0(x,0) = \sum_{n=1}^{n=\infty} c_n J(a_n x).$$

Based on the work of D. HILBERT, E. W. HOBSON and W. H. YOUNG[1] we know how to identify the possibility of the development in question and calculate the coefficients c_n for a broad class of functions.

I apply the method first to the case of the function $T(\tau) = h = const.$ and I find:

$$u_0 = \frac{h}{4}(x^2 - 1)$$

and

$$u(x,\tau) = \frac{h}{4}\left[x^2 - 1 - 8\sum_{n=1}^{\infty} \frac{J(a_n x)}{a_n^3 J'(a_n)} \cdot e^{-a_n^2 \tau} \right]. \qquad (4)$$

I prove the uniform convergence of this series and its formal derivatives $\frac{\partial u}{\partial \tau}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ for all values $0 \le x \le 1$ and $\tau \ge 0$ with the exception of the point ($x = 1$, $\tau = 0$) where the functions $\frac{\partial u}{\partial \tau}$ and $\frac{\partial^2 u}{\partial x^2}$ are discontinuous. This is due to the fact that in the case considered the function $T(\tau)$ does not vanish for $\tau = 0$, which is necessary if we are to satisfy the equation (II) and the conditions (α) and (β).

I then prove that the function (4) tends uniformly to u_0, i.e. to the function of POISEUILLE, if the time τ increases to infinity. In the same manner, the derivatives $\frac{\partial u}{\partial \tau}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ tend respectively to $\frac{\partial u_0}{\partial \tau}$, $\frac{\partial u_0}{\partial x}$ and $\frac{\partial^2 u_0}{\partial x^2}$. The figure (Fig. 3 herein) shows the curves of the distribution of velocities for several values of τ. The limit curve is the parabola of POISEUILLE.

I construct by the same method the solutions $A(x, \tau, k)$ and $B(x, \tau, k)$ of our problem, corresponding respectively to the cases:

$$T(\tau) = \cos k\tau - 1$$

and

$$T(\tau) = -\sin k\tau$$

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[1] D. HILBERT. Nachr. Ges. Gött. 1904 math.-phys. p. 231-241 and Th. der linearen Integralgleichungen. Leipzig, 1924, p. 54.
E. W. HOBSON. Representation of a function by Bessel series of functions. Proc. of the London Math. Soc. 2 ser. vol. VII, 1909, p. 387.
W. H. YOUNG. On series of Bessel functions. Ibid. vol. 18, 1920, p. 184.

With the aid of these two functions $A(x, \tau, k)$ and $B(x, \tau, k)$ I express the solutions of the problem for more general cases. I limit myself to the following two important cases.

First case. The function T(τ) is periodic.

The result that I have obtained can be summarized as follows.

»Let T(τ) be a periodic function of period ω; if:

1. T(0) = 0,

2. T(τ) has a first derivative T '(τ),

3. the functions T(τ) and T '(τ) have bounded variations on the interval (0, ω),

4. $c_n = \frac{2}{\omega}\int_0^{\omega} T(s)\cos\frac{2\pi ns}{\omega}ds$ *and* $s_n = \frac{2}{\omega}\int_0^{\omega} T(s)\sin\frac{2\pi ns}{\omega}ds$, *the function:*

$$u(x,\tau) = \sum_{n=1}^{n=\infty} c_n A\left(x,\tau,\frac{2\pi n}{\omega}\right) - s_n B\left(x,\tau,\frac{2\pi n}{\omega}\right)$$

is a regular integral of equation (II) satisfying the conditions (α), (β) and (γ).»

The movement represented by the function u is not periodic although the pressure (the function $T(\tau)$) varies periodically; but this movement is periodic in the limit for an infinite time τ. By this it is meant that the function u can be decomposed into two parts, one of which is periodic and the other tends uniformly to zero as τ increases indefinitely. The movement limits at the same period ω and follows the variations of the pressure.

Second case. The function T(τ) has a bounded limit for infinite τ.

I prove the following theorem:

»Let T(τ) be a function defined for all values $\tau \geq 0$*; if:*

1. *T(0) = 0,*
2. *T(τ) tends to a specified limit L for* $\tau \to \infty$,
3. *the integral* $\int_0^{\infty} \left|T(\tau) - L\right| d\tau$ *exists,*
4. *T(τ) is continuous and has bounded variations on the interval (0, ∞),*
5. *T(τ) has a first derivative T '(τ) with bounded variation on the interval (0, ∞),*
6. $c(k) = \frac{2}{\pi}\int_0^{\infty} \left|T(s) - L\right| \cos ks\, ds$, *the function:*

$$u(x,\tau) = \int_0^{\infty} c(k) A(x,\tau,k)\ dk$$

is a regular integral of equation (II) satisfying the conditions (α), (β) and (γ).»

In this general case the limit of the motion is also that of POISEUILLE, as in the case already studied. In particular we have:

»The functions u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ *and* $\frac{\partial u}{\partial \tau}$ *tend for infinite τ uniformly to* $\frac{L}{4}(x^2 - 1)$, $\frac{L}{2}x$, $\frac{L}{2}$ *and 0, respectively.»* [1]

II. Flow through a pipe of arbitrary cross-section.

We now use Cartesian coordinates as being more convenient. Let us have the pipe axis, as before, along the axis *OZ* of the coordinates. Let *C* be the contour of the cross-section of the pipe in the plane *OXY*, or in any arbitrary plane perpendicular to the pipe axis. The points (x, y) located within or on the contour *C*, form a closed region which we refer to as *R*.

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[1] I am indebted to MR. ZYGMUND for the simplification of the proof of this theorem.

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Assuming $v_x = 0$ and $v_y = 0$ one arrives at, as before, the conclusions $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 0$ and $\frac{\partial v_z}{\partial z} = 0$. The problem reduces to a single equation:

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} - \frac{1}{\nu}\frac{\partial v_z}{\partial t} = T(t) \qquad \text{(I)}$$

with one unknown function v_z of three variables *x*, *y* and *t*. The function $T(t)$ is given and has the same meaning as before, ν is a constant ($\nu = \frac{\mu}{\sigma}$). The function v_z must be *regular* for $t \geq 0$ and for any point (x, y) of the region *R*. The boundary conditions are here:

$v_z = 0$ for $t \geq 0$ and for every point (x, y) on the contour *C*, (α)

$v_z = 0$ for $t = 0$ and for every point (x, y) of the region *R*. (β)

Let:

$$v_z = u - \nu \int_0^t T(s)\,ds$$

The equation becomes:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\nu}\frac{\partial u}{\partial t} = 0 \qquad \text{(II)}$$

with the conditions:

$u = \nu \int_0^t T(s)\,ds$ for $t \geq 0$ and (x, y) situated on *C*, (α)

$u = 0$ for $t = 0$ and all (x, y) of the area *R*. (β)

I study the more general problem where the function $u(x, y, t)$ reduces for $t = 0$ to a given *initial* function $\varphi(x, y)$ and for the points (x, y) situated on *C* it has the values of the given *boundary* function $f(x, y, t)$.

First I take care of the uniqueness of the solutions and I reach the following theorem:

»If u_1 and u_2 are regular solutions of equation (II) corresponding to boundary functions $f_1(x, y, t)$ and $f_2(x, y, t)$, respectively, and to the same initial function $\varphi(x, y)$ and if, moreover, the functions $f_1(x, y, t)$ and $f_2(x, y, t)$ have the same values for $0 \leq t \leq t_1$ and for (x, y) situated on C, then the functions u_1 and u_2 coincide for $0 \leq t \leq t_1$. In particular, if $t_1 = \infty$, the functions u_1 and u_2 are identical.»

I express the solutions of the problem with the aid of the function $G(x, y, t, \xi, \eta, \tau)$ which I define in the following manner:[1]

1. $$G(x, y, t, \xi, \eta, \tau) = w(x, y, t, \xi, \eta, \tau) - g(x, y, t, \xi, \eta, \tau)$$

where:

$$w = \frac{1}{t-\tau} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4\nu(t-\tau)}}$$

and g is a solution of the adjoint equation:

$$\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \eta^2} + \frac{1}{\nu}\frac{\partial g}{\partial \tau} = 0$$

regular for $0 \le \tau \le t$ and for points (ξ, η) in the region R,

2. $g = w$ if $0 \le \tau \le t$ and if the point (ξ, η) is located on C,
3. $g = 0$ if $\tau = t$ and if the point (ξ, η) belongs to R.

If for the region R the function G exists, the solution of the problem has the form:

$$u(x,y,t) = \frac{1}{4\pi\nu}\iint_R \phi(\xi,\eta) G(x,y,t,\xi,\eta,0)\, d\xi d\eta + \frac{1}{4\pi}\int_0^t d\tau \int_C f(\xi,\eta,\tau)\frac{dG}{dn} ds$$

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[1] Cf. A. SOMMERFELD. Zur analytischen Theorie der Wärmeleitung. Math. Ann. vol. 45, 1894, p. 263. See also: H. WEBER. Part. Different. Gleich. 1919 vol. II.

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Applying this general formula one obtains for the velocity v_z of our flow the following expression:

$$v_z = \frac{\nu}{4\pi}\int_0^t (t-\tau) T(\tau)\, d\tau \int_C \frac{dG}{dn} ds - \nu \int_0^t T(\tau)\, d\tau$$

Warsaw, the 26th of April 1930.

Szymański P (1930) Sur l'écoulement non permanent du fluide visqueux dans le tuyau. *Compte Rendu du III Congrès International de Mécanique Appliquée*, Stockholm, 1930, pp. 249-254 (in French).