On Boussinesq and Coriolis coefficients and implications for the Joukowsky equation

Alan E Vardy
University of Dundee, UK & Dundee Tunnel Research

Arris S Tijsseling
Department of Mathematics and Computer Science
Eindhoven University of Technology, The Netherlands

ABSTRACT

The purpose of this paper is to show how derivatives of the Boussinesq and Coriolis coefficients, β and α, can be handled formally in 1-D analyses of unsteady flow. In the case of low Mach number flows typical of liquid flows in many pipes, it is usual to disregard differences between these coefficients and unity, thereby simplifying expressions such as the Joukowsky equation. When this is deemed to be unacceptable – e.g. in moderate and high Mach number flows – a different approach is usually followed, namely allowing for the actual values of the coefficients, but disregarding derivatives of them. It is shown herein that this approach is not only unnecessary, but is actually less accurate than disregarding the coefficients altogether (i.e. using plug-flow approximations). Mathematically, the new result is obtained by deriving expressions that relate derivatives of β and α to derivatives of the principal flow parameters (pressure p, density ρ and mean velocity U). Because these relationships involve derivatives, they do not enable actual values of β and α to be deduced. However, it is shown rigorously that inertial waves do not change the product $\rho^2U^2(\beta-1)$ and so, if β is known a priori before a wave-induced velocity change, its value after the change can be deduced.

Keywords:
one-dimensional, unsteady, compressible pipe-flow; momentum correction factor; energy correction factor

NOMENCLATURE

A	cross-sectional area – m^2
c	sound speed – m/s
F_τ	shear force per unit length, N/m
M	Mach number
MOC	method of characteristics
p	cross-sectional mean pressure – Pa
RHS	right-hand side of equations
t	time coordinate – s
U	mean velocity – m/s
u	local velocity – m/s
x	distance coordinate – m

Greek symbols

α	Coriolis coefficient – see Eq.2.3
β	Boussinesq coefficient – see Eq.2.2
Δ	finite interval
λ	coefficient in MOC Eqs.
ρ	fluid density – kg/m^3
τ	shear stress – Pa
φ	Un-named coefficient – see Eq.2.4
ψ	see Eq.A1.11

Suffices

wall	inner surface of pipe/duct wall

1 INTRODUCTION

For many years, the authors have corresponded about the influence of neglecting non-uniform velocity distributions in simulations of unsteady flows in ducts and pipes, as is standard practice in papers presented at this series of conferences on Pressure Surges. It is not possible to allow exactly for the non-uniformity in one-dimensional representations of such flows, but their existence is sometimes acknowledged by introducing correction coefficients for terms describing momentum flux and energy flux – as is done below. These are often referred to as Boussinesq and Coriolis coefficients, β and α respectively. However, there are several difficulties with this approach, notably:

(i) It is fundamentally impossible to deduce exact values of β and α without knowing the true velocity distribution over the flow cross-section. This can sometimes be known with good accuracy in the case of steady flows, but it is not known in 1-D analyses of unsteady flows, except perhaps as an initial condition.

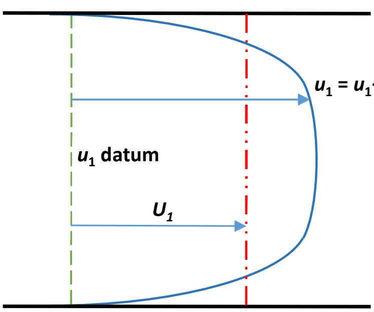
(ii) The true values of these coefficients can vary hugely; β ranges from unity to infinity and α ranges from minus to plus infinity. The infinite extremes exist when the mean velocity is zero, but both positive and negative velocities exist locally in the cross-section.

(iii) The 1-D equations of momentum and energy contain not only β and α themselves, but also their spatial derivatives and, in some formulations, their temporal derivatives too. Hitherto, it has always been assumed that it would be impossible to allow for these in a meaningful way even if approximate methods of estimating instantaneous values of β and α could be found. In all papers that the authors have studied, these derivatives have simply been neglected.

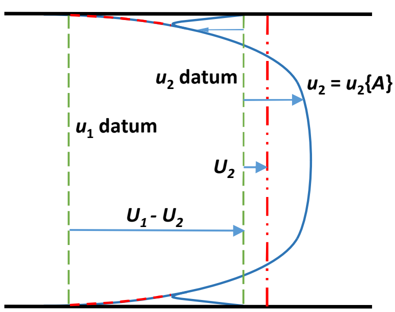
For completeness, it is pointed out that, although the focus of the present paper is on unsteady flows – and, in particular, on pressure waves – variations in β and α also occur in steady flows. Obvious examples include regions of flow up to a few tens of pipe diameters after an inlet, bend or junction. A less-widely acknowledged instance relates to velocity changes implied by density changes that inevitably accompany pressure changes due to skin friction, say.

A few years ago, the authors chanced upon a simple method of eliminating the derivative $\partial\beta/\partial x$ from momentum equations developed using area-weighted formulations of the 1-D equations of unsteady flow. That is, the third of the above complications could be overcome rigorously (Vardy & Tijsseling, 2022). It shows that momentum equation and MOC equations derived using the method are a very close approximation to standard plug-flow equations derived by completely disregarding the non-uniformity of velocity distributions. This is, of course, excellent news for analysis and designers using bespoke or standard tools in which this simplification is implicit.

The resulting analysis had an even more unexpected bonus in that it went a long way towards eliminating the first of the above complications because it yielded an explicit relationship between changes in β and changes in the mass flow rate. Direct use can be made of this to infer values of β when two conditions are fulfilled. First, the value of β needs to be known a priori for one mass flow rate. In practice, this will commonly be possible when steady flow conditions exist before the arrival of a wavefront, but it will rarely be possible otherwise. Second, all changes in the velocity profile must be due exclusively to the influence of pressure waves. That is, the time scales must be too short for lateral vorticity diffusion to exert significant influence. Clearly, this limits the direct usefulness of the relationship in analyses that include long periods between successive strong wavefronts, but it is nevertheless highly informative in understanding the wavefronts themselves. The implications of this limitation can be inferred from Fig.1, which shows a velocity distribution before and ***shortly*** after the passage of a wavefront causing a change in mean velocity from U_1 to U_2. In both parts of the Figure, the continuous (blue) line assumes the existence of a no-slip condition at the wall. The broken (red+blue) line in Fig.1b depicts a hypothetical variation in which the ***change*** in velocity is not subjected to the no-slip condition. The sole purpose of the hypothetical line is to demonstrate that, during the short time of passage of a sudden wavefront, the shape of the velocity profile changes negligibly except very close to the wall.



(a) Initial velocity distribution



(b) Distribution after step change

Fig-1 Flow reversal within a cross-section caused by a sudden wavefront.
[*The (red) dash-dot lines correspond to plug flow*]

This paper reports further developments that have been made since the work reported by Vardy & Tijsseling, 2022). An extension of the original logic is used to enable explicit expressions to be derived for the derivatives $\partial\alpha/\partial x$ and $\partial\beta/\partial t$ and the first of these is then used to derive an enhanced version of the flow-weighted form of the momentum equation. The second could be used to eliminate $\partial\beta/\partial t$ from the energy equation, but, for brevity, this is not done herein. Energy equations are used extensively in the first author's work, but they are used only occasionally in papers in this conference series.

There are two appendices. The first develops explicit relationships between β and α and the mass flow rate. The second introduces a conundrum that exercised the authors' meagre brains for a considerable time before the penny finally dropped. It is an especially satisfying way in which to end a Forum paper because it casts doubt on the reliability of a fundamental assumption that is made almost universally in studies of unsteady flows in ducts and pipelines. To avoid undue concern or over-expectation, we declare at this point that the uncertainty is not sufficient to challenge the practical use of existing methods, but it is likely to provide food for thought for academics who, like us, have not previously considered it.

2 DEFINITIONS

It is convenient to precede the main derivations with some basic integrals that are used more than once. This has the spin-off benefit of giving a family resemblance between various definitions and hence providing intuitive guidance on their implications.

2.1 Base parameters

The following four integrals may alternatively be interpreted as definitions of (i) the mean velocity U, (ii) the Boussinesq coefficient β, the Coriolis coefficient α and an un-named coefficient φ. Herein, the integration variable A is always used to denote the cross-sectional area of flow and δA is the cross-sectional area of a typical stream tube.

Mean velocity U:
$$\int_A u dA = A.U \tag{2.1}$$

Boussinesq coefficient β:
$$\int_A u^2 dA = \beta.AU^2 \tag{2.2}$$

Coriolis coefficient α:
$$\int_A u^3 dA = \alpha.AU^3 \tag{2.3}$$

Un-named coefficient φ:
$$\int_A u^4 dA = \varphi.AU^4 \tag{2.4}$$

For future reference, it is pointed out that the velocity is not the only parameter that varies over the cross-section. So too do parameters such as pressure, density and temperature. Nevertheless, following the usual practice in pressure-surge studies, no account is taken of these additional variations herein. No *formal* justification is advanced for neglecting them, but a short descriptive assessment is now given, focussing initially on the case of single-phase liquids in straight conduits. As a preliminary remark, it is noted that, in most one-dimensional analyses of pressure surges, it is assumed to be acceptable to neglect the influence of *axial* variations in density and temperature, at least locally. The focus herein is on *cross-sectional* variations that, in many cases (although not all) will be far smaller than axial ones.

Pressure: Cross-sectional variations in *pressure* can occur because of hydrostatic effects (i.e. gravity), geometric curvature, temperature variations or relatively high-frequency disturbances. The first three of these possible influences on pressure are commonly found to be negligible even in the case of steady or quasi-steady flows even though local *axial* variations in pressure are then far smaller than in wave-like flows. It does not seem necessary to discuss them further herein. The fourth (i.e. high-frequency disturbances arising, for instance, from, say, turbulence fluctuations or pipe hoop vibrations) is not necessarily negligible, however. Indeed, account is sometimes taken of them in detailed studies (e.g. Yen 1973; Fenton et al 2005). Pressure changes due to individual fluctuations disperse rapidly (in line with Huygen's Principle), but the cumulative effect of on-going phenomena such as turbulence can nevertheless be readily detectable. Herein, account is taken of this implicitly by defining the variable 'p' in equations to be a cross-sectional mean pressure.

Density: Cross-sectional variations in *density* can occur because of variations in pressure or temperature. Proportional variations due to pressure change are approximately equal to the ratio of cross-sectional pressure differences and the fluid bulk modulus. This ratio is tiny in pipe systems of interest to the pressure-surge community. Indeed, the effect is so small that it is conventionally neglected even in relation to *axial* variations of density induced by pressure changes that are hugely greater than cross-sectional variations due to, say, turbulent fluctuation. Likewise, at constant pressure, proportional cross-sectional variations in density due to temperature are equal to the product of the thermal expansivity of the fluid and the cross-sectional temperature differences. Implausibly large temperature differences would be required to cause proportional density variations of practical relevance to the present study.

Velocity: In contrast with these variations in pressure and density, proportional cross-sectional variations of axial velocity components are always large – indeed, larger than the mean velocity of flow, even in steady flows (except in the special case of super-cooled liquids). Inevitably, therefore, these will dwarf any corresponding consequences of cross-sectional variations in density, pressure and temperature. Indeed, it is to be expected that such consequences will be dwarfed even by uncertainties in calculated values of β and α arising from unknowable details such as the details of turbulence.

The authors consider that, for single-phase liquids, this short discussion is more than adequate for justifying neglecting cross-sectional variations in density and temperature and for defining 'p' to be a mean pressure. Furthermore, it is assumed herein that broadly similar reasoning is also adequate in the case of gas flows – at least for the stated purposes of this paper – even though the proportional variations in density, pressure and temperature will be much larger than those for liquids. Nevertheless, for completeness, it is acknowledged that neglecting them – and also neglecting consequence of non-axial components of velocity – clearly violates the formal laws of physics in all directions except axial. That is, the principle of Galilean invariance is not respected. Readers wishing to explore this matter in more detail are referred to a recent paper by Gray & Miller (2023).

2.2 Derivatives of U, β and α

Use is made below of derivatives of U, β and α. In most cases, differentiation is made with respect to the spatial coordinate x. However, corresponding derivatives with respect to time t can be inferred from these equations by simply replacing x by t.

∂U/∂x:

$$\int_A \frac{\partial u}{\partial x} dA = \frac{\partial}{\partial x}\int_A u\, dA = A\frac{\partial U}{\partial x} \qquad (2.5)$$

$\partial\beta/\partial x$:

$$\int_A \frac{\partial u^2}{\partial x} dA = \frac{\partial}{\partial x}\int_A u^2\, dA = A\frac{\partial}{\partial x}(\beta U^2)$$
$$= 2\beta AU\frac{\partial U}{\partial x} + AU^2\frac{\partial \beta}{\partial x} \tag{2.6}$$

$\partial\alpha/\partial x$:

$$\int_A \frac{\partial u^3}{\partial x} dA = \frac{\partial}{\partial x}\int_A u^3\, dA = A\frac{\partial}{\partial x}(\alpha U^3)$$
$$= 3\alpha AU^2\frac{\partial U}{\partial x} + AU^3\frac{\partial \alpha}{\partial x} \tag{2.7}$$

3 CONTINUITY

Several variations on the continuity equation are presented below. The first of these (Eq.3.1) is applicable when plug flow conditions are assumed. It is widely used in analyses of unsteady flows in pipes except that it is commonly simplified by neglecting the convective term $U\partial\rho/\partial x$. This is not done here, partly because it is not appropriate at moderate or large Mach numbers and partly because it would hide a revealing complication that is discussed in Appendix-1. The second (Eq.3.2a) is applicable for any typical streamtube in which the velocity is u. In simulations that neglect cross-sectional variations, it can be expressed for the whole cross-section using the mean velocity U instead of the local velocity u, thereby yielding plug-flow equations. However, the focus herein is on more general cases with non-uniform velocity distributions.

The remaining versions of the continuity equation are integral formulations using different weighting factors for the various streamtubes. All are mathematically valid, although some are used more commonly than others. The chosen weighting factors are δA, $u\delta A$ and $u^2\delta A$. The first two of these are referred to in the literature as area-weighted and flow-weighted, respectively. The development of the first one is presented in full, but, for brevity, only the final forms of the others are presented.

Continuity (Plug flow):

$$\frac{\partial\rho}{\partial t} + U\frac{\partial\rho}{\partial x} + \rho\frac{\partial U}{\partial x} = 0 \tag{3.1}$$

Continuity (Streamtube): (δA-weighted)

$$\frac{\partial\rho}{\partial t}\delta A + u\frac{\partial\rho}{\partial x}\delta A + \rho\frac{\partial u}{\partial x}\delta A = 0 \tag{3.2a}$$

Continuity (Full section): (δA-weighted)

$$\frac{\partial\rho}{\partial t}\int_A dA + \frac{\partial\rho}{\partial x}\int_A u\, dA + \rho\int_A \frac{\partial u}{\partial x} dA = 0 \tag{3.2b}$$

Continuity: (δA-weighted)

$$\frac{\partial\rho}{\partial t} + U\frac{\partial\rho}{\partial x} + \rho\frac{\partial U}{\partial x} = 0 \tag{3.2c}$$

Continuity: ($u.\delta A$-weighted)

$$\frac{\partial\rho}{\partial t} + \beta U\frac{\partial\rho}{\partial x} + \beta\rho\frac{\partial U}{\partial x} + \frac{1}{2}\rho U\frac{\partial\beta}{\partial x} = 0 \tag{3.3}$$

Continuity: ($u^2.\delta A$-weighted)

$$\beta\frac{\partial\rho}{\partial t} + \alpha U\frac{\partial\rho}{\partial x} + \alpha\rho\frac{\partial U}{\partial x} + \frac{1}{3}\rho U\frac{\partial\alpha}{\partial x} = 0 \tag{3.4}$$

By inspection, the true area-weighted formulation (Eq.3.2c) is identical to the usual plug-flow formulation (Eq.3.1) even though it allows explicitly for the velocity distribution. However, the other two formulations are more complex. In part, this is because they involve the Boussinesq and Coriolis coefficients, but even more important, they involve spatial derivatives of these parameters. These complications are highly restrictive because it is not possible to infer suitable values of any of these terms from one-dimensional considerations alone. This is true even in steady flows, let alone in general, unsteady flows. Fortunately, however, as shown below, it is possible to work-around some of these limitations in a rigorous manner. As a precursor to this, it is useful to re-express Eqs.3.3 and 3.4 to highlight their potential use for eliminating $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ from other equations. Using Eq.3.2c, these lead respectively to:

$\partial\beta/\partial x$:

$$\rho U \frac{\partial \beta}{\partial x} = -2(\beta - 1)\frac{\partial(\rho U)}{\partial x} \tag{3.5}$$

$\partial\alpha/\partial x$:

$$\rho U \frac{\partial \alpha}{\partial x} = -3(\alpha - \beta)\frac{\partial(\rho U)}{\partial x} \tag{3.6}$$

4 MOMENTUM

The sequence followed for the continuity equation in the preceding Section is now repeated for the momentum equation. Once again, the derivation of the area-weighted case is presented fully, but only the final forms of the $u.\delta A$-weighted and $u^2.\delta A$-weighted cases are given (NB: the last of these is not used herein. It is included for completeness and for record purposes).

Each of the following equations includes the influence of shear stresses even though the main thrust of the paper is restricted to inviscid flow behaviour. This is done to serve as a reminder that some conclusions of the paper are not <u>*exactly*</u> true for real flows (although they are nearly true for strong waves of special interest in studies of pressure surges in pipes). The parameter $F_{\tau,wall}$ is the shear force per unit length on the pipe wall. The interpretation of all other terms in F_τ is given after all formulations of the equations have been presented.

Momentum (Plug flow):

$$\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = \frac{F_{\tau,wall}}{A} \tag{4.1}$$

Momentum (Streamtube): (δA-weighted)

$$\frac{\partial p}{\partial x}\delta A + \rho\frac{\partial u}{\partial t}\delta A + \rho u\frac{\partial u}{\partial x}\delta A = \delta F_\tau \tag{4.2a}$$

Momentum (Full section): (δA-weighted)

$$\frac{\partial p}{\partial x}\int_A dA + \rho\int_A \frac{\partial u}{\partial t}dA + \rho\int_A u\frac{\partial u}{\partial x}dA = \int_A \frac{\partial F_\tau}{\partial A}dA = F_{\tau,wall} \tag{4.2b}$$

Momentum: (δA-weighted)

$$A\frac{\partial p}{\partial x} + \rho A\frac{\partial U}{\partial t} + \beta A\rho U\frac{\partial U}{\partial x} + \frac{1}{2}A\rho U^2\frac{\partial \beta}{\partial x} = F_{\tau,wall} \tag{4.2c}$$

Momentum: (δA-weighted)

$$\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial t} + \beta\rho U\frac{\partial U}{\partial x} + \frac{1}{2}\rho U^2\frac{\partial \beta}{\partial x} = \frac{F_{\tau,wall}}{A} \tag{4.2d}$$

Momentum: ($u.\delta A$-weighted)

$$\frac{\partial p}{\partial x} + \beta\rho\frac{\partial U}{\partial t} + \alpha\rho U\frac{\partial U}{\partial x} + \frac{1}{2}\rho U\frac{\partial \beta}{\partial t} + \frac{1}{3}\rho U^2\frac{\partial \alpha}{\partial x} = \frac{1}{AU}\int_A u\frac{\partial F_\tau}{\partial A}dA \equiv \frac{F_{U\tau}}{AU} \tag{4.3}$$

Momentum: ($u^2.\delta A$-weighted)

$$\beta\frac{\partial p}{\partial x} + \alpha\rho\frac{\partial U}{\partial t} + \varphi\rho U\frac{\partial U}{\partial x} + \frac{1}{3}\rho U\frac{\partial \alpha}{\partial t} + \frac{1}{4}\rho U^2\frac{\partial \varphi}{\partial x} = \frac{1}{AU^2}\int_A u^2\frac{\partial F_\tau}{\partial A}dA \equiv \frac{F_{U^2\tau}}{AU^2} \tag{4.4}$$

where δF_τ denotes the *net* shear force per unit length acting on the lateral surfaces of a streamtube and $F_{U\tau}$, introduced solely for clarity, is defined as the integral immediately preceding it. In general, the shear stress varies over the pipe cross-section and this must be allowed for in the evaluation of integrals over the cross-section. Fortunately, however, in the case of the area-weighted formulation of the equation, the net result can be shown to be identical to that used in the plug-flow formulation. That is, only the shear stress on the pipe wall itself contributes to the overall integral. The corresponding integral for the flow-weighted formulation is less straightforward. It depends on variation of velocities and shear stresses over the whole cross-section and so it cannot be expressed as a function of U and $F_{\tau,wall}$ alone. This matter is discussed more fully by Brunone et al (1995) and by Vardy & Tijsseling (2022).

In contrast with the continuity equation, the area-weighted formulation (Eq.4.2d) is ***not*** the same as the plug flow one (Eq.4.1). The product of $U\partial U/\partial x$ is multiplied by β and there is an additional term in $\partial\beta/\partial x$. This matter is discussed more fully in Section 4.1. First, however, it is noted that the flow-weighted formulation (Eq.4.3) enables a useful expression for $\partial\beta/\partial t$ to be developed in a similar manner to the development of expressions for $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ from the continuity relationships. By subtracting Eq.4.1 from Eq.4.3 and using Eq.3.5 to eliminate $\partial\beta/\partial x$, we obtain (after a little manipulation):

$\boldsymbol{\partial\beta/\partial t}$:

$$\rho U\frac{\partial \beta}{\partial t} = -2(\beta-1)\rho\frac{\partial U}{\partial t} + 2(\alpha-1)U\frac{\partial(\rho U)}{\partial x} + \frac{2F_{U\tau}}{AU} - \frac{2F_{\tau,wall}}{A} \tag{4.5}$$

By inspection, there is a strong family resemblance between the expressions for $\partial\beta/\partial x$, $\partial\alpha/\partial x$ and $\partial\beta/\partial t$ in Eqs.3.5, 3.6 and 4.5. However, although all three of these can be used eliminate inconvenient derivatives of β and α from other equations, the use of Eq.4.5 to eliminate $\partial\beta/\partial t$ would carry a serious penalty in simulations where shear stresses are not neglected. This is because there is no reason to suppose that the difference between the two shear force terms in the equation is negligible.

The authors originally hoped that the use of a $u^2.\delta A$-weighted formulation of the momentum equation would enable a convenient expression to be deduced in a similar manner for the derivative $\partial\alpha/\partial t$. Unfortunately, this is not possible because the equation derived in this way – i.e. Eq.4.4 – also includes the derivative $\partial\varphi/\partial x$.

4.1 Practical form of the area-weighted momentum equation

Although a few examples exist in which account has been taken of the Boussinesq and Coriolis coefficients, the authors are not aware of any cases in which their derivatives have been retained. Instead, they have simply been discarded in the expectation (or hope) that they would be less important than the influence of the coefficients themselves. However, it has been shown by Vardy & Tijsseling (2022) that this practice can be seriously misleading. The reasoning presented in that paper can be summarised by comparing three forms of the momentum equation, namely:

(i) Simple plug-flow approximation:

Plug flow: $$\frac{\partial p}{\partial x} + \rho \frac{\partial U}{\partial t} + \rho U \frac{\partial U}{\partial x} = \frac{F_{\tau,wall}}{A} \quad (4.5a)$$

(ii) Area-weighted formulation, discarding the term in $\partial\beta/\partial x$:

δA-weighted:
(Neglecting $\partial\beta/\partial x$) $$\frac{\partial p}{\partial x} + \rho \frac{\partial U}{\partial t} + \beta\rho U \frac{\partial U}{\partial x} = \frac{F_{\tau,wall}}{A} \quad (4.5b)$$

(iii) Area-weighted formulation, using Eq.3.5 to eliminate $\partial\beta/\partial x$:

δA-weighted:
(No simplifications) $$\left[1 - (\beta - 1)\frac{U^2}{c^2}\right]\frac{\partial p}{\partial x} + \rho \frac{\partial U}{\partial t} + \rho U \frac{\partial U}{\partial x} = \frac{F_{\tau,wall}}{A} \quad (4.5c)$$

The third of these is nominally exact, in the sense that no approximations are made regarding the velocity distribution over the cross-section (NB: note that β is a function of x and t). It can be used as a base of reference from which to compare the relative performances of the first two formulations, both of which use approximations. It is shown in Section 5.1 below that the plug-flow approximation can be significantly more accurate than the approximation that allows for β, but discards $\partial\beta/\partial x$. This outcome is highly satisfactory for analysts of pressure surges because it justifies the use of plug-flow approximations – as is almost standard practice in the literature. Nevertheless, it was initially a surprise for the authors. Fortunately, it is easily explained with the benefit of hindsight by inspecting the coefficient of the first term in Eq.4.5c. For simulations such as pressure surge in liquid-filled pipelines, the Mach number ($M = U/c$) is very small, and its square is even smaller. Accordingly, the coefficient of $\partial p/\partial x$ will be very close to unity unless $\beta >> 1$. Furthermore, it is shown in Appendix 1 that the coefficient remains almost unchanged during the passage of a wave. For completeness, it is also shown that disregarding derivatives of β and α in the *flow*-weighted formulation leads to greater errors than either of the above approximations.

5 MOC

Analyses of transient flows in pipelines are commonly undertaken using the 1-D Method of Characteristics (MOC). Equation 5.1 shows the resulting equations when the pressure and density are deemed to satisfy $c^2 = K/\rho$, where K is the bulk modulus of a liquid, or $c^2 = \gamma p/\rho$, where γ is the ratio of the principal specific heats of a perfect gas. The interpretations of the parameter λ and the right-hand side terms (RHS) in the various formulations are listed in Table 1and the characteristic directions in which the equations are applicable are given in Table 2.

MOC equations: $$\lambda \frac{dp}{dt} + \rho c \frac{dU}{dt} = RHS \quad (5.1)$$

TABLE 1. Definitions of *λ*, *RHS* & d*x*/d*t*

Formulation	***λ***	***RHS***
Plug flow:	± 1	$c\dfrac{F_{\tau,w}}{A}$
δ*A*-weighted: Replacing ∂*β*/∂*x*	$\pm\sqrt{[1-(\beta-1)M^2]}$	$c\dfrac{F_{\tau,w}}{A}$
δ*A*-weighted: Neglecting ∂*β*/∂*x*	$-½(\beta-1)M$ $\pm\sqrt{[¼(\beta-1)^2M^2+1]}$	$c\left[-½\rho U^2\dfrac{\partial\beta}{\partial x}+\dfrac{F_{\tau,w}}{A}\right]$
*u*δ*A*-weighted: Neglecting ∂*β*/∂*x* & ∂*α*/∂*x*	$-½\left(\dfrac{\alpha}{\beta^2}-1\right)M$ $\pm\sqrt{\left[¼\left(\dfrac{\alpha}{\beta^2}-1\right)^2M^2+\dfrac{1}{\beta^2}\right]}$	$-½\lambda\rho c^2U\dfrac{\partial\beta}{\partial x}$ $+\dfrac{c}{\beta}\left[-½\rho U\dfrac{\partial\beta}{\partial t}-⅓\rho U^2\dfrac{\partial\alpha}{\partial x}+\dfrac{F_{U\tau}}{AU}\right]$

TABLE 2. Definition of d*x*/d*t*

Formulation	**d*x*/*dt***
Plug flow:	$U \pm c$
δ*A*-weighted: (No simplifications)	$U+\lambda c$
δ*A*-weighted: (Neglecting ∂*β*/∂*x*)	$\beta U+\lambda c$
*u*δ*A*-weighted: (Neglecting ∂*β*/∂*x* & ∂*α*/∂*x*)	$(\alpha/\beta)U+\lambda c$

The equations are presented in this manner for clarity of comparisons and they are shown in a sequence of increasing complexity. The second of these, in which Eq.3.5 has been used to eliminate $\partial\beta/\partial x$ as above, is the reference case. The third and fourth are the conventional forms in which $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ appear in their own right, but are then neglected. If this is done, the right-hand side terms in all of the formulations include only shear-stress terms.

Attention is drawn to three features seen in the Table. Firstly, only the plug-flow version is consistent with the standard form of the Joukowsky equation, namely $\Delta p = \rho c\Delta U$. The accurate area-weighted expression shows that, strictly, when account is taken of velocity distributions, a more correct form of the Joukowsky equation is $\sqrt{[1-(\beta-1)M^2]}.\Delta p = \rho c\Delta U$. This change will have negligible influence in practical studies of pressure surges in liquid-filled pipelines, although there are some instances where it could be significant in gas flows at moderate Mach numbers. Secondly, the expression in square brackets is identical to the coefficient of $\partial p/\partial x$ in the corresponding momentum equation 4.5c. As stated at the end of Section 4.1, it is almost unaffected by waves and so is may sensibly be regarded as *locally* constant during the integration of MOC equations. This is another valuable property – although its direct use will usually be limited by a lack of knowledge of appropriate values of β. Thirdly, the directions of applicability of the characteristic equations depend strongly on the value of λ.

5.1 Numerical example

Vardy & Tijsseling (2022) have given numerical examples of changes during the passage of a wavefront. They illustrated a range of cases, each starting from a steady flow in which the value of β is deemed to be known. This enabled the accuracy of the plug-flow and conventional area-weighted formulations to be assessed quantitatively, leading to the conclusions stated above. However, they did not assess the conventional *flow*-weighted formulation. This is done in Fig.2.

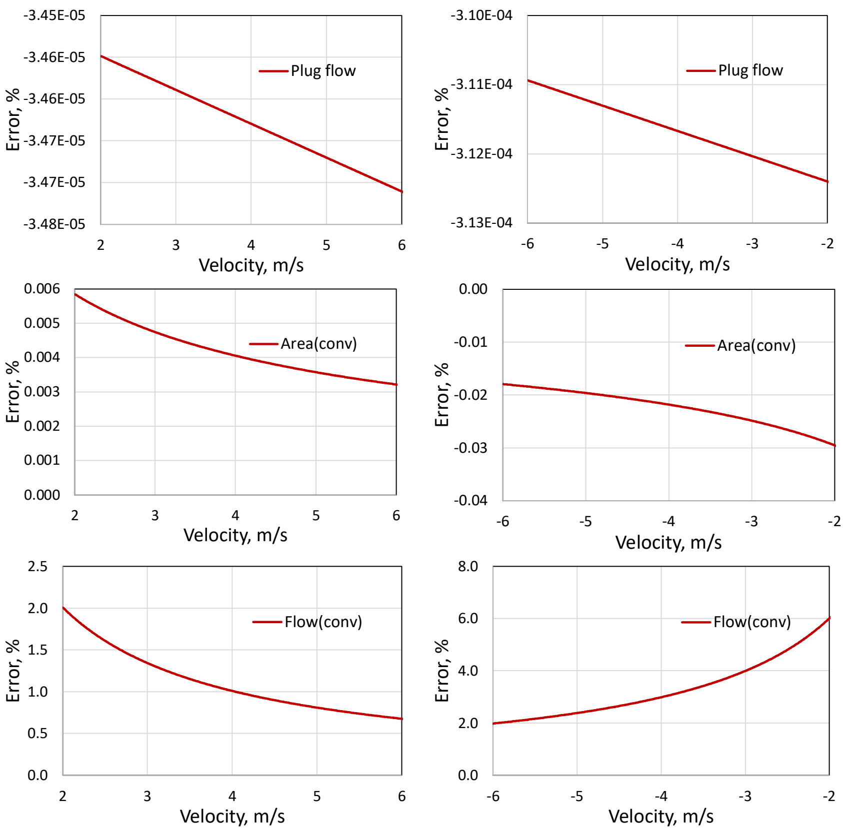


Fig-2 Errors in calculated pressure change during the passage of a wavefront.
(LH column: *Velocity change from +2 m/s to +6 m/s;*
RH column: *Velocity change from –6 m/s to –2 m/s*)

The Figure shows percentage errors in calculated pressure changes during the passage of a wavefront. In the left-hand column, the velocity magnitude increases linearly from 2 m/s to 6 m/s and in the right-hand column, it decreases from 6 m/s to 2 m/s. In both cases, the initial velocity profile is equivalent to a steady turbulent flow for which $\beta = 1.02$ although no account is taken of viscous phenomena during the passage of the wavefront. Using a result presented in Appendix 1, changes in the value of β during the passage of the wavefront are deemed to satisfy $(\beta - 1)\,\rho\, U^2$ = constant, so it reduces in the first case, but increases in the second. The Figure shows differences between the true change in pressure and the changes predicted by each of the three approximate methods. By inspection, the plug-flow representation is highly accurate and the solution based on the conventional area-weighted formulation is also well within acceptable tolerances for engineering purposes. However, the same is not true for the conventional flow-weighted solution in the case of deceleration even though the Mach number is small (less than 0.02 at all times).

6 CONCLUSIONS

The use of the Boussinesq and Coriolis coefficients in 1-D analyses of unsteady flows in pipes has been re-visited with a view to improving their use in flows where it is deemed unacceptable to approximate them to unity. In such cases, it is standard practice to estimate numerical values of the coefficients, but to disregard terms in the equations that include their derivatives. It has been shown that this practice is unsatisfactory and a much-improved methodology has been developed. The authors are not aware of any prior studies in which the new approach has been used.

It has been shown that formal expressions for $\partial\beta/\partial x$ & $\partial\alpha/\partial x$ can be inferred from alternative forms of the continuity equation and that a corresponding expression for $\partial\beta/\partial t$ can be inferred from these together with alternative forms of the momentum equation. By using these expressions to eliminate these derivatives from alternative forms of the momentum equation, it has been shown that the simple, widely-used plug-flow equation is actually a good approximation even in relatively high Mach-number flows.

The new formulation shows that, strictly, the conventional Joukowsky equation should be modified by a coefficient that depends on β and M. Fortunately, it is found that (a) the coefficient is almost unaffected by the passage of waves and (b) it will almost never differ from unity by enough to justify its inclusion in studies of liquid flows in pipelines.

For completeness, it is reiterated that the whole of the paper relates only to wave-induced changes in velocity. No account is taken of viscosity-induced effects that cause much slower changes than the sudden wavefronts that are the focus herein.

ACKNOWLEDGEMENT

The authors wish to give credit to an (unknown) reviewer of the paper (Vardy & Tijsseling 2022). That reviewer showed that a direct link can be developed between the present work and previous work by Kruisbrink & Tijsseling (2008). Consequences of her/his derivation are presented in Appendix 1.

REFERENCES

BRUNONE B, GOLIA UM & GRECO M (1995) *Effects of two-dimensionality on pipe transients modeling.* ASCE Journal of Hydraulic Engineering, **121** (12), 906-912

FENTON JD *On the energy and momentum principles in hydraulics*, Proc. 31st IAHR Congress, Seoul, Korea, 625-636

GRAY WG & MILLER CT (2023) *On the equations of open channel flow.* Journal of Hydraulic Research, **61** (1), 1-17

KRUISBRINK A.CH.H & TIJSSELING AS (2008) *The momentum flux coefficient for modelling steady and unsteady head losses in turbulent pipe flow.* Proc 10th Int Con. on Pressure Surges, Edinburgh, UK, BHR Group, 379-399

VARDY AE & TIJSSELING AS (2022) *On Boussinesq and Coriolis coefficients and their derivatives in transient pipe flows.*
Applied Mathematical Modelling, **111**, 454-470

YEN BC (1973) *Open-channel flow equations revisited,* Journal of Engineering Mechanics, ASCE, **5**, 979–1009

APPENDIX 1 VARIATION OF *β* AND *α* DURING THE PASSAGE OF A WAVEFRONT

In the main body of the paper, Eqs. 3.5 and 3.6 have been used to eliminate $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ from the equations of flow. However, these equations also give valuable information in their own right.

A1.1 Boussinesq coefficient

First, it is shown that changes in β caused solely by a wavefront depend only on the variation in mass flow rate induced by the wavefront. This is done by developing Eq.3.5 as follows:

Eq.3.5:

$$\frac{\partial\beta}{\partial x} = -2(\beta-1)\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x} \tag{A1.1}$$

$$\frac{1}{\beta-1}\frac{\partial\beta}{\partial x} = -2\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x} \tag{A1.2}$$

$$\frac{1}{\beta-1}\frac{\partial(\beta-1)}{\partial x} = -2\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x} \tag{A1.3}$$

On integration, this gives

$$\ln(\beta-1) = -2\ln(\rho U) + f\{t\} \tag{A1.4}$$

and so

$$(\beta-1)\rho^2U^2 = \mathrm{e}^{f\{t\}} \tag{A1.5}$$

The right-hand side of Eq.A1.5 is a function of time only. Therefore, at any particular instant, spatial variations caused solely by the inertial consequences of waves must satisfy:

$$(\beta-1)\rho^2U^2 = Constant \tag{A1.6}$$

This shows that, provided that a suitable value of β can be estimated for one particular value of ρU (the mass flow rate per unit area), Eq.A1.6 enables the corresponding value to be determined for any other value of ρU. However, this is, of course, valid only along changes that are sufficiently compact for the influence of lateral diffusion to be neglected

– e.g. close to strong wavefronts, but not during long periods of readjustment between successive wavefronts.

Use has been made of Eq.A1.6 in the development presented in Section 5.1 above.

A1.2 Coriolis coefficient

A similar process is now used to derive an equivalent expression describing variations in α. In this case, however, the result is less simple, but its potential use in numerical simulations is nevertheless straightforward. The development of this relationship begins with Eq.3.6.

Eq.3.6:
$$\frac{\partial \alpha}{\partial x} = -3(\alpha - \beta)\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x} \tag{A1.7}$$

$$\frac{1}{(\alpha - \beta)}\frac{\partial \alpha}{\partial x} = -3\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x} \tag{A1.8}$$

Using Eq.3.5:
$$\begin{aligned}\frac{1}{(\alpha - \beta)}\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial x}\right) \\ = -3\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x} \\ + 2\frac{(\beta - 1)}{(\alpha - \beta)}\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial x}\end{aligned} \tag{A1.9}$$

If it could be assumed that the ratio $(\beta-1)/(\alpha-\beta)$ is independent of 'x', then, following the same reasoning as above, this would lead to:

$$(\alpha - \beta)\rho^{\psi}U^{\psi} = Constant \tag{A1.10}$$

where ψ is defined as

$$\psi = \frac{3\alpha - 5\beta + 2}{\alpha - \beta} \tag{A1.11}$$

Unfortunately, there are no grounds for assuming that the ratio $(\beta-1)$ / $(\alpha-\beta)$ is independent of 'x' or even that it is nearly so. Therefore, the authors acknowledge freely that Eq. A1.10 is not valid. The purpose of presenting it herein is to highlight the difficulty in finding equations to describe the behaviour of α that are broadly analogous to those obtained above for β (especially Eq. A1.6). Furthermore, even if Eq.A1.10 could be assumed to be approximately valid, the fact that ψ is a function of α and β, not a simple constant, would greatly complicate its use in numerical analyses. It should also be noted that both α and β vary strongly when a wavefront causes the mean velocity of a well-established flow to become small. In such cases, the direct use of either of Eqs. A1.6 and A1.10 in a numerical solution is likely to become unstable. Indeed, both α and β become infinite when the mean velocity passes through zero and, moreover, the sign of α then reverses.

Notwithstanding these major deficiencies, one final point is of possible interest, namely that, in the particular case of $\alpha = 3\beta - 2$, which is commonly approximately true in quasi-steady flows in pipes, Eq.A1.11 reduces to $\psi = 2$, so that – using Eq.A1.6 – it follows that $(\alpha-1)\rho^2 U^2 = Constant$. The authors feel intuitively that this simple result could have a role to play in some circumstances, but they have not yet identified such a case.

APPENDIX 2 A CONUNDRUM

In the derivation of Eq.A1.6, the continuity and momentum equations are applied to individual streamtubes and the result is then integrated over the cross-section. Intriguingly, a slightly different result was obtained by Kruisbrink and Tijsseling (2008) who also considered a step change in velocity. The following analysis is broadly similar to theirs.

Consider an arbitrary velocity distribution defined by $u_1 = u_1\{A\}$ with a mean velocity U_1 and a Boussinesq coefficient β_1. If a uniform step change ΔU is imposed over the whole cross-section, the distribution will be $u_2\{A\} = u_1\{A\} + \Delta U$, for which the mean velocity is U_2 and the Boussinesq coefficient is β_2. Now express the product $\beta A U^2$ as follows (*N.B. Each step in the development is presented to minimise any risk of ambiguity*):

$$\beta_2 A U_2{}^2 = \int_A u_2{}^2\, dA \tag{A2.1}$$

$$\beta_2 A U_2{}^2 = \int_A (u_1 + \Delta U)^2\, dA \tag{A2.2}$$

$$\begin{aligned}\beta_2 A U_2{}^2 = \int_A u_1{}^2\, dA + \int_A (2\Delta U . u_1)\, dA \\ + \int_A (\Delta U)^2\, dA\end{aligned} \tag{A2.3}$$

$$\beta_2 A U_2{}^2 = A\left[\beta_1 U_1{}^2 + 2\Delta U . U_1 + (\Delta U)^2\right] \tag{A2.4}$$

$$\beta_2 A U_2{}^2 = A\left[(\beta_1 - 1) U_1{}^2 + (U_1 + \Delta U)^2\right] \tag{A2.5}$$

$$\beta_2 A U_2{}^2 = A\left[(\beta_1 - 1) U_1{}^2 + U_2{}^2\right] \tag{A2.6}$$

and so:

$$(\beta_2 - 1) U_2{}^2 = (\beta_1 - 1) U_1{}^2 \tag{A2.7}$$

That is, the imposed step does not change the product $(\beta - 1)\, U^2$.

Although Eqs.A1.6 and A2.7 have a close family resemblance, they are not exactly equivalent. – because wave-induced changes in velocity necessarily imply changes in pressure and hence in density, however small. When the authors first realised this, they assumed that at least one of the derivations must contain an error. However, careful re-development failed to reveal one, so it became necessary to explore the seemingly ridiculous possibility that both might be correct [*cf: Sherlock Holmes observed that, when all possibilities have been exhausted, it is necessary to consider the impossible*]. Remarkably, with the benefit of hindsight, it turns out that both equations are indeed correct. However, each is exact only when the assumptions on which it is based are respected. The implications of this statement in the context of pressure surge analysis are spelt out in the following paragraphs.

A strong clue to the dilemma can be obtained by noting that the derivation in this Appendix makes no use whatsoever of physical equations. No concept of flowrate is necessary for this purpose. If desired, the derivation can be regarded as a purely mathematical exercise yielding a relationship between mean-squares and square-means. In a context of fluid mechanics, however, it can be interpreted as a relationship between outcomes obtained when a particular flow state is expressed relative to two co-ordinate axes moving at a relative velocity of ΔU. As a simple example, imagine an idealistic case in which a flow along a straight pipe is exactly steady and is also exactly uniform. Then describe the flow relative to two observers who are moving relative to each other. Clearly, this does not change the flow itself even though the velocities relative to the two observers are different. Nevertheless, it does satisfy all of the conditions necessary to satisfy the derivation presented in this Appendix.

In contrast, any finite change in velocity prescribed in the development of Eq.A1.6 necessarily implies the existence of a wave and the equation shows that the product $(\beta - 1)$ U^2 does *not* remain constant, but instead varies inversely with the square of the density. Of course, analysts of pressure surges in liquid-filled pipelines may (reasonably) choose to simplify equations by disregarding the influence of density changes, but that will be merely a choice, not an assertion that such changes are truly irrelevant. In summary:
• Eq. A2.7 is valid for a change of axes describing the flow state at a single flow cross-section;
• Eq. A1.6 is valid for a two flow states expressed relative to the same axes.