The *abc*-Conjecture and the *n*-conjecture

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1 The *abc*-Conjecture

1.1 Definition

The *abc*-conjecture states that for three natural numbers a, b and c, if $a+b=c$ and $\gcd(a,b)=1$ at least one of the three does not have too (many) high powers in its prime decomposition. In order to formalize this vague definition, first the radical $r(N)$ of a natural number N needs to be defined:

Definition 1.1 *The radical of a natural number is:*

$$r(N) = \prod_{p|N, for\ p\ prime} p.$$

The formal description of the *abc*-conjecture now is as follows:

Conjecture 1.2 *The abc-conjecture:*
given any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that for every triple of positive integers a,b,c satisfying $a+b=c$ and $\gcd(a,b)=1$ it holds that $c \leq C(\varepsilon) \cdot r(a \cdot b \cdot c)^{1+\varepsilon}$.

In order to compare *abc*-triples the quality is introduced:

Definition 1.3 *The quality of an abc-triple, i.e. a triple of positive integers a,b,c satisfying $a+b=c$ and $\gcd(a,b)=1$, is:*

$$Q(a,b,c) = \frac{\log c}{\log r(a \cdot b \cdot c)}.$$

Considering the quality, the conjecture translates to the following: for any real $\eta > 1$, there is only a finite number of *abc*-triples satisfying $Q(a,b,c) > \eta$. For random a,b,c one can expect a quality of $Q \approx \frac{1}{3}$. To see this, take $a, b \approx N$ for N large, then $c \approx 2N$. In general it will hold that $r(a \cdot b \cdot c) \approx a \cdot b \cdot c$, so $Q(a,b,c) = \frac{\log c}{\log r(a \cdot b \cdot c)} \approx \frac{\log(2N)}{\log(2N^3)} \approx \frac{1}{3}$.
But there exist infinitely many *abc*-triples with $Q > 1$. Take $a=1$, $b=c-1$ and $c=9^n$, it is clear that $a+b=c$ and $\gcd(a,b)=1$. Furthermore, note that b is divisible by 8. This can be shown inductively; for $n=1$ we have $b=8$ so obviously $8|b$. Now assume that for $n=k$ we have $b=8\tilde{b}$, then $c=9^k=8\tilde{b}+1$. But then for $n=k+1$ we have $c=9^{k+1}=9\cdot(8\tilde{b}+1)=(9\tilde{b}+1)\cdot 8+1$ so $8|b$. The quality of such an *abc*-triple can then easily be bounded from below: $Q = \frac{\log c}{\log r(a \cdot b \cdot c)} \geq \frac{\log(8\tilde{b}+1)}{\log(2\tilde{b} \cdot 3)} > 1$.

1.2 Finding *abc*-triples

There are several strategies for finding good *abc*-triples, i.e. with high quality, namely Jaap Top's method [1], continued fractions [2], linear forms in logarithms [3] and brute force. In this section these strategies will be discussed briefly.

1.2.1 Jaap Top's method

Jaap Top's method tries to avoid large prime factors, the method works as follows: take coprime $n_1 = \prod p_{1,i}^{m_{1,i}}$, $n_2 = \prod p_{2,i}^{m_{2,i}}$ and $n_3 = \prod p_{3,i}^{m_{3,i}}$ of comparable size, say N. Now find an integral solution to the equation $a_1n_1+a_2n_2-a_3n_3 = 0$, where $a_i << N$. Finding this solution can be done by brute force. To generate a solution, list all options $in_1+jn_2+kn_3, 0 < i, j, k \leq M$ for a properly chosen upper bound M. There are M^3 different combinations for i, j, k and the total sum $in_1+jn_2+kn_3$ is bounded from above by $3MN$. In order to make sure at least two options are equal, we need at least $M^3 \geq 3MN \Leftrightarrow M \geq \sqrt{3N}$, so it suffices to choose $M = \lceil\sqrt{3N}\rceil$. Now given $b_1n_1+b_2n_2+b_3n_3 = c_1n_1+c_2n_2+c_3n_3$, the equation $(b_1-c_1)n_1+(b_2-c_2)n_2 + (b_3 - c_3)n_3$ with appropriate signs leads to $a_1n_1 + a_2n_2 = a_3n_3$. With this method *abc*-triples with $Q \approx 1$ are expected, since $\max|a_in_i| \approx \sqrt{3N}N$ and $r(a_1n_1a_2n_2a_3n_3) \leq r(a_1a_2a_3)r(n_1n_2n_3)$, where $r(a_1a_2a_3)$ is bounded by $\sqrt{3N}^3$ and $r(n_1n_2n_3)$ is small and can be kept constant for $N \to \infty$, so $Q(a, b, c) = \frac{\log(\max|a_in_i|)}{\log(r(a_1n_1a_2n_2a_3n_3))} \approx \frac{\log(\sqrt{3N}N)}{\log(\sqrt{3N})^3} \to 1$. When it happens that $r(a_1, a_2, a_3)$ is small and / or $\gcd(a_1a_2a_3, p_{1,1}\cdots p_{3,q}) > 1$, one may expect Q slightly larger than 1. It is however not known how this could be forced.

As an example, take $n_1 = 7^3$, $n_2 = 3^6$, $n_3 = 2^{10}$. We find for instance $2\cdot 7^3+44\cdot 3^6-30\cdot 2^{10} = 1 \cdot 7^3 - 37 \cdot 3^6 + 28 \cdot 2^{10}$. This yields the *abc*-triple $(a, b, c) = (7^3, 3^{10}, 2^{11} \cdot 29)$, with quality $Q(a, b, c) \approx 1.54708$. The quality is this high since miraculously $44 + 37 = 3^4$, if this would not be a prime power, the quality would be approximately $Q \approx \frac{\log(2^{11}\cdot 29)}{\log(2\cdot 3\cdot 7\cdot 29\cdot 81)} \approx 0.95586$.

1.2.2 Lattice Basis Reduction

Another possibility to find an integral solution to $a_1n_1 + a_2n_2 + a_3n_3 = 0$ is by using lattice basis reduction, due to Dokchitser [1]. Here a lattice is a discrete subgroup of $\mathbb{R}^n$ generated from a basis by all linear combinations with integral coefficients. Define $\theta \equiv -(n_1n_2^{-1}) \mod n_3$, $\psi = -\frac{n_1+n_2\theta}{n_3}$ and $u = \frac{a_2-a_1\theta}{n_3}$. Note that the modular inverse n_2^{-1} exists because n_2 and n_3 are coprime. Furthermore note that θ, ψ and u are integers (trivial for θ, the others are shown below). Now $\{(1, \theta, \psi), (0, n_3, -n_2)\}$ forms a lattice basis. Define $\Gamma = \begin{pmatrix} 1 & 0 \\ \theta & n_3 \\ \psi & -n_2 \end{pmatrix}$, this gives $\Gamma \begin{pmatrix} a_1 \\ u \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ (also shown below). Now take $\Gamma^* = \begin{pmatrix} 1 & 0 \\ \theta & n_3 \end{pmatrix}$ as basis and use the LLL-reduction algorithm ([4], see Appendix A, in this 2-dimensional case it is actually the Euclidean Algorithm) to reduce Γ^* to Γ^*_{red} where "reduced" is as defined in A.1. Determine $T = \Gamma^{*-1}\Gamma^*_{red}$, and use it to reduce Γ. This gives $\Gamma T = \Gamma_{red}$. It holds that $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \Gamma \begin{pmatrix} a_1 \\ u \end{pmatrix} = \Gamma T \begin{pmatrix} v \\ w \end{pmatrix} = \Gamma_{red} \begin{pmatrix} v \\ w \end{pmatrix}$ for integers v and w. Now loop over v, w in some range to find *abc*-triples. Since Γ_{red} is a reduced basis, all "small" lattice points will be found.

Again as example, take $n_1 = 7^3$, $n_2 = 3^6$, $n_3 = 2^{10}$. This gives $\Gamma_{red} = \begin{pmatrix} 13 & -25 \\ 29 & 23 \\ -25 & -8 \end{pmatrix}$.

Now $\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ yields $\Gamma_{red} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 13 & -25 \\ 29 & 23 \\ -25 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 81 \\ -58 \end{pmatrix}$ which again gives the *abc*-triple $(a, b, c) = (7^3, 3^{10}, 2^{11} \cdot 29)$.

Claim 1.4
ψ and u as defined above are integers.

Proof:
For ψ to be an integer, we must have $n_3|(n_1 + n_2\theta)$. Since $\theta \equiv -(n_1 n_2^{-1}) \mod n_3$, we have $n_2\theta \equiv -n_1 \mod n_3$. Then $n_1 + n_2\theta \equiv 0 \mod n_3$ and thus $n_3|(n_1 + n_2\theta)$, so that $\psi \in \mathbb{Z}$.
Since $a_1n_1 + a_2n_2 + a_3n_3 = 0$, it holds that $a_1n_1 + a_2n_2 \equiv 0 \mod n_3$ and therefore $a_1n_1n_2^{-1} + a_2 \equiv 0 \mod n_3$. Now since $a_2 - a_1\theta \equiv a_2 + a_1n_1n_2^{-1} \mod n_3$ it follows that $n_3|(a_2 - a_1\theta)$, so that $u \in \mathbb{Z}$.

Claim 1.5
$\Gamma \begin{pmatrix} a_1 \\ u \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Proof:
$\begin{pmatrix} 1 & 0 \\ \theta & n_3 \\ \psi & -n_2 \end{pmatrix} \begin{pmatrix} a_1 \\ u \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1\theta + un_3 \\ a_1\psi - n_2u \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1\theta + a_2 - a_1\theta \\ a_1\frac{-n_1-n_2\theta}{n_3} - n_2\frac{a_2-a_1\theta}{n_3} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \frac{-n_1a_1-n_2a_2}{n_3} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$,
using $a_1n_1 + a_2n_2 + a_3n_3 = 0$.

1.2.3 Continued Fractions

Continued fractions can be used to approximate real numbers by rational numbers [5]. A continued fraction of a given real number x can be constructed as follows: write x as $x = \lfloor x \rfloor + (x - \lfloor x \rfloor) := \lfloor x \rfloor + \{x\}$. Now this can be written as $x = \lfloor x \rfloor + \frac{1}{\left\lfloor \frac{1}{\{x\}} \right\rfloor + \left\{ \frac{1}{\{x\}} \right\}}$ and so on. For example, the continued fraction for π starts with:

$$\pi \approx 3 + \frac{1}{7 + \frac{1}{15+\frac{1}{1}}} = \frac{355}{113}.$$

Here $3+\frac{1}{7+\frac{1}{15+\frac{1}{1}}}$ is usually denoted as $[3, 7, 15, 1]$. Truncations of continued fractions are called convergents. Let $\alpha = [a_0, a_1, \ldots]$ and denote the n'th truncation as $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$, here the a_n are called partial quotients. Then it holds that $\frac{1}{(a_{n+1}+2)q_n^2} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}$ [5]. So it is clear that these convergents approximate the real number very well and in particular if the truncation is taken right before a relatively large partial quotient. A rational number is called a best approximation to a real number if all rational numbers which approximate this number closer have larger numerator and denominator. Furthermore, if an approximation is a convergent of a real number, it is exactly a best approximation and vice versa. Let $y = \sqrt[n]{A}$, with A rational, with a good approximation $\frac{p}{q}$, say $\frac{p}{q} = \sqrt[n]{A} + \varepsilon$. Then $|\varepsilon| < \frac{1}{q^2}$, so

$c = q^n A - p^n \in O(q^{n-2})$ and therefore one may expect quality $Q \approx \frac{\log(q^n A)}{\log(pqO(q^{n-2}))} \approx 1$.
An example; take $y = \sqrt[5]{109}$, the continued fraction starts with $[2, 1, 1, 4, 77733]$, so take $[2, 1, 1, 4] = 2 + \frac{1}{1+\frac{1}{1+\frac{1}{4}}} = \frac{23}{9}$. It follows that $\left(\frac{23}{9}\right)^5 = 109 + \frac{2}{9^5}$, so $(a, b, c) = (2, 3^{10} \cdot 109, 23^5)$ which has quality $Q(a, b, c) \approx 1.62991$. This is the record up to now, due to Eric Reyssat. Here the quality is this high since miraculously the partial quotient is high and 9 is a square.

1.2.4 Linear Forms in Logarithms

Another way to generate *abc*-triples is via linear forms in logarithms [6]. Given $a + b = c$, suppose that a is small compared to b and c. This implies that $\frac{a}{c}$ is close to zero. Write $1 - \frac{b}{c} = \frac{a}{c}$ with $\frac{b}{c} = \prod_{i=1}^{r} p_i^{e_i}$, then $-\frac{a}{c} \approx \log(\frac{b}{c}) = \sum_{i=1}^{r} e_i \log(p_i) = \Lambda$. The idea is that the p_i's are fixed but the e_i's not, so that Λ is a linear form in the variables e_i, with $\log(p_i)$ as coefficients. Such a Λ is called a linear form in logarithms.
Now we can write:

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ K\log(p_1) & \cdots & K\log(p_{r-1}) & K\log(p_r) \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ K\Lambda \end{pmatrix}.$$

The columns of the matrix form the basis of a lattice. Now if the *abc*-triple has a high quality, there will be a lattice point with untypically small distance to the origin, for properly chosen K.
In order to find this point, the LLL-algorithm is applied. In this LLL-reduced basis, the vectors with short distances to the origin will appear, and linear combinations with small coefficients are also close to the origin. The entries in this vector represent the coefficients in Λ in terms of the original basis. Let $(x_1, x_2, \ldots, x_r)$ be the shortest non-zero vector in the LLL-reduced basis of Γ with respect to $||\cdot||_2$. For r-dimensional lattices Γ, it holds in general that $d(\Gamma) := \min_{x \in \mathbb{Z}^r, x \neq 0} ||\Gamma x||_2 \approx (\det(\Gamma))^{\frac{1}{\dim(\Gamma)}} \approx (\det(\Gamma))^{\frac{1}{r}}$, unless the lattice is "distorted". So one would expect $||(x_1, x_2, \ldots, x_r)||_2 \approx K^{\frac{1}{r}}$ and therefore also $|x_i| \approx K^{\frac{1}{r}}$. Now define a partition $\zeta \subseteq \{1, 2, \ldots, k\}$ such that for every $i \in \zeta$ it holds that $x_i \geq 0$ and for every $i \in \{1, 2, \ldots, r\} \setminus \zeta$ it holds that $x_i < 0$. Then we can write

$$K \log(p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}) = K \log\left(\prod_{i \in \zeta} p_i^{x_i} \prod_{i \in \{1,2,\ldots,r\} \setminus \zeta} p_i^{x_i}\right) \approx K^{\frac{1}{r}}.$$

Then it holds that

$$\overbrace{\prod_{i \in \zeta} p_i^{x_i}}^{A:=} - \overbrace{\prod_{i \in \{1,2,\ldots,r\} \setminus \zeta} p_i^{-x_i}}^{B:=} \approx K^{-\frac{r-1}{r}} \prod_{i \in \{1,2,\ldots,r\} \setminus \zeta} p_i^{-x_i}.$$

So for the quality $Q(A,B,A-B)$ we may expect

$$Q \approx \frac{\log(B)}{\log(K^{-\frac{r-1}{r}}) + B + \log\left(\prod_{i=1}^{r} p_i\right)} \approx \frac{K^{\frac{1}{r}} \log\left(\prod_{i\in\{1,2,\ldots,r\}\setminus\zeta} p_i\right)}{K^{\frac{1}{r}} \log\left(\prod_{i\in\{1,2,\ldots,r\}\setminus\zeta} p_i\right) - \frac{r-1}{r}\log(K) + \log\left(\prod_{i=1}^{r} p_i\right)} \overset{(*)}{\approx} 1.$$

With $(*)$ since $K^{\frac{1}{r}}$ is the dominant term.

For example, take $p_1 = 3$, $p_2 = 109$, $p_3 = 23$ and

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ [K\log(p_1)] & [K\log(p_2)] & [K\log(p_3)] \end{pmatrix}.$$

After applying LLL for $K = 620000$ the reduced basis

$$\tilde{\Gamma} = \begin{pmatrix} 10 & 29 & 218 \\ 1 & -353 & -31 \\ -5 & -16 & 437 \end{pmatrix}$$

remains. One would typically expect $d(\Gamma) \approx (\det(\Gamma))^{\frac{1}{\dim(\Gamma)}} \approx K^{\frac{1}{3}} \approx 85$. In this case $d(G) = |(10,1,-5)|_2 \approx 11$, which is significantly smaller. To construct the *abc*-triple, simply reverse the process described earlier: $\Lambda = 10\log(3) + \log(109) - 5\log(23)$, it appears that $23^5 - 3^{10} \cdot 109 = 2$ and so our *abc*-triple is $(a,b,c) = (2, 3^{10} \cdot 109, 23^5)$ with $Q \approx 1.62991$.

1.2.5 Brute Force

There are several algorithms to find all *abc*-triples with $a,b,c \leq N$ and $Q > 1$. One of these algorithms, used in "Reken mee met ABC" (`http://www.rekenmeemetabc.nl/` or `http://www.abcathome.com/`) is as follows; first of all, suppose that we have an *abc*-triple with $Q > 1$, then it holds that $r(abc) < c \leq N$. Sort a, b and c by the size of their radical, such that we have $r(x) < r(y) < r(z)$. Then also $r(xyz) = r(abc) < c \leq N$ and $r(y)^2 < r(y)r(z) \leq r(x)r(y)r(z) \leq N \Rightarrow r(y) < \sqrt{N}$ and $r(x) < \frac{N}{r(y)^2}$. Now to find all *abc*-triples with $a,b,c \leq N$ and $Q > 1$ the following steps are needed:

1. Find all radicals $r < \sqrt{N}$, these are the possible radicals for y
2. Find all numbers with these radicals, these are the possible numbers for y
3. For every radical $r(y)$ find all radicals $r < \frac{N}{r(y)^2}$, these are the possible radicals for x
4. Find all numbers with these radicals, these are the possible numbers for x
5. Define $z_1 = |x-y|$ and $z_2 = x+y$
6. For $z = z_1$ and $z = z_2$, compute $r(z)$ and check whether $r(xyz) < \max\{x,y,z\}$

Here a "radical" is a number which has no powers in its prime decomposition. The first 5 steps of this algorithm use only additions, multiplications and comparisons so are computationally fast. The factorization in the last step is computationally slow, but the first steps ensure that the amount of radicals to compute is kept minimal. So far, all triples up to $N = 10^{12}$ have been checked.

2 p-adic numbers

The methods given in 1.2.3 and 1.2.4 can be generalized to the p-adic numbers [7]. In order to make this plausible, first the p-adic numbers will be defined.

2.1 Definition

The real numbers can be defined as a completion of the rational numbers through convergent sequences. In order to do this, convergence needs to be defined. We will use the Cauchy criterion for this:

Definition 2.1 *A sequence (a_n) is convergent if and only if for every $\varepsilon > 0$ there exists a number $n_0 \in \mathbb{N}$ such that for all $n,m > n_0$ it holds that $|a_n - a_m| < \varepsilon$.*

Note that the concept of convergence only depends on the absolute value. The idea of p-adic numbers is based on defining a different absolute value. Before this is done, an absolute value is formally defined:

Definition 2.2 *An absolute value on $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{Q}_+$, which satisfies the following conditions:*

1. $|x| = 0 \Leftrightarrow x = 0$
2. $|xy| = |x||y|$ *for all* $x, y \in \mathbb{Q}$
3. $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{Q}$.

Now to define the p-adic absolute value, we will first define the order of a number with respect to a prime p.

Definition 2.3 *The order with respect to a prime p of an integer $n \in \mathbb{Z}\backslash\{0\}$ or rational number $\frac{a}{b}$ with $a \in \mathbb{Z}, b \in \mathbb{Z}\backslash\{0\}$ is:*

- $\text{ord}_p(n) = k$ *with k such that $n = p^k n'$, $n' \in \mathbb{N}$ and $p \nmid n'$*
- $\text{ord}_p(0) = \infty$
- $\text{ord}_p(\frac{a}{b}) = \text{ord}_p(a) - \text{ord}_p(b)$.

Note that this is well defined: $\text{ord}_p(\frac{a}{b})$ is unique, because common divisors of a and b cancel out. Note also that $\text{ord}_p(xy) = \text{ord}_p(x) + \text{ord}_p(y)$ holds for all $x, y \in \mathbb{Q}$ (proof is obvious).

Definition 2.4 *The p-adic absolute value is defined to be:* $|q|_p = p^{-\text{ord}_p(q)}, q \in \mathbb{Q}$.

If this absolute value is to be used to complete the rational numbers, it must be checked that it meets the requirements of an absolute value.

Theorem 2.5 *The p-adic value is an absolute value on* $\mathbb{Q}$*, i.e. satisfies Definition 2.2.*

Proof:

- Take $x = 0$, then $|0|_p = p^{-\infty} = 0$. Next take $|x|_p = 0$ then $p^{-\text{ord}_p(x)} = 0$ therefore $\text{ord}_p(x) = \infty$ and thus $x = 0$.
- Let $x, y \in \mathbb{Q}$ be given. $|xy|_p = p^{-\text{ord}_p(xy)} = p^{-(\text{ord}_p(x)+\text{ord}_p(y))} = p^{-\text{ord}_p(x)}p^{-\text{ord}_p(y)} = |x|_p|y|_p$.
- Let $x, y \in \mathbb{Q}$ be given. $|x+y|_p = p^{-\text{ord}_p(x+y)} \overset{(i)}{\leq} p^{-\min\{\text{ord}_p(x),\text{ord}_p(y)\}} = \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$.

(i) First write $x = p^k x'$ and $y = p^l y'$, now without loss of generality assume $k \geq l$. Then $(x + y) = p^l(p^{k-l}x' + y')$ with $p \nmid (p^{k-l}x' + y')$. Now it is easy to see that $\text{ord}_p(x + y) \geq \min\{\text{ord}_p(x), \text{ord}_p(y)\}$.

The p-adic absolute value acts against the instinct, for example $|\frac{1}{4}|_2 = 4$ and $|4|_2 = \frac{1}{4}$. So in fact, if the number is "more divisible" by p, the absolute value becomes smaller.
Now that we have a notion of convergence, it remains to show that this leads to a field essentially larger than $\mathbb{Q}$ itself. To show this, we need to define completeness:

Definition 2.6 *A field* $\mathbb{K}$ *is complete with respect to* $|\cdot|$ *if for every Cauchy sequence* (x_n) *of elements in* $\mathbb{K}$ *there is an* $L \in \mathbb{K}$ *such that* $\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n > n_0 : |x_n - L| < \varepsilon$.

It follows that $\mathbb{Q}$ is not complete with respect to the normal absolute value, take for example the sequence $x_1 := 1$ and $x_{n+1} := \frac{x_n}{2} + \frac{1}{x_n}$. This forms a Cauchy sequence of rational numbers and it converges to an irrational number, namely $\sqrt{2}$. Now adding all limits of Cauchy sequences to $\mathbb{Q}$ we get $\mathbb{R}$. The same can be done with respect to the p-adic absolute value.

Theorem 2.7 $\mathbb{Q}$ *is not complete with respect to* $|\cdot|_p$

Before we prove this, first the following lemmas are given:

Lemma 2.8 *If* $x_n^3 \equiv 3 \mod 2^n$ *then either* $x_n^3 \equiv 3 \mod 2^{n+1}$ *or* $(x_n + 2^n)^3 \equiv 3 \mod 2^{n+1}$ $(n \geq 1)$.

Proof: Let $x_n^3 \equiv 3 \mod 2^n$ be given, so $x_n^3 = 3 + 2^n k$. Now if k is even, it holds that $x_n^3 = 3 + 2^{n+1}l$ so $x_n^3 \equiv 3 \mod 2^{n+1}$. If k is odd then $(x_n + 2^n)^3 \equiv 2^{3n} + 3x_n 2^{2n} + 3x_n^2 2^n + x_n^3 \mod 2^{n+1} \equiv 2^n(3x_n^2 + k) + 3 \mod 2^{n+1} \equiv 3 \mod 2^{n+1}$, since both x_n and k are odd.

Lemma 2.9 *A sequence (x_n) of rational numbers is a Cauchy sequence with respect to $|\cdot|_p$ if and only if $\lim_{n\to\infty} |x_{n+1} - x_n|_p = 0$.*

Proof: If $m = n+k > n$, then $|x_m - x_n|_p = |x_{n+k} - x_{n+k-1} + x_{n+r-1} - x_{n+k-2} + \cdots + x_{n+1} - x_n|_p \leq max\{|x_{n+k} - x_{n+k-1}|_p, |x_{n+k-1} - x_{n+k-2}|_p, \ldots, |x_{n+1} - x_n|_p\}$.

Lemma 2.10 *Given $x_n^2 = a \mod p^{n+1}$, it is always possible to find an x_{n+1} such that $x_{n+1} \equiv x_n \mod p^{n+1}$ and $x_{n+1}^2 \equiv a \mod p^{n+2}$.*

Proof: Let x_n be given. Take k such that $x_n^2 + 2kp^{n+1} \equiv a \mod p^{n+2}$. Such a k exists, since $x_n^2 \equiv a \mod p^{n+1}$. It then holds that $x_{n+1} \equiv x_n \mod p^{n+1}$ and $x_{n+1}^2 \equiv x_n^2 + 2kp^{n+1} + kp^{2(n+1)} \mod p^{n+2} \equiv x_n^2 + 2kp^{n+1} \mod p^{n+2} \equiv a \mod p^{n+2}$.

Now we are ready to prove Theorem 2.7:

Proof: The proof is given in two distinct cases, first for $p = 2$ and then for $p \neq 2$.
Let $p = 2$, we will construct a Cauchy sequence that tends to $\sqrt[3]{3}$. In order to do this, we will construct a sequence (x_n) which satisfies $x_n^3 \equiv 3 \mod 2^n$. Take $x_0 = 1$ and apply Lemma 2.8 to inductively define x_n. In order for this sequence to be a Cauchy sequence tending to $\sqrt[3]{3}$, it must hold that $|x_{n+1} - x_n|_2 \to 0$ (Lemma 2.9) and $|x_n^3 - 3|_2 \to 0$. Now because of the way the sequence is constructed, we either have $x_{n+1} = x_n$ or $x_{n+1} = x_n + 2^n$, so $|x_{n+1} - x_n|_2 \leq \frac{1}{2^{n+1}} \to 0$. Furthermore since $x_n^3 \equiv 3 \mod 2^n$ we know that $|x_n^3 - 3|_2 \leq \frac{1}{2^n} \to 0$. And this completes the proof for the case $p = 2$.
Let $p \neq 2$, again we will construct a Cauchy sequence, but this time with a more general limit not in $\mathbb{Q}$. Take some $a \in \mathbb{Z}$ such that a is not a square in $\mathbb{Q}$, $p \nmid a$ and a is a quadratic residue modulo p. Now choose x_0 to be a solution to $x_0^2 \equiv a \mod p$ and choose x_n to satisfy $x_{n+1} \equiv x_n \mod p^{n+1}$ and $x_{n+1}^2 \equiv a \mod p^{n+2}$. By Lemma 2.10 x_{n+1} exists. Now again we must show that $|x_{n+1} - x_n|_p \to 0$ and $|x_n^2 - a|_p \to 0$. By construction it holds that $|x_{n+1} - x_n|_p \leq \frac{1}{p^{n+1}} \to 0$ and $|x_n^2 - a|_p \leq \frac{1}{p^{n+1}} \to 0$. This completes the proof.

Now that we know that $\mathbb{Q}$ is not complete with respect to the p-adic absolute value, we can make a completion. To do this, we will add the limits of all Cauchy sequences to $\mathbb{Q}$. We could define these limits first, but this can be avoided by just adding the sequence itself, including the limit.

Definition 2.11 *Let $\mathcal{C}_p$ denote all Cauchy sequences with respect to $|\cdot|_p$ in $\mathbb{Q}$.*

It follows that $\mathcal{C}_p$ has a commutative ring structure if the sum and product are defined as $(x_n) + (y_n) = (x_n + y_n)$ and $(x_n)(y_n) = (x_n y_n)$ respectively. Unfortunately, $\mathcal{C}_p$ is not a field, since it has zero divisors. Also, $\mathcal{C}_p$ can contain different sequences which should have the same limit. These sequences can be identified by taking a quotient of $\mathcal{C}_p$.

Definition 2.12 *Take $\mathcal{N}_p \subset \mathcal{C}_p$ as the ideal of sequences that tend to zero:*
$\mathcal{N}_p = \{(x_n) | \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \forall n > n_0 : |x_n|_p < \varepsilon\}$.

This ideal forms a maximal ideal of $\mathcal{C}_p$:

Theorem 2.13 *$\mathcal{N}_p$ is a maximal ideal of $\mathcal{C}_p$.*

Proof: First let I be the ideal generated by $(x_n) \notin \mathcal{N}_p$ and $\mathcal{N}_p$. If $\mathcal{N}_p$ is maximal, I must contain all of $\mathcal{C}_p$. To show this, it suffices to show that I contains the unit element $u_n = \{1, 1, 1, \ldots\}$. Since (x_n) is Cauchy, there exist $r, N > 0$ such that for $n \geq N$ it holds that $|x_n|_p \geq r > 0$. Now define a sequence (y_n) by $y_n = 0$ for $n < N$ and $y_n = \frac{1}{x_n}$ if $n \geq N$. This forms a Cauchy sequence, since for $n > N$ it holds that $|y_{n+1} - y_n|_p = |\frac{1}{x_{n+1}} - \frac{1}{x_n}|_p = \frac{|x_{n+1}-x_n|_p}{|x_{n+1}x_n|_p} \leq \frac{|x_{n+1}-x_n|_p}{r^2} \to 0$. It follows that $u_n - (x_n)(y_n) \in \mathcal{N}_p$, but then the unit element can be written as a multiple of (x_n) and an element in $\mathcal{N}_p$ and thus is in I. Thus we conclude that $\mathcal{N}_p$ is maximal.

Since $\mathcal{N}_p$ is a maximal ideal of $\mathcal{C}_p$, it is now possible to define the field of p-adic numbers, namely as follows:

Definition 2.14 *Define the field of p-adic numbers as $\mathbb{Q}_p = \mathcal{C}_p/\mathcal{N}_p$.*

By Ostrowski's Theorem (presented below without a proof) there exists no other nontrivial completion of the rational numbers, other than the p-adic numbers and the real numbers.

Theorem 2.15 *(Ostrowski) Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to one of the absolute values $|\cdot|_p$, where either p is prime or $p = \infty$.*

2.2 Representation of p-adic numbers

Now that $\mathbb{Q}_p$ is formally defined, it is useful to define a representation for p-adic numbers. The p-adic expansion of a positive integer is the same as the representation in base p. For example, $37 = 2 \cdot 5^0 + 2 \cdot 5^1 + 1 \cdot 5^2 \stackrel{5}{=} .221$. This can be generalized to positive rationals in an intuitive manner. Given $\frac{a}{b}$, write both a and b in powers of p and divide formally. In essence, for every p one can write any positive rational number as $\sum_{n \geq n_0} a_n p^n$, where $n_0 \in \mathbb{Z}$ is such that $\frac{a}{b} = p^{n_0} \frac{a'}{b'}$, with $p \nmid a'b'$. The corresponding p-adic representation is $\frac{a}{b} \stackrel{p}{=} a_{n_0} a_{n_0+1} \ldots$. If the p-adic representation of x is given, one can find the representation of $-x$ by using that the two must add up to zero. For example, $-37 \stackrel{5}{=} .32344444\ldots$, since $.221 + .3234444\ldots \stackrel{5}{=} .000000\ldots$. This can also be seen by writing $3 + 2 \cdot 5 + 3 \cdot 5^2 + 4 \cdot 5^3 \frac{1}{1-5} = 88 + \frac{4}{-4} \cdot 5^3 = 88 - 125 = -37$. An example of a number which is in $\mathbb{Q}_5$ but not in $\mathbb{Q}$ is $\sqrt{-1}$. The p-adic representation can be found by solving $\sqrt{-1} = \sum_{k=0}^{\infty} a_k 5^k$. In this example, we have $\sqrt{-1} \equiv a_0 \mod 5$, so $a_0^2 \equiv -1 \mod 5 \equiv 4 \mod 5 \Leftrightarrow a_0 = 2$ or 3. Now we know either $\sqrt{-1} \equiv 2 + a_1 \cdot 5 \mod 5^2$ or $\sqrt{-1} \equiv 3 + a_1 \cdot 5 \mod 5^2$, so $(2 + a_1 \cdot 5)^2 \equiv -1 \mod 5^2 \Leftrightarrow a_1 = 1$ or $(3 + a_1 \cdot 5)^2 \equiv$

$-1 \mod 5^2 \Leftrightarrow a_1 = 3$, etcetera. So in general, we can write $\mathbb{Q}_p = \{\alpha = \sum_{n=n_0}^{\infty} a_n p^n | a_n \in \{0, 1, \ldots, p-1\}, n_0 = \text{ord}_p(\alpha)\}$.

2.3 Finding *abc*-triples via *p*-adic numbers

As stated before, the methods given in 1.2.3 and 1.2.4 also work for *p*-adic numbers. In this section the *p*-adic version of these methods will be discussed briefly.

2.3.1 Approximation of *p*-adic Numbers

The method of continued fractions is based on approximations of real numbers by rational numbers. If a *p*-adic version of this method is to be given, approximation of *p*-adic numbers [3] must be defined first. Let p be prime and let $\alpha \in \mathbb{Q}_p$. Write $\alpha = \sum_{i=k}^{\infty} a_i p^i$ and $\alpha_m = \sum_{i=k}^{m-1} a_i p^i$, for $m \in \mathbb{Z}$, $a_i \in \{0, 1, \ldots, p-1\}$ and $k = \text{ord}_p(\alpha)$.

Definition 2.16 *The ordered pair of rational integers (x, y) is called a p-adic approximation to α of order m if $|x - y\alpha|_p = p^{-m}$. The set $\Gamma_m = \{(x, y) \in \mathbb{Z}^2 : |x - y\alpha|_p \leq p^{-m}\}$ is called the mth approximation lattice of α.*

With this definition, the following holds:

Lemma 2.17 $\{(p^m, 0), (\alpha_m, 1)\}$ *is a basis for* Γ_m.

Proof: In other words, to be proven is that:

(i) $(p^m, 0) \in \Gamma_m$ and $(\alpha_m, 1) \in \Gamma_m$

(ii) For every $\beta, \gamma \in \mathbb{Z}$ it holds that $\beta(p^m, 0) + \gamma(\alpha_m, 1) = 0 \Rightarrow \beta = \gamma = 0$

(iii) $x \in \Gamma_m \Rightarrow$ there exist $\beta, \gamma \in \mathbb{Z}$ such that $x = \beta(p^m, 0) + \gamma(\alpha_m, 1)$

(i) and (ii) are trivial, for (iii): Let $x \in \Gamma_m$, and write $x = (x_1, x_2) \in \mathbb{Z}^2$. Then $|x_1 - x_2\alpha|_p \leq p^{-m}$, so $x_1 \equiv x_2\alpha \mod p^m \equiv x_2\alpha_m \mod p^m \Leftrightarrow x_1 = x_2\alpha_m + kp^m$, for $k \in \mathbb{Z}$. This implies that $x = (x_2\alpha_m + kp^m, x_2) \Leftrightarrow \gamma = x_2$, $\beta = k$.

This lattice can be reduced with respect to a convex norm Φ by the following algorithm (which is a variant of the Euclidian algorithm):

Algorithm 2.18 *Let $\{(x, y), (z, u)\}$ be a basis of Λ.*

(i) Compute the minimal $k \in \mathbb{Z}$ for which $\Phi(x + kz, y + ku)$ is minimal. Put $(x, y) := (x + kz, y + ku)$.

(ii) If $\Phi(x,y) < \Phi(z,u)$ then interchange (x,y) and (z,u), and go to (i); else stop.

So in order to find *abc*-triples, choose $\alpha \in \mathbb{Q}_p$ and $m \in \mathbb{N}$. Then construct the mth order p-adic approximation lattice and reduce with the algorithm. For an mth approximation lattice Γ_m, it holds for $(x,y) \in \Gamma_m$ that $|x - y\alpha|_p \leq p^{-m}$, so $x - y\alpha \equiv 0 \mod p^m$. Now it is easy to find a $l \in \mathbb{Z}$ such that $x = lp^m + y\alpha$.
If one chooses $\alpha = \sqrt[n]{k}$ then one may expect a quality of $Q \approx 1$. This because one may expect that $d(\Gamma_m) \approx \det(\Gamma_m)^{\frac{1}{\deg(\Gamma_m)}} = p^{\frac{m}{2}}$, so $|x - y\alpha|_p \leq p^{-m} \Rightarrow x - y\sqrt[n]{k} \equiv 0 \mod p^m \Leftrightarrow x^n = ky^n + \xi p^m$, which forms our $a + b = c$, has quality $Q = \frac{\log(c)}{\log(r(abc))} \approx \frac{\log(p^{\frac{mn}{2}})}{\log(xyk\xi p)} \overset{(*)}{\approx} \frac{\frac{mn}{2}\log(p)}{\frac{mn}{2}\log(p)} = 1$. With $(*)$ because $|x| \approx |y| \approx p^{\frac{m}{2}}$ and $|\xi p^m| \approx p^{\frac{mn}{2}} \Leftrightarrow |\xi| \approx p^{\frac{mn}{2}-m}$.
For example in $\mathbb{Q}_3$, take $\alpha = \sqrt[5]{2}$. Then $\alpha \overset{3}{=} .21200000001\ldots$ (a miracle), so with $m = 10$ it follows that $\{(p^m, 0), (\alpha_m, 1)\}$ is a basis of Γ_m. Now $\alpha_m = 23$, so $(23, 1)$ yields $|23 - \sqrt[5]{2}|_3 \leq 3^{-10}$. This implies $23^5 \equiv 2 \mod 3^{10}$, so $23^5 - 2 = k3^{10}$, which leads to the *abc*-triple $(a, b, c) = (2, 3^{10}109, 23^5)$.

2.3.2 Linear Forms in p-adic Logarithms

To use linear forms in logarithms, first the p-adic logarithm must be defined [7, Chapter 4.3]. Without worrying about convergence, define the p-adic logarithm as follows:

Definition 2.19 $\log_p(1-x) = \sum_{k=1}^{\infty} -\frac{1}{k}x^k$.

The p-adic logarithm converges on the open ball $B(1,1)$ and the properties $\log_p(ab) = \log_p(a) + \log_p(b)$, $\log_p(a^n) = n\log_p(a)$ for $n \in \mathbb{N}_+$ and $a = b^c \Leftrightarrow c = \frac{\log_p(b)}{\log_p(a)}$ hold. But there are subtleties, for roots of unity the p-adic logarithm is 0. $\mathbb{Q}_p$ contains exactly $p-1$ $(p-1)$-roots of unity and no others for p odd. So in $\mathbb{Q}_p$ $\log_p(a) = \log_p(b) \Rightarrow a = b$ no longer holds, but $\log_p(a) = \log_p(b) \Rightarrow a = bu$ with u a $(p-1)$th root of unity.

Now the method in p-adics [6] is as follows. Let $a + b = c$ and suppose that $c - a = b \equiv 0 \mod p^m$. Then $\frac{b}{c} \approx \log_p \frac{a}{c} \equiv 0 \mod p^m$, write $\frac{a}{c} = \prod_{i=1}^{k} p_i^{e_i}$ so that $\frac{b}{c} \approx \sum_{i=1}^{k} e_i \log_p(p_i)$. Define for $i = 1, \ldots, k-1$: $\theta_i = \frac{\log_p(p_i)}{\log_p(p_k)} \mod p^m$.
Now $\Gamma = \{(1, 0, \ldots, 0, \theta_1), (0, 1, 0, \ldots, 0, \theta_2), \ldots, (0, \ldots, 0, 1, \theta_{k-1}), (0, \ldots, 0, p^m)\}$ forms the basis of a lattice by Lemma 2.17. It holds for $x \in \Gamma$ that $|x_k - \sum_{i=1}^{k-1} x_i\theta_i|_p \leq p^{-m}$ and therefore $x_k - \sum_{i=1}^{k-1} x_i\theta_i \equiv 0 \mod p^m$. This implies $\sum_{i=1}^{k} x_i \log_p(p_i) \equiv 0 \mod p^m$ and so $\prod_{i=1}^{k} p_i^{x_i} \equiv 1 \mod p^m$. Now there exists a partition $\zeta \subseteq \{1, \ldots, k\}$ such that for every $i \in \zeta$ it holds that $x_i \geq 0$ and for every $i \in \{1, \ldots, k\} \backslash \zeta$ it holds that $x_i < 0$. Then $\prod_{i \in \zeta} p_i^{x_i} = \prod_{i \in \{1,\ldots,k\}\backslash\zeta} p_i^{x_i} + l \cdot p^m$,

which forms our abc-triple.

With such an abc-triple, it holds that $r(abc) \approx l < \frac{\max\left\{\prod_{i\in\zeta} p_i^{x_i}, \prod_{j\in\{1,\ldots,k\}\setminus\zeta} p_j^{x_j}\right\}}{p^m} \approx \frac{c}{p^m}$. Now the quality may expected to be $Q \approx \frac{\log(c)}{\log(c)-m\log(p)} \approx 1$.

As example, consider $\theta = \frac{\log_3(23)}{\log_3(2)} \mod 3^7 \stackrel{3}{=} .2012101 \mod 3^7$. Now $\{(1,\theta),(0,3^7)\}$ forms the basis of a lattice. The LLL-reduction algorithm returns a reduced basis $\{(5,1),(-82,421)\}$. It follows that $1 \equiv 5\frac{\log_3(23)}{\log_3(2)} \mod 3^7$, so $1 \cdot \log_3(2) \equiv 5 \cdot \log_3(23) \mod 3^8$, and therefore $2 = 23^5 + k \cdot 3^8$ for $k \in \mathbb{Z}$. This gives the abc-triple $2 + 3^{10}109 = 23^5$.

3 Discussion of methods

All methods obviously have advantages and disadvantages. Some properties are given in the following table:

Table 1: Properties of given methods

Method	Expected Quality	Advantages	Disadvantages
Jaap Top	$Q \approx 1$	Avoids large prime factors	Primes and powers fixed, loops over $\sqrt{3N}$ combinations ($N \approx c$)
Lattice Basis Reduction	$Q \approx 1$	Avoids large prime factors, can generate several triples for one basis reduction	Primes and powers fixed
Continued Fractions	$Q \approx 1$ for $\alpha = \sqrt[n]{k}$	Works fast	No expected results, except when strategies for certain numbers to approximate are used
Linear Forms in Logarithms	$Q \approx 1$	Several prime factors can be chosen, powers are not fixed	Primes are fixed, only one term can have other factor, only works when a is small
Brute Force	$Q \approx \frac{1}{3}$	Every triple will be found	Time consuming
p-adic Continued Fractions	$Q \approx 1$	Works fast	No expected results, except when strategies for certain numbers to approximate are used
p-adic Linear Forms in Logarithms	$Q \approx 1$	Several prime factors can be chosen, powers are not fixed	Primes are fixed, only one term can have other factor, only works when $\lvert a\rvert_p$ is small

There are several aspects on which methods can be compared, for instance the triples which are found, the expected quality and the computation time needed. When comparing only the triples that can be found by the methods, it is safe to state that no method finds triples that other methods could not find. If one knows the *abc*-triple, for every method it is possible to choose the variables in such a way that it will find this triple, but perhaps not efficiently.
Since the expected quality for all methods is $Q \approx 1$ with the exception of brute force and continued fractions without number strategies, they do not significantly differ on this aspect. So the computation time needed remains. For Jaap Top's method, the time needed to find an *abc*-triple for a set of coprime n_1, n_2, n_3 of size N will be $O(\sqrt{N})$, since it will make at most $\sqrt{N}$ comparisons. The Lattice Basis Reduction will find an *abc*-triple for a set of coprime n_1, n_2, n_3 of size N in $O(n^6 \log^3(N))$ since the LLL-reduction algorithm will find a reduced basis in $O(n^6 \log^3(N))$ time [4], here n is the dimension of the lattice. Furthermore,

the Lattice Basis Reduction method may find more *abc*-triples for one set of n_1, n_2, n_3 since it loops over linear combinations of vectors in the reduced basis. For Continued Fractions it is harder to state the computation time for an *abc*-triple, since it is not clear from the choice of the number which convergent should be chosen. According to [2] it is not relevant in general if the continued fraction is truncated before a large a_i or even if a convergent with large denominator is chosen. The Linear Forms in Logarithms method will find *abc*-triples in $O(n^6 \log^3(N))$ time, where $N = \max_i p_i$ since it also uses LLL.

4 The n-conjecture

The *abc*-conjecture has been generalized by Browkin-Brzezinski in [2] to the n-conjecture as follows:
Let $a_1, a_2, \ldots, a_n \in \mathbb{Z} n \geq 3$, satisfy:

(i) $\gcd(a_1, a_2, \ldots, a_n) = 1$

(ii) $a_1 + a_2 + \cdots + a_n = 0$

(iii) no proper subsum of (ii) is equal to 0.

Define $Q_n(a_1, a_2, \ldots, a_n) = \dfrac{\log\left(\max\limits_{1\leq i\leq n}(|a_i|)\right)}{\log\left(r(a_1 \cdot a_2 \cdots a_n)\right)}$ as the quality of such $a_1, \ldots, a_n$.

Conjecture 4.1 *The n-conjecture states that, for given $n \geq 3$, $\limsup\{Q_n\} = 2n - 5$.*

In order to find such $a_1, \ldots, a_n$, the method given in 1.2.2 can be generalized.

4.1 Lattice Basis Reduction for the n-conjecture

Let $n_1, n_2, \ldots, n_r \approx N$ with $r(n_1 \cdot n_2 \cdots n_r) = P$. Define $\theta_i = -n_i n_{r-1}^{-1} \mod n_r$, $\psi_i = -\frac{n_i + n_{r-1}\theta_i}{n_r}$ and $u = \frac{a_{r-1} - a_1\theta_1 - \ldots - a_{r-2}\theta_{r-2}}{n_r}$ for $i = 1, 2, \ldots, r-2$. Now take

$$\Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \theta_1 & \cdots & \theta_{r-2} & n_r \\ \psi_1 & \cdots & \psi_{r-2} & -n_{r-1} \end{pmatrix}$$

as $(r-1)$-dimensional lattice in $\mathbb{Z}$. It holds that

$$\Gamma \begin{pmatrix} a_1 \\ \vdots \\ a_{r-2} \\ u \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r \end{pmatrix}.$$

Reduce Γ with LLL and loop over $a_1, \dots, a_{r-2}, u$ to find $a_1, \dots, a_r$ that satisfy the conditions in the n-conjecture. Note that this method will work with only the demand $\gcd(n_r, n_{r-1}) = 1$, other pairs may have larger common divisors.
Again one may expect $d(\Gamma) \approx N^{\frac{1}{r-1}}$, so $|a_i| \approx N^{\frac{1}{r-1}}$. Now $r(a_1 \cdot n_1 \cdot a_2 \cdot n_2 \cdots a_r \cdot n_r) \approx N^{\frac{r}{r-1}} P$ and $a_i \cdot n_i \approx N^{\frac{1}{r-1}+1} = N^{\frac{r}{r-1}}$, so for the quality one may expect $Q_n \approx \frac{\frac{r}{r-1}\log(N)}{\frac{r}{r-1}\log(N)+\log(P)} \approx 1$.
This result gives reason to expect that the bound in the n-conjecture can be tightened under some demands. For the case of the $n = 4$ we will look further into this.

4.2 The 4-conjecture

In particular the 4-conjecture, or *abcd*-conjecture, states that $\limsup\{Q_4\} = 3$. One can find *abcd* satisfying the conditions straight from *abc*-triples, because if $a + b = c$ with $Q(a, b, c) = q_0$ then $a^3 + 3abc + b^3 = c^3$ with $Q(a^3, 3abc, b^3, c^3) \approx 3q_0$ [2]. But with this method it is not necessary that the greatest common divider for every pair equals 1, for instance in $2 \cdot 7 \cdot 11^2 + 5 \cdot 11 \cdot 13 = 2 \cdot 3 \cdot 5 \cdot 53 + 3^2 \cdot 7 \cdot 13$. If one adds the demand that the pairwise greatest common divider equals 1, perhaps the bound can be tightened such that $\limsup\{Q_4\} = 1$. The bound of 1 is proposed because of the argument given in 4.1. In order to strengthen this proposition, examples are generated.

4.2.1 *abcd*-examples without extra gcd demands

The conditions proposed by Browkin-Brzezinski for the *abcd*-conjecture are as follows:
Let $a, b, c, d \in \mathbb{Z}$ satisfy:

(i) $\gcd(a, b, c, d) = 1$

(ii) $a + b + c + d = 0$

(iii) $(a + b)(a + c)(a + d)(b + c)(c + d) \neq 0$

Without loss of generality, assume $d > a, b, c$. The *abcd*-conjecture states that $\limsup\{Q_4(a, b, c, d)\} = \limsup\left\{\frac{\log(d)}{\log(r(a,b,c,d))}\right\} = 3$.
Application of the method given in 4.1 results in the following *abcd*-examples:

Table 2: *abcd*-examples with $\gcd(a, b, c, d) = 1$

1.	4.49948	$2^{22} + 3 \cdot 5^7$	=	$3^{11} \cdot 5^2 + 2^2$
2.	4.34929	$2^{15} \cdot 3^4 + 5^5$	=	$3^{12} \cdot 5 + 2^7$
3.	4.28368	$2^2 \cdot 3^{12}$	=	$2^{14} + 3^3 \cdot 5^7 + 5$
4.	4.2797	2^{21}	=	$3^{11} + 2^{10} \cdot 3 \cdot 5^4 + 5$
5.	4.02628	$3^{11} \cdot 5 + 1$	=	$2^{15} \cdot 3^3 + 2^3 \cdot 5^3$
6.	3.99316	$2^{18} \cdot 3^5 \cdot 5^2 + 7^{10}$	=	$2^6 \cdot 3 \cdot 5^{10} + 7^2$
7.	3.7938	$3^{17} \cdot 5 + 5^4 \cdot 7^2$	=	$2^{13} \cdot 3^2 + 2^4 \cdot 7^9$
8.	3.78752	$5^8 + 2^5 \cdot 3^4$	=	$2^{17} \cdot 3 + 1$
9.	3.78558	5^8	=	$2^{10} \cdot 3 \cdot 5^3 + 3^8 + 2^6$
10.	3.78558	$5^8 + 2^3 \cdot 5^2$	=	$2^{12} \cdot 3^4 + 3^{10}$
11.	3.67888	3^6	=	$2^9 + 2^3 \cdot 3^3 + 1$
12.	3.64435	$7^7 + 3^2$	=	$2^{14} \cdot 7^2 + 2^8 \cdot 3^4$
13.	3.64435	$7^7 + 3^4$	=	$2^{10} \cdot 7^3 + 2^3 \cdot 3^{10}$
14.	3.63982	$2^6 \cdot 3^{11} \cdot 5^2 + 7^2$	=	$2^9 \cdot 3 \cdot 5^4 + 7^{10}$
15.	3.62521	$2^{21} \cdot 5^3 + 3^2$	=	$2^3 \cdot 3^8 \cdot 7^3 + 5^{12}$
16.	3.55308	$3^{11} + 5$	=	$2^{14} \cdot 3 + 2^{10} \cdot 5^3$
17.	3.55308	$2^{17} + 2^{10} \cdot 3^2 \cdot 5$	=	$3^{11} + 5$
18.	3.53146	$2^{36} \cdot 11$	=	$3^3 \cdot 5^{12} \cdot 7 + 2^8 \cdot 3^2 \cdot 7^{10} + 5^2 \cdot 11^9$

19.	3.53045	$3^8 \cdot 5^2 + 1$	$=$	$2^5 \cdot 3^5 + 2 \cdot 5^7$
20.	3.51618	$2 \cdot 5^7$	$=$	$3^6 + 2^7 \cdot 3^5 \cdot 5 + 1$
21.	3.51618	$2 \cdot 5^7$	$=$	$2 \cdot 3^7 + 3^5 \cdot 5^4 + 1$
22.	3.51618	$2 \cdot 5^7$	$=$	$3^{10} + 2^4 \cdot 3^5 \cdot 5^2 + 1$
23.	3.49212	$3^{16} + 2^6 \cdot 3^5 \cdot 5^4 \cdot 17^2$	$=$	$2^{25} \cdot 5 \cdot 17 + 1$
24.	3.48168	$3^3 + 2 \cdot 3^5$	$=$	$2^9 + 1$
25.	3.48168	$3^4 + 2^4 \cdot 3^3$	$=$	$2^9 + 1$
26.	3.46452	$3^5 \cdot 5^2 + 2^3 \cdot 5^6$	$=$	$2^{17} + 3$
27.	3.45992	$2^{18} + 3^2 \cdot 7^5$	$=$	$3^{10} \cdot 7 + 2^6$
28.	3.39881	$2^7 \cdot 5^2 + 3^{13} \cdot 7^2$	$=$	$2^3 \cdot 5^{10} + 3^3$
29.	3.39881	$2^3 \cdot 5^{10}$	$=$	$2 \cdot 3^6 + 3^{13} \cdot 7^2 + 5 \cdot 7^3$
30.	3.39242	$2^7 \cdot 3^{10} + 5^{10} \cdot 7$	$=$	$2^{23} \cdot 3^2 + 5^2 \cdot 7^5$
31.	3.38496	$2^5 \cdot 5^5 + 3^2$	$=$	$2^{13} \cdot 5 + 3^{10}$
32.	3.32531	$3^4 \cdot 7^3 + 2^6 \cdot 7^7$	$=$	$3^3 \cdot 5^9 + 2^5 \cdot 5$
33.	3.32531	$2^{10} \cdot 3^3 + 2^6 \cdot 7^7$	$=$	$3^3 \cdot 5^9 + 5^2$
34.	3.30796	$2^{12} \cdot 3^2 \cdot 5^4 + 2^3 \cdot 3^{14} \cdot 5 + 11^2$	$=$	11^8
35.	3.27619	$2^3 \cdot 3 \cdot 7^5 \cdot 11^3 + 11$	$=$	$2^{29} + 3^5 \cdot 7^2$
36.	3.26105	$2^{16} + 3 \cdot 5^2$	$=$	$2 \cdot 3^8 \cdot 5 + 1$
37.	3.24677	$2^2 \cdot 5^6 + 5$	$=$	$2^7 \cdot 3^3 + 3^{10}$
38.	3.23005	$2^{26} \cdot 3^2 \cdot 11^2 + 5^3 \cdot 7 \cdot 11^5$	$=$	$3^{21} \cdot 7 + 2^2 \cdot 5^2$
39.	3.22302	$2^9 \cdot 11^3 + 2 \cdot 3^4 \cdot 7^3 + 7^4 \cdot 11^5$	$=$	3^{18}
40.	3.21942	$2^{16} \cdot 11 + 3^3$	$=$	$2^8 \cdot 3^7 + 11^5$
41.	3.21077	$2^{11} \cdot 3^3 + 2^2$	$=$	$3^7 \cdot 5^2 + 5^4$
42.	3.20616	$2^{15} \cdot 5^2 + 5^4 \cdot 7$	$=$	$7^7 + 2^5$
43.	3.20616	$2^{15} \cdot 5^2 + 2^5 \cdot 5^3 + 7^3$	$=$	7^7
44.	3.19545	$2^{12} \cdot 3^2 + 5^6$	$=$	$2^3 \cdot 3^8 + 1$
45.	3.19459	$3^6 \cdot 5^6 + 2 \cdot 3^2 \cdot 7^7 + 1$	$=$	$2^{20} \cdot 5^2$
46.	3.18814	$2^{11} \cdot 5^2 + 1$	$=$	$2^6 \cdot 3^2 + 3^4 \cdot 5^4$
47.	3.17192	$3^{18} + 5^2$	$=$	$2^{13} \cdot 17 + 2 \cdot 3^6 \cdot 5^6 \cdot 17$
48.	3.16261	$3^{15} \cdot 5 + 2^7 \cdot 11^5$	$=$	$2^{23} \cdot 11 + 3^3 \cdot 5^5$
49.	3.15333	$2^{20} \cdot 5^3 + 2^5 \cdot 3^{12} + 13$	$=$	$3^6 \cdot 5^6 \cdot 13$
50.	3.13739	$2^{12} + 2 \cdot 3^{13} \cdot 5^2$	$=$	$3^2 \cdot 5 \cdot 11^6 + 1$
51.	3.13577	$2^2 \cdot 3^{14} + 2 \cdot 5^4 \cdot 7$	$=$	$5^8 \cdot 7^2 + 1$
52.	3.13577	$2^2 \cdot 3^7 + 2^2 \cdot 3^{14} + 1$	$=$	$5^8 \cdot 7^2$
53.	3.13577	$5^8 \cdot 7^2 + 2^9$	$=$	$2^2 \cdot 3^{14} + 3^3 \cdot 7^3$
54.	3.13577	$5^8 \cdot 7^2$	$=$	$2^2 \cdot 3^{14} + 5^4 \cdot 7 + 2 \cdot 3^7$
55.	3.13573	$2 \cdot 3^7 \cdot 5^4 \cdot 7 + 1$	$=$	$2^2 \cdot 3^{14} + 5^4 \cdot 7$
56.	3.13168	$2^2 \cdot 5^{13} \cdot 7$	$=$	$2 \cdot 3^6 \cdot 5^3 + 7^{10} \cdot 11^2 + 11^2$
57.	3.12475	$2 \cdot 3^{10}$	$=$	$7^6 + 2^6 \cdot 7 + 1$
58.	3.12475	$2 \cdot 3^{10}$	$=$	$3^2 \cdot 7^2 + 7^6 + 2^3$
59.	3.12475	$2^9 + 7^6$	$=$	$2 \cdot 3^{10} + 3^2 \cdot 7$
60.	3.10163	$2^{23} + 2^7 \cdot 3^{10}$	$=$	$3^6 \cdot 5^5 \cdot 7 + 5$
61.	3.09635	$3 \cdot 5^2 \cdot 7^7 + 11^{10}$	$=$	$2^{14} \cdot 3^3 \cdot 7 + 2 \cdot 5^{10} \cdot 11^3$
62.	3.09482	2^8	$=$	$2^2 \cdot 3 + 3^5 + 1$
63.	3.09482	$2^4 + 3^5$	$=$	$2^8 + 3$
64.	3.09482	$3^5 + 3^2 + 2^2$	$=$	2^8
65.	3.08798	$2 \cdot 3^2 \cdot 7^7$	$=$	$2^{16} \cdot 3^2 \cdot 5^2 + 5^7 + 7^2$
66.	3.08189	3^{15}	$=$	$2^{14} \cdot 5^3 \cdot 7 + 2^3 \cdot 5^3 + 3^5 \cdot 7^2$
67.	3.06118	$2^{18} \cdot 7^2 + 3^2 \cdot 5^4$	$=$	$2^3 \cdot 3^{11} \cdot 5 + 7^8$
68.	3.05726	$3^8 \cdot 5$	$=$	$2^{15} + 2^2 \cdot 3^2 + 1$
69.	3.05726	$3^8 \cdot 5 + 3$	$=$	$2^{15} + 2^3 \cdot 5$
70.	3.05726	$3^8 \cdot 5$	$=$	$2^{15} + 5^2 + 2^2 \cdot 3$
71.	3.05726	$2^{15} + 3^2 \cdot 5$	$=$	$3^8 \cdot 5 + 2^3$
72.	3.04899	$2^{24} \cdot 3 \cdot 7^3 + 3^2 \cdot 5^{11} + 2^4 \cdot 11^7$	$=$	$5^5 \cdot 7^8$
73.	3.04582	$2^3 \cdot 3 \cdot 5^9 + 5^2 \cdot 11^4$	$=$	$2^{22} + 3^{16}$
74.	3.03795	$5^3 \cdot 7 + 3^3 \cdot 5^2 \cdot 7^5$	$=$	$2^{13} + 2^6 \cdot 3^{11}$
75.	3.01497	$2^{24} \cdot 3 \cdot 5^2 \cdot 11 + 2^5 \cdot 3 \cdot 5^3 \cdot 7 + 1$	$=$	7^{12}
76.	3.01497	$2^{24} \cdot 3 \cdot 5^2 \cdot 11 + 3^8 + 2^7 \cdot 5 \cdot 11^2$	$=$	7^{12}
77.	3.0103	$5^2 + 2^3 \cdot 5^3$	$=$	$2^{10} + 1$
78.	3.0103	$5^4 + 2^4 \cdot 5^2$	$=$	$2^{10} + 1$
79.	3.00992	$5^{10} + 2^2 \cdot 5 \cdot 7$	$=$	$2^2 \cdot 3^{10} + 3^4 \cdot 7^6$
80.	3.00992	5^{10}	$=$	$2^{17} \cdot 5 \cdot 7 + 3^{14} + 2^7 \cdot 3^2 \cdot 7^3$
81.	3.00839	$2^{28} \cdot 7^2 + 3 \cdot 5^6 \cdot 11$	$=$	$2^8 \cdot 7^2 + 3^3 \cdot 5^2 \cdot 11^7$
82.	3.00607	$2^{25} \cdot 5 \cdot 7 \cdot 11 + 3^{11} + 2^3 \cdot 11^5$	$=$	$3^3 \cdot 5^{10} \cdot 7^2$
83.	3.00131	$2^3 \cdot 3^3 \cdot 7^7 \cdot 19^2 + 7$	$=$	$2 \cdot 3^{19} \cdot 5^2 + 5^{14}$

These examples were generated by B.M.M. de Weger.

These results indeed show that there are examples with $Q_4 > 3$, and therefore support that $\limsup\{Q_4\} = 3$, but note that in each example there is a pair for which the gcd is strictly greater than 1.

4.2.2 *abcd*-examples with extra gcd conditions

If the condition is added that every pairwise gcd $= 1$, the conditions are as follows:
Let $a, b, c, d \in \mathbb{Z}$ satisfy:

(i) $\gcd(a,b) = \gcd(a,c) = \gcd(a,d) = \gcd(b,c) = \gcd(b,d) = \gcd(c,d) = 1$

(ii) $a + b + c + d = 0$

Without loss of generality, assume $d > a, b, c$. The *abcd*-conjecture with pairwise gcd $= 1$ states that $\limsup\{Q_4(a,b,c,d)\} = \limsup\left\{\frac{\log(d)}{\log(r(a,b,c,d))}\right\} = 1$.
Application of the method given in 4.1 results in the following *abcd*-examples:

Table 3: *abcd*-examples with $\gcd(a,b) = \gcd(a,c) = \ldots = \gcd(c,d) = 1$ and $Q > 1.3$

1.	1.60791	$3^{16} + 7^3 + 13^4$	$=$	$5^4 \cdot 41^3$
2.	1.47837	53^{10}	$=$	$3 \cdot 5^{12} \cdot 6329 + 11^{15} + 7 \cdot 29^9 \cdot 41^2$
3.	1.46744	$7^2 \cdot 11^{10} + 13^3$	$=$	$3^{16} \cdot 11801 + 5^{17}$
4.	1.45713	$3^{20} + 5^{18}$	$=$	$23^4 \cdot 47 + 7^2 \cdot 29^6 \cdot 131$
5.	1.41832	$7^{10} + 19^{11}$	$=$	$3 \cdot 11^3 \cdot 53 \cdot 547 + 5 \cdot 13^{12}$
6.	1.37733	$11^7 + 41$	$=$	$5^8 + 3^3 \cdot 29^4$
7.	1.37471	$19^{11} + 41^4 \cdot 59$	$=$	$5 \cdot 13^{12} + 29 \cdot 197$
8.	1.36004	$3^{20} + 5^5 \cdot 41^3$	$=$	$13^7 \cdot 59 + 23$
9.	1.3541	$3^{31} \cdot 11 + 7 \cdot 307 + 13^4 \cdot 67^4$	$=$	$31^9 \cdot 257$
10.	1.3524	$31^8 + 37^2 \cdot 71^2$	$=$	$3^{17} \cdot 13 + 11^9 \cdot 19^2$
11.	1.34834	$3^{19} \cdot 5^2 \cdot 97 + 31^4$	$=$	$7^{13} \cdot 29 + 11^6 \cdot 17^3$
12.	1.34807	3^{18}	$=$	$5^5 + 7^4 \cdot 11^5 + 19^3 \cdot 107$
13.	1.3394	$3^{13} \cdot 101 + 19^{11} + 7 \cdot 67^2 \cdot 181$	$=$	$5 \cdot 13^{12}$
14.	1.33849	$13^5 + 17^6 \cdot 19$	$=$	$3^6 + 5^{10} \cdot 47$
15.	1.33819	$13^9 \cdot 17 + 23^3$	$=$	$11^8 \cdot 29^2 + 3^5 \cdot 53^2$
16.	1.33721	$3^{21} \cdot 5^3 \cdot 13 + 17^9$	$=$	$29^7 \cdot 239 + 11^4 \cdot 31^6$
17.	1.33459	$19^7 \cdot 2411 + 59^3$	$=$	$3^{19} \cdot 43^2 + 5^{14}$
18.	1.3323	$5^9 \cdot 23^2$	$=$	$3^5 \cdot 11^6 + 29^6 + 53^4$
19.	1.32579	$19^{11} + 41^4 \cdot 233$	$=$	$5 \cdot 13^{12} + 7 \cdot 17^4 \cdot 29^2$
20.	1.32446	$3^3 \cdot 17^9$	$=$	$5^{15} + 7^6 \cdot 71 \cdot 73 + 13^{10} \cdot 23$
21.	1.32218	$3^{14} \cdot 173 + 5 \cdot 13^{12}$	$=$	$11^5 \cdot 6173 + 19^{11}$
22.	1.32217	59^5	$=$	$3^3 + 5^{10} \cdot 73 + 7^5 \cdot 11^2$
23.	1.32153	$5^{11} + 43^3$	$=$	$7^6 \cdot 19^2 + 23^5$
24.	1.31769	$17^{13} \cdot 197 + 5^2 \cdot 29^2 \cdot 67^5$	$=$	$3^{36} \cdot 13 + 31^2 \cdot 1931$
25.	1.31716	$5^{10} \cdot 13^3 + 7^7 + 29^9$	$=$	$11^8 \cdot 67777$
26.	1.3148	$19^{11} + 31^3 \cdot 5647$	$=$	$3^9 \cdot 7 \cdot 11 + 5 \cdot 13^{12}$
27.	1.31384	$11^6 + 19^2 + 29^5 \cdot 347$	$=$	$3^6 \cdot 5^{10}$
28.	1.30996	$7^2 \cdot 17^5$	$=$	$3^3 \cdot 11^5 + 13^7 + 19^5$
29.	1.30738	$3^{29} + 29^4 \cdot 31^2 \cdot 59 + 5 \cdot 37^7$	$=$	$7^{11} \cdot 11^2 \cdot 17^2$
30.	1.30732	$23 \cdot 61^7 \cdot 659$	$=$	$11^{16} + 5 \cdot 13^8 + 41^8 \cdot 211$
31.	1.30667	$3^3 \cdot 17^8$	$=$	$5^{11} \cdot 797 + 7^{10} \cdot 23^2 + 13^4$
32.	1.30588	$5^4 \cdot 7^{16} \cdot 41 + 37^4$	$=$	$11^7 \cdot 2857 + 31^{10} \cdot 1039$
33.	1.30497	$3^{11} \cdot 11^3 + 5^{17}$	$=$	$53^4 \cdot 311^2 + 7^3 \cdot 67$
34.	1.30259	$3^{11} \cdot 5^2 \cdot 7 + 11^4 + 41^8$	$=$	$19^7 \cdot 8933$

For complete Table see Appendix C.1. These examples were generated with the Mathematica notebook given in Appendix B.1 with as primebase all subsets of length 4 of the first 20 primes, with powers up to $\frac{50\log(2)}{\log(p)}$ and linear combinations of the reduced lattice base vectors with coefficients up to 2.

These results make it more plausible that indeed with the extra gcd demands $\limsup\{Q_4\} = 1$. Perhaps it can be stated that in the general case $\limsup\{Q_4\} = 1$, with the exception of some special cases, for instance when $a^3 + 3abc + b^3 = c^3$. In this case there would be certain patterns in the *abcd*-examples with $Q_4 > 3$.

4.3 Patterns in the *abcd*-examples

As noted before, one can construct *abcd*-examples by using *abc*-triples with $a^3 + b^3 + 3abc = c^3$. In general, Table 2 consists of the forms:

(i) $P_1^{A_1}(P_2^{A_2} \pm P_3^{A_3}) = P_4^{A_4}(P_5^{A_5} \pm P_6^{A_6})$ with (either $\gcd(P_1, P_2) \neq 1$ or $\gcd(P_1, P_3) \neq 1$) and (either $\gcd(P_4, P_5) \neq 1$ or $\gcd(P_4, P_6) \neq 1$) and $\gcd(P_1, P_4) = 1$

(ii) $P_1^{A_1} + P_2^{A_2}(P_3^{A_3} \pm P_4^{A_4}) = \pm 1$ with $\gcd(P_1, P_2) = 1$ and (either $\gcd(P_2, P_3) \neq 1$ or $\gcd(P_2, P_4) \neq 1$

(iii) $P_1^{A_1} + P_2^{A_2} + P_1^{A_4} P_2^{A_5} P_3^{A_6} = P_3^{A_3}$ with $\gcd(P_1, P_2) = \gcd(P_1, P_3) = \gcd(P_2, P_3) = 1$.

Here $P_i^{A_i}$ represents $p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$ for some primes $p_1, \dots, p_n$ and some $a_1, \dots, a_n \in \mathbb{N}$. Also interesting about Table 2 is that every tuple contains two's, which is not necessary. Since $a + b + c + d = 0$, one expects either zero or two two's. Obviously with the pairwise gcd $= 1$ demand no two's occur, but it is not clear why this does not occur with the general gcd $= 1$ demand and $Q > 3$.
In Table 3 there are 6 cases in which $5 \cdot 13^{12}$ and 19^{11} occur. In the full list of 25857 examples, this occurs 568 times.

4.4 Polynomial Identities

According to [8], Granville conjectures that every counterexample to the conjecture $\limsup\{Q_4\} = 1$ comes from at most finitely many polynomial families. As example the polynomial identity $(x+1)^5 - (x-1)^5 = 10(x^2+1)^2 - 8$ is given. Indeed, in this polynomial the terms are relatively prime, except for the last two. This is in agreement with the conjecture that $\limsup\{Q_4\} = 1$ holds for pairwise relatively prime terms. If one takes $x = 11^k - 1$, the polynomial gives *abcd*-examples with an expected quality $Q \approx \frac{5}{3}$. The conjecture $\limsup\{Q_4\} = 1$ can be formulated as: for each $\varepsilon > 0$ only for a finite number of cases we have $r(abcd)^{1+\varepsilon} < max\{|a|, |b|, |c|, |d|\}$. In this case Granville also states that the amount of polynomial families grows to infinity if ε shrinks to 0.
This can be seen by looking at a general case for polynomials such as $(x + 1)^5 - (x - 1)^5 = 10(x^2 + 1)^2 - 8$; consider $(x+1)^{2m+1} - (x-1)^{2m+1} = 2(2m+1)(x^2 + \frac{2m-1}{3})^m + O(x^{2m-4})$ for $m \equiv 2 \mod 3$. These polynomials give an expected quality $Q \approx \frac{m}{m-2}$ if one takes $x = p^k - 1$ for p some prime. Indeed, as ε shrinks to zero, larger m may be taken.
Darmon and Granville [9] give more polynomial identities which form exceptions:

- $2 = (1 + 6t^3)^3 + (1 - 6t^3)^3 - (6t^2)^3$ with gcd ≥ 2 and $Q \approx \frac{9}{8}$
- $\left(\frac{x_n+y_n}{2}\right)^4 + \left(\frac{x_n-y_n}{2}\right)^4 + y_n^4 = 2 \cdot 7^{2n}$, where $\left(\frac{x_n+y_n\sqrt{-3}}{2}\right) = \left(\frac{5+\sqrt{-3}}{2}\right)^n$ with gcd ≥ 2 and $Q \approx \frac{4}{3}$

5 Open Problems

These results all suggest that the *abcd*-conjecture is as stated in 4.2.2. Furthermore, there are several open problems, some of which are summarized below:

1. Why does every generated *abcd*-example with $Q > 3$ contain two 2's?
2. Does a pattern exist within the $abcd(e)$-examples with total gcd $= 1$ which explains their high quality?
3. Do other polynomials exist besides $a^3 + b^3 + 3abc = c^3$ that generate *abcd*-examples with $Q > 3$?
4. Why do more than 2 percent of the *abcd*-examples with pairwise gcd $= 1$ contain $5 \cdot 13^{12}$ and 19^{11}?
5. Do more experiments, with e.g. other "base" and more primes together in one term

In general, we can conjecture the following:

5.1 Strong n-conjecture

Let $a_1, a_2, \ldots, a_n \in \mathbb{Z} n \geq 3$, satisfy:

(i) $a_1, a_2, \ldots, a_n$ are pairwise coprime

(ii) $a_1 + a_2 + \cdots + a_n = 0$

(iii) no proper subsum of (ii) is equal to 0.

Define $Q_n(a_1, a_2, \ldots, a_n) = \dfrac{\log\left(\max\limits_{1 \leq i \leq n}(|a_i|)\right)}{\log\left(r(a_1 \cdot a_2 \cdots a_n)\right)}$ as the quality of such $a_1, \ldots, a_n$.

Conjecture 5.1 *The strong n-conjecture states that, for given $n \geq 3$, $\limsup\{Q_n\} = 1$.*

A LLL-reduction Algorithm

The LLL Algorithm [4] takes as input a lattice basis $\{b_1, \ldots, b_n\}$ and outputs an LLL-reduced basis $\{\tilde{b_1}, \ldots, \tilde{b_n}\}$ for the same lattice. It is based on the Gram-Schmidt orthogonalization process, which inductively defines $b'_i = b_i - \sum_{j=1}^{i-1} \mu_{ij} b'_j$, $\mu_{ij} = \frac{(b_i, b'_j)}{(b'_j, b'_j)}$. LLL extends this algorithm such that the following holds:

Definition A.1 *A basis $\{b_1, \ldots, b_n\}$ for a lattice is called LLL-reduced if $|\mu_{ij}| \leq \frac{1}{2}$ for $1 \leq j \leq i \leq n$ and $|b_i + \mu_{ii-1} b_{i-1}|^2 \geq y |b_{i-1}|^2$ for $1 < i \leq n$, with $y \in (\frac{1}{4}, 1)$.*

This implies certain useful properties, such as the following:

Proposition A.2 *Let $b_1, \ldots, b_n$ be a reduced basis for L and $b'_1, \ldots, b'_n$ be defined as above. Then $|b_j|^2 \leq 2^{i-1} |b'_i|^2$ for $1 \leq j \leq i \leq n$,*

Proposition A.3 *Let L be a lattice with reduced basis $b_1, \ldots, b_n$, then $|b_1|^2 \leq 2^{n-1} |x|^2$ for every $x \in L$, $x \neq 0$,*

and

Proposition A.4 *Let L be a lattice with reduced basis $b_1, \ldots, b_n$, and $x_1, \ldots, x_t \in L$ linearly independent. Then $|b_j|^2 \leq 2^{n-1} max |x_1|^2, \ldots, |x_t|^2$ for $j = 1, \ldots t$.*

This implies that if $\lambda_1, \ldots \lambda_n$ denote the successive minima of $|\cdot|^2$ on L, then $2^{1-i} \lambda_i \leq |b_i|^2 \leq 2^{n-1} \lambda_i$ for $1 \leq i \leq n$, so $|b_i|^2$ is a good approximation of λ_i.

The algorithm is as follows, where $B_j = |b'_j|^2$:

$$\left.\begin{array}{l} b'_i := b_i; \\ \left.\begin{array}{l} \mu_{ij} := \frac{(b_i, b'_j)}{B_j}; \\ b'_i := b'_i - \mu_{ij} b'_j \end{array}\right\} \text{for } j = 1, 2, \ldots, i-1; \end{array}\right\} \text{for } i = 1, 2, \ldots, n; \; k := 2;$$

(1) perform (*) for $l = k - 1$;
if $B_k < (\frac{3}{4} - \mu_{kk-1}^2) B_{k-1}$, go to (2);
perform (*) for $l = k-2, k-3, \ldots, 1$;
if $k = n$, terminate;
$k := k + 1$;
go to (1);
(2) $\mu := \mu_{kk-1}; B := B_k + \mu^2 B_{k-1}; \mu_{kk-1} := \mu \frac{B_{k-1}}{B}; B_k := \frac{B_{k-1} B_k}{B}; B_{k-1} := B;$

$$\begin{pmatrix} b_{k-1} \\ b_k \end{pmatrix} := \begin{pmatrix} b_k \\ b_{k-1} \end{pmatrix};$$

$$\begin{pmatrix} \mu_{k-1j} \\ \mu_{kj} \end{pmatrix} := \begin{pmatrix} \mu_{kj} \\ \mu_{k-1j} \end{pmatrix} \text{ for } j = 1, 2, \ldots, k-2;$$

$$\begin{pmatrix} \mu_{ik-1} \\ \mu_{ik} \end{pmatrix} := \begin{pmatrix} 1 & \mu_{kk-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\mu \end{pmatrix} \begin{pmatrix} \mu_{ik-1} \\ \mu_{ik} \end{pmatrix} \text{ for } i = k+1, k+2, \ldots, n;$$

if $k > 2$, then $k := k - 1$;

go to (1);

(*) If $|\mu_{kl}| > \frac{1}{2}$, then:

$$\begin{cases} r := \text{ integer nearest to } \mu_{kl}; b_k := b_k - rb_l; \\ \mu_{kj} := \mu_{kj} - r\mu_{lj} \text{ for } j = 1, 2, \ldots, l-1; \\ \mu_{kl} := \mu_{kl} - r. \end{cases}$$

B Mathematica Notebooks

Below the notebooks which are used to generate $abcd(e)$-examples with pairwise gcd equal to 1 are given. It is trivial how to adapt these notebooks to the case of a total gcd so these are not given.

B.1 *abcd*

```
qu[x_] :=
  Log[Max[Abs[x]]]/
   Log[Times @@ FactorInteger[Times @@ Abs[x]][[All, 1]]];
doks4[base_, pri_, stop1_, stop2_] :=
  Module[{st, abssols, sols, t1, i1, i2, i3, i4, n, theta1, theta2,
     psi1, psi2, m, mr, x1, x2, x, ex, q, sv, t2},
   st = Floor[stop1 Log[2]/Log[pri]];
   abssols = {};
   sols = {};
   t1 = TimeUsed[];
   For[i1 = 1, i1 <= st[[1]], i1++,
    For[i2 = 1, i2 <= st[[2]], i2++,
     For[i3 = 1, i3 <= st[[3]], i3++,
      For[i4 = 1, i4 <= st[[4]], i4++,
       n = base pri^{i1,i2,i3,i4,i5};
       theta1 = Mod[-n[[1]] PowerMod[n[[-2]], -1, n[[-1]]], n[[-1]]];
       theta2 = Mod[-n[[2]] PowerMod[n[[-2]], -1, n[[-1]]], n[[-1]]];
       psi1 = -(( n[[1]] + n[[-2]] theta1)/n[[-1]]);
       psi2 = - ((n[[2]] + n[[-2]] theta2)/n[[-1]]);
       m =
        Append[IdentityMatrix[Length[n] - 1], {psi1, psi2, -n[[-2]]}];
       m[[-2]] = {theta1, theta2, n[[-1]]};
       mr = Transpose[LatticeReduce[Transpose[m]]];
       For[x1 = 0, x1 <= stop2, x1++,
        For[x2 = If[x1 == 0, 0, -stop2], x2 <= stop2, x2++,

         For[x3 = If[x1 == 0 && x2 == 0, 1, -stop2], x3 <= stop2,
          x3++,
          x = {x1, x2, x3};
          ex = (mr.x) n;

          If[GCD[ex[[1]], ex[[2]]] == 1 &&
             GCD[ex[[1]], ex[[3]]] == 1 &&
             GCD[ex[[1]], ex[[4]]] == 1 &&
             GCD[ex[[2]], ex[[3]]] == 1 &&
             GCD[ex[[2]], ex[[4]]] == 1 &&
             GCD[ex[[3]], ex[[4]]] ==
               1 && (ex[[1]] + ex[[2]]) (ex[[1]] +
                  ex[[3]]) (ex[[1]] + ex[[4]]) != 0, q = N[qu[ex]];
           If[q > 1, sv = {Sort[Abs[ex]], q};
            If[! MemberQ[abssols, sv], abssols = Append[abssols, sv];
             sols = Append[sols, {ex, q}]]]]]]]]]]];
    t2 = TimeUsed[] - t1;
    Print["doks4[", pri, "]:_", i1, "/", st[[1]], "_", t2,
     "_sec.,_aantal_opl:_", Length[sols]];];
   Return[Sort[sols, #1[[2]] > #2[[2]] &]]];
```

B.2 *abcde*

```
qu[x_] :=
  Log[Max[Abs[x]]]/
   Log[Times @@ FactorInteger[Times @@ Abs[x]][[All, 1]]];
doks5[base_, pri_, stop1_, stop2_] :=
  Module[{st, abssols, sols, t1, i1, i2, i3, i4, i5, n, theta1,
     theta2, theta3, psi1, psi2, psi3, m, mr, x1, x2, x3, x4, x, ex, q,
      sv, t2},
   st = Floor[stop1 Log[2]/Log[pri]];
   abssols = {};
   sols = {};
```

```
t1 = TimeUsed[];
For[i1 = 1, i1 <= st[[1]], i1++,
 For[i2 = 1, i2 <= st[[2]], i2++,
  For[i3 = 1, i3 <= st[[3]], i3++,
   For[i4 = 1, i4 <= st[[4]], i4++,
    For[i5 = 1, i5 <= st[[5]], i5++,
     n = base pri^{i1,i2,i3,i4,i5};
     theta1 = Mod[-n[[1]] PowerMod[n[[-2]], -1, n[[-1]]], n[[-1]]];
     theta2 = Mod[-n[[2]] PowerMod[n[[-2]], -1, n[[-1]]], n[[-1]]];
     theta3 = Mod[-n[[3]] PowerMod[n[[-2]], -1, n[[-1]]], n[[-1]]];
     psi1 = -(( n[[1]] + n[[-2]] theta1)/n[[-1]]);
     psi2 = - ((n[[2]] + n[[-2]] theta2)/n[[-1]]);
     psi3 = - ((n[[3]] + n[[-2]] theta3)/n[[-1]]);
     m =
      Append[IdentityMatrix[Length[n] - 1], {psi1, psi2,
        psi3, -n[[-2]]}];
     m[[-2]] = {theta1, theta2, theta3, n[[-1]]};
     mr = Transpose[LatticeReduce[Transpose[m]]];
     For[x1 = 0, x1 <= stop2, x1++,
      For[x2 = If[x1 == 0, 0, -stop2], x2 <= stop2, x2++,

       For[x3 = If[x1 == 0 && x2 == 0, 1, -stop2], x3 <= stop2,
         x3++,

        For[x4 =
          If[x1 == 0 && x2 == 0 && x3 == 0, 1, -stop2],
         x4 <= stop2, x4++,
         x = {x1, x2, x3, x4};
         ex = (mr.x) n;

         If[GCD[ex[[1]], ex[[2]]] == 1 &&
           GCD[ex[[1]], ex[[3]]] == 1 &&
           GCD[ex[[1]], ex[[4]]] == 1 &&
           GCD[ex[[1]], ex[[5]]] == 1 &&
           GCD[ex[[2]], ex[[3]]] == 1 &&
           GCD[ex[[2]], ex[[4]]] == 1 &&
           GCD[ex[[2]], ex[[5]]] == 1 &&
           GCD[ex[[3]], ex[[4]]] == 1 &&
           GCD[ex[[3]], ex[[5]]] == 1 &&
           GCD[ex[[4]], ex[[5]]] == 1 &&
            (ex[[1]] + ex[[2]]) (ex[[1]] +
                ex[[3]]) (ex[[1]] + ex[[4]]) (ex[[1]] +
                ex[[5]]) (ex[[2]] + ex[[3]]) (ex[[2]] +
                ex[[4]]) (ex[[2]] + ex[[5]]) (ex[[3]] +
                ex[[4]]) (ex[[3]] + ex[[5]]) (ex[[4]] + ex[[5]]) !=
            0, q = N[qu[ex]];
          If[q > 1, sv = {Sort[Abs[ex]], q};
           If[! MemberQ[abssols, sv],
            abssols = Append[abssols, sv];
            sols = Append[sols, {ex, q}]]]]]]]]]]]]]];
 t2 = TimeUsed[] - t1;
 Print["doks5[", pri, "]: ", i1, "/", st[[1]], " ", t2,
  " sec., aantal opl: ", Length[sols]];];
Return[Sort[sols, #1[[2]] > #2[[2]] &]];
```

C n-examples

C.1 $abcd$-examples with gcd condition

Table 4: $abcd$-examples with $\gcd(a,b) = \gcd(a,c) = \ldots = \gcd(c,d) = 1$ (continued)

35.	1.29782	$5^{21} \cdot 337^2 + 3 \cdot 7^3 \cdot 31^{11}$	=	$11^{15} \cdot 47 \cdot 409 + 17^6 \cdot 19 \cdot 29^2$
36.	1.29686	$3^6 + 17^2 + 53^4$	=	$7^2 \cdot 11^5$
37.	1.29375	5^{10}	=	$3^6 + 7^6 \cdot 83 + 29$
38.	1.29157	47^8	=	$5^{17} \cdot 31 + 3^4 \cdot 7^{11} + 37 \cdot 4019$
39.	1.2913	$3^{22} + 5 \cdot 19^7 \cdot 307$	=	$13^9 + 23^6 \cdot 97^2$
40.	1.28896	$11^{14} + 3^5 \cdot 13^7 \cdot 67 + 19^{10} \cdot 9643$	=	$37^3 \cdot 53^7$
41.	1.2883	$5^{20} \cdot 1223 + 53^7 \cdot 193 + 3^7 \cdot 61^4$	=	$7^3 \cdot 13^2 \cdot 17^{10}$
42.	1.28826	$23^9 \cdot 1123$	=	$7^{11} \cdot 53^2 + 11^{10} \cdot 37 + 3^2 \cdot 5^3 \cdot 13^{11}$
43.	1.28704	$7^{13} \cdot 13 + 23 \cdot 71^5$	=	$5^{13} + 3^4 \cdot 19 \cdot 61^5$
44.	1.28242	$5 \cdot 13^{12}$	=	$3^9 \cdot 7 \cdot 1093 + 17^4 \cdot 193 + 19^{11}$
45.	1.28188	$11 \cdot 17^4 + 5^5 \cdot 31^5 + 3^2 \cdot 13^2 \cdot 47^4$	=	7^{13}
46.	1.28125	$3^{24} \cdot 11087 + 13^3$	=	$7^9 \cdot 17 + 5^2 \cdot 11^{11} \cdot 439$
47.	1.27965	$29^2 + 37^7$	=	$3^3 \cdot 17^4 + 7^3 \cdot 23^5 \cdot 43$
48.	1.27859	$7 + 17 \cdot 19^5 \cdot 29$	=	$5^{13} + 3^4 \cdot 13^2$
49.	1.27853	$5 \cdot 19^{12}$	=	$3^2 \cdot 7^{16} \cdot 37 + 17^4 \cdot 97231 + 29^6$
50.	1.27732	29^6	=	$3^4 \cdot 7^3 + 5^5 \cdot 11^4 \cdot 13 + 17^3$
51.	1.27485	$3^{20} \cdot 109^2 + 7^4 \cdot 10733$	=	$5 \cdot 17^3 + 23^{10}$
52.	1.26987	$3^{10} \cdot 53^2 + 7^5 + 37^5$	=	$5 \cdot 19^6$
53.	1.26951	$7^{16} \cdot 52919 + 17 \cdot 53^2$	=	$3^{29} \cdot 5^4 \cdot 41 + 31^6 \cdot 37$
54.	1.26921	$7^9 \cdot 17^3 + 3^2 \cdot 19^2 \cdot 43^5$	=	$5^{14} \cdot 109 + 13^9$
55.	1.26863	$3^{12} \cdot 5 + 7^9 + 11^4$	=	$17^2 \cdot 53^3$

56.	1.26748	$23^2 \cdot 71^6$	=	$3^{21} \cdot 6451 + 11^{11} + 5 \cdot 47^2$
57.	1.26694	$3^{14} \cdot 43 + 19^{11}$	=	$7^3 \cdot 113567 + 5 \cdot 13^{12}$
58.	1.26614	23^7	=	$5^{11} \cdot 41 + 3 \cdot 11^4 + 7^2 \cdot 31^5$
59.	1.26537	$3^{28} \cdot 61 \cdot 643 + 11^{13} \cdot 41^2 + 7^5 \cdot 23$	=	$5^3 \cdot 17^{11} \cdot 223$
60.	1.26465	$5^2 \cdot 5099 + 7^{18} \cdot 11$	=	$3^{31} \cdot 29 + 17^2 \cdot 31^4 \cdot 79$
61.	1.26415	$5^2 \cdot 17^7 \cdot 29 + 19^7 \cdot 103^2 + 43^7$	=	$3^{21} \cdot 31^2$
62.	1.26409	$7^2 \cdot 23^8 \cdot 307$	=	$3^{22} + 5^4 \cdot 13 \cdot 29^7 + 41^7 \cdot 73^2$
63.	1.26115	$5^{12} \cdot 19^2 \cdot 389$	=	$11^7 \cdot 601 + 13^8 + 17^{11}$
64.	1.26009	$3 \cdot 13^5 \cdot 8581 + 7^4 \cdot 29^7 + 41$	=	23^{10}
65.	1.25948	11^{15}	=	$3^{23} \cdot 44357 + 7 \cdot 13^5 + 61^4 \cdot 311^2$
66.	1.25914	$11^3 \cdot 19^5$	=	$3^3 \cdot 5^{11} + 7^{11} + 13 \cdot 127$
67.	1.25875	$3^{25} \cdot 7 + 5 \cdot 13 \cdot 43^2 + 67^7 \cdot 71$	=	$11^{12} \cdot 139$
68.	1.25844	$7^3 \cdot 23 \cdot 31^{10}$	=	$3^{19} \cdot 13^2 \cdot 1697 + 5^2 \cdot 19^{13} + 61^9 \cdot 463$
69.	1.25622	$13^9 \cdot 32359 + 31^3 \cdot 173^2$	=	$3^{29} \cdot 5 + 11^3$
70.	1.25613	$3^{10} \cdot 109 + 7$	=	$5 + 23^5$
71.	1.25568	$3^{17} \cdot 5^4 + 29^7$	=	$11^8 \cdot 457 + 7^3 \cdot 37^2$
72.	1.25504	$23^7 \cdot 1801$	=	$3^4 \cdot 5 \cdot 7^6 \cdot 17 + 11^8 + 19^{10}$
73.	1.25454	$3^{19} \cdot 5^6 \cdot 11$	=	$7^7 \cdot 53 \cdot 1637 + 41^8 + 61^8$
74.	1.25414	$3^7 \cdot 7 \cdot 11 + 17^3 \cdot 109 \cdot 311 + 19^{11}$	=	$5 \cdot 13^{12}$
75.	1.25356	$17^5 \cdot 197 + 19^{11}$	=	$5 \cdot 13^{12} + 3 \cdot 11 \cdot 23 \cdot 53^3$
76.	1.25343	$3^2 \cdot 11 \cdot 317 + 5 \cdot 13^{12}$	=	$19^{11} + 37^2 \cdot 349^2$
77.	1.25294	$23^{10} + 29^2 \cdot 137^2$	=	$3^{20} \cdot 109^2 + 7^6 \cdot 353$
78.	1.25267	7^{21}	=	$3 \cdot 17^2 \cdot 79 + 5^3 \cdot 13^2 \cdot 31^9 + 19 \cdot 43^5 \cdot 3167$
79.	1.24996	$3^{12} \cdot 61^2$	=	$7^{11} + 13^2 \cdot 31^2 + 53^2$
80.	1.24901	$5^{18} + 11^8 \cdot 31 \cdot 163$	=	$3^{15} + 7^{11} \cdot 2477$
81.	1.24821	$47^9 \cdot 53^2$	=	$5^4 \cdot 13^3 + 37^8 \cdot 79 \cdot 11329 + 3^7 \cdot 41$
82.	1.24791	$5 \cdot 11^{12} \cdot 13 \cdot 109$	=	$3^{15} \cdot 47^2 + 19^5 \cdot 67^2 + 29^2 \cdot 31^9$
83.	1.24735	$3 \cdot 5^{19} \cdot 7^2 \cdot 11^2 + 17 \cdot 29^2$	=	$13^4 + 53^7 \cdot 288803$
84.	1.24696	$3^{12} + 7^{15}$	=	$5^3 \cdot 11^7 \cdot 1949 + 23 \cdot 283$
85.	1.24375	$3^{20} \cdot 11 + 23 \cdot 61^5$	=	$13^3 \cdot 17^6 + 41^6$
86.	1.24176	$19^4 \cdot 12973 + 29^{10}$	=	$5^{16} \cdot 797 + 3^2 \cdot 7^{16}$
87.	1.2405	$13^3 \cdot 79^2 + 17^{10} \cdot 2819 + 7 \cdot 41^7$	=	$5^4 \cdot 71^7$
88.	1.24028	$3^{14} + 7 \cdot 11^9 \cdot 271 + 5^5 \cdot 23^4$	=	$31^6 \cdot 71^2$
89.	1.24009	$3^{15} + 37^6 + 53^2$	=	$5^9 \cdot 1321$
90.	1.2397	$3^2 \cdot 11^{11} \cdot 163 + 5^2 \cdot 13^{12} \cdot 2593 + 23^{10} \cdot 317^2$	=	$7^{17} \cdot 29^3$
91.	1.23968	$3^8 \cdot 41 + 19^{12} \cdot 112129 + 29^2$	=	$11^2 \cdot 67^8 \cdot 5051$
92.	1.23959	$5^{11} + 3^2 \cdot 7^6 \cdot 157$	=	$11^8 + 29^4$
93.	1.23955	$19^{11} \cdot 53^2$	=	$3^{27} \cdot 179^2 + 7^4 \cdot 11^{13} + 29 \cdot 47^4 \cdot 2377$
94.	1.23876	$7^{18} \cdot 17^3$	=	$3^2 \cdot 5^9 \cdot 2647 + 13^3 \cdot 43^7 \cdot 97 + 47^{10} \cdot 151$
95.	1.23876	$3^{32} + 11$	=	$5 \cdot 19^8 \cdot 21821 + 31^5 \cdot 997$
96.	1.23829	$13 \cdot 17^9 \cdot 23^2$	=	$5^4 \cdot 2311 + 7 \cdot 19^{11} + 3^2 \cdot 47^6$
97.	1.23791	$7^{13} \cdot 37 \cdot 127$	=	$13^8 + 11 \cdot 31^6 \cdot 71^2 + 67^8$
98.	1.23764	$11^{10} \cdot 13^3 + 59^4 \cdot 19471$	=	$3 \cdot 5^{19} + 53$
99.	1.23734	$17^{11} + 23 \cdot 31^6$	=	$3 \cdot 19^{10} + 37^2 \cdot 43^5 \cdot 79$
100.	1.23694	$11^{10} \cdot 701 + 13^3 \cdot 137$	=	$19^4 + 3 \cdot 67^7$
101.	1.23678	$5^2 + 19^2 + 43^5 \cdot 61 \cdot 127$	=	$3^3 \cdot 59^6$
102.	1.23564	$5 \cdot 13^{10} \cdot 137 \cdot 557 + 11 \cdot 19^6$	=	$3^{13} \cdot 233^2 + 47^{10}$
103.	1.23518	$3^{12} + 7^{13} \cdot 11 \cdot 197$	=	$13^{10} \cdot 1523 + 19^4 \cdot 23$
104.	1.23436	$5^8 + 7^2 + 17^2$	=	$3 \cdot 19^4$
105.	1.23293	$3^6 \cdot 5^{10} \cdot 11 + 17^2 \cdot 37 \cdot 41$	=	$7 + 23^8$
106.	1.23232	$3 \cdot 5^{22} \cdot 547$	=	$7^{22} + 23^2 \cdot 37^3 \cdot 211 + 53^7 \cdot 2237$
107.	1.23198	$11^9 \cdot 43^2 + 3^2 \cdot 13^3 \cdot 173$	=	$5^4 \cdot 17^8 + 67^3$
108.	1.2318	$5 \cdot 13^{12} + 17^2 \cdot 61 \cdot 1667$	=	$3^{14} \cdot 41 + 19^{11}$
109.	1.23137	$13^8 \cdot 3719 + 5^5 \cdot 17^4$	=	$3^{25} + 11^8 \cdot 101^2$
110.	1.23134	$3^{22} \cdot 17 \cdot 53 \cdot 103$	=	$7^4 \cdot 631 + 5^2 \cdot 19^{11} + 31^4$
111.	1.23077	$3^4 \cdot 37^8$	=	$5^{15} + 7^{11} \cdot 1193 + 11^{10} \cdot 73 \cdot 149$
112.	1.23066	$17^6 + 11 \cdot 19^8 \cdot 1439$	=	$3 \cdot 5 \cdot 7^{10} + 23 \cdot 43^8$
113.	1.22974	$11^{13} + 59^5 \cdot 229^2$	=	$7^{11} \cdot 61^2 + 3^4 \cdot 19^8 \cdot 47$
114.	1.2296	41^5	=	$3^3 \cdot 7^3 + 13^6 \cdot 19 + 17^6$
115.	1.22955	$3^{11} + 19^8 \cdot 29^2$	=	$5 \cdot 7^9 \cdot 13^2 + 11^9 \cdot 6043$
116.	1.22938	$5^8 \cdot 13 + 11^6$	=	$3^{10} \cdot 7 + 23^5$
117.	1.22938	$3^{26} + 5^{15} \cdot 23 \cdot 227 + 47^3 \cdot 113^2$	=	$11^{10} \cdot 79^2$
118.	1.22897	$5^{16} \cdot 7^2 \cdot 1601 + 19^{10} \cdot 29 \cdot 37 + 61^9 \cdot 653$	=	$3^{37} \cdot 17$
119.	1.22882	$17^8 \cdot 23 \cdot 37$	=	$5^{11} + 19^5 \cdot 89^2 + 7^3 \cdot 29^7$
120.	1.22863	$19^{11} \cdot 239 + 3^2 \cdot 29^9 \cdot 20101$	=	$7^2 \cdot 13^{15} + 23^{10} \cdot 59^2$
121.	1.22856	$7^7 \cdot 41^3 + 11^3 \cdot 61^6$	=	$3^{29} + 19 \cdot 79 \cdot 211$
122.	1.22824	$7^5 \cdot 29^2 + 11^2 \cdot 19^5$	=	$3^4 + 5 \cdot 13^7$
123.	1.22757	$3^{19} \cdot 67 + 11^{13} + 37^7 \cdot 839$	=	$7 \cdot 19^8 \cdot 31^2$
124.	1.22706	$61^6 \cdot 3907$	=	$3^{28} + 5^4 \cdot 11^{11} + 7 \cdot 17^5 \cdot 97^2$
125.	1.22668	$3 \cdot 47^7 \cdot 39217 + 7^3 \cdot 13^2 \cdot 59$	=	$5^{24} + 11^2 \cdot 29^6$
126.	1.22635	$41^5 \cdot 83^2 + 13^3 \cdot 71^4$	=	$5 \cdot 11^8 + 31^8$
127.	1.22612	$11^6 \cdot 13 \cdot 1051 + 5^2 \cdot 37^7 \cdot 7027$	=	$3^{34} + 29^5$
128.	1.2258	$3^{14} \cdot 5^6 \cdot 47 + 23^3$	=	$37^8 + 7^2 \cdot 61 \cdot 67^2$
129.	1.22508	$17^8 \cdot 113^2 + 3^4 \cdot 5 \cdot 29^7$	=	$11^{11} + 23^8 \cdot 1223$
130.	1.22495	$3^3 \cdot 5^2 \cdot 31^6$	=	$11^{10} \cdot 23 + 17^6 \cdot 29 + 71^5$
131.	1.22449	$3^{18} \cdot 19^2 + 5^{19} \cdot 97^2$	=	$7^{15} \cdot 103 \cdot 367 + 17^4 \cdot 991$
132.	1.22445	$3 \cdot 31^{10} \cdot 73^2$	=	$13^7 \cdot 277^2 + 5^2 \cdot 37^{10} \cdot 109 + 53^4 \cdot 157^2$
133.	1.22405	$3^7 \cdot 11^2 + 23^5$	=	$13^6 + 37^4$
134.	1.22394	$3^5 \cdot 7^4 + 13^9 \cdot 23 + 5^3 \cdot 29^6$	=	$41^6 \cdot 67$
135.	1.22391	$5^{18} \cdot 133979 + 3 \cdot 11^{11}$	=	$7^{17} \cdot 13^3 + 23^6 \cdot 6361$

136.	1.22378	$7^{15} \cdot 197 + 17^{10} \cdot 53$	=	$3^{21} \cdot 41^2 + 5^2 \cdot 19^9 \cdot 127$
137.	1.22353	$3^6 \cdot 11^2 + 5^3 \cdot 29^6$	=	$7 \cdot 13^9 + 41^4 \cdot 43$
138.	1.2227	$7^{16} + 37^7 \cdot 677$	=	$5^{12} \cdot 1051 + 13^4 \cdot 23^7$
139.	1.22267	$3^{32} \cdot 11 + 5^4 \cdot 19^6 + 61^5 \cdot 166597$	=	$13^{10} \cdot 53^3$
140.	1.22252	$3^{18} \cdot 90499 + 31^9 \cdot 1117$	=	$7 \cdot 11^{15} + 41^9$
141.	1.22237	$5^5 \cdot 23^2 + 37^8 \cdot 79 \cdot 11329$	=	$11^4 \cdot 13 + 47^9 \cdot 53^2$
142.	1.22167	$5 \cdot 13^{12}$	=	$7^3 \cdot 5591 + 19^{11} + 29^4 \cdot 233$
143.	1.22141	$11^{10} \cdot 181$	=	$3^{23} \cdot 29 + 5^{13} \cdot 1609 + 17^7$
144.	1.22137	$5 \cdot 13^{12}$	=	$3^{10} \cdot 2797 + 19^{11} + 7^3 \cdot 23 \cdot 197$
145.	1.22115	11^{12}	=	$3^{17} \cdot 19 \cdot 1277 + 5 \cdot 7^8 \cdot 13^2 + 47^5$
146.	1.22068	$5 \cdot 13^{12}$	=	$11^3 \cdot 17^2 \cdot 433 + 19^{11} + 47^2 \cdot 71$
147.	1.22042	$3^{13} \cdot 11 \cdot 19 + 31^6$	=	$5^{13} + 7^3 \cdot 41$
148.	1.21999	$3^3 \cdot 11^6$	=	$5^8 + 19^6 + 17^2 \cdot 37^2$
149.	1.21999	$5 \cdot 13^{12}$	=	$3 \cdot 17^2 + 19^{11} + 31^2 \cdot 283 \cdot 613$
150.	1.21953	$11^6 \cdot 47 \cdot 673 + 3^4 \cdot 17^8 + 23^9$	=	$5^2 \cdot 7^{13}$
151.	1.21952	$5^{19} \cdot 13^4$	=	$7 \cdot 17^6 + 3 \cdot 29^9 \cdot 12517 + 47^4 \cdot 883$
152.	1.21945	$5^4 \cdot 7^9 \cdot 79 + 11^{15} \cdot 43 \cdot 59 + 3^3 \cdot 13^3$	=	$41^9 \cdot 32371$
153.	1.21945	$5^5 \cdot 7^{13} + 11^5 \cdot 23 \cdot 26177$	=	$3^7 \cdot 83^2 + 13^{13}$
154.	1.21941	$3^{16} + 5^4 \cdot 7^5 + 11^6$	=	$13^5 \cdot 149$
155.	1.2193	$11^9 \cdot 43^2 + 7^2 \cdot 13^3 \cdot 29$	=	$3^3 \cdot 73 + 5^4 \cdot 17^8$
156.	1.21911	$5 \cdot 13^{12}$	=	$11^5 \cdot 947 + 19^{11} + 3 \cdot 23^3 \cdot 389$
157.	1.21885	$11^8 + 67^5$	=	$3^{15} \cdot 109 + 5^6 \cdot 29$
158.	1.21877	$5 \cdot 17^7 \cdot 59^2 + 11^2 \cdot 37^9$	=	$7^{14} \cdot 29^2 + 19^{10} \cdot 2473$
159.	1.21849	$3^{21} + 23 \cdot 37^3 + 53$	=	$5^2 \cdot 13^2 \cdot 19^5$
160.	1.2182	$3^{20} \cdot 109^2 + 5^7 \cdot 337$	=	$7^3 \cdot 1699 + 23^{10}$
161.	1.21815	$7^9 \cdot 113^2 + 19^2 \cdot 29^2$	=	$3^{17} \cdot 67 + 47^7$
162.	1.2178	$3^9 \cdot 8539 + 19^{11}$	=	$11^3 \cdot 1021 + 5 \cdot 13^{12}$
163.	1.21772	$5^2 \cdot 29^2 \cdot 47^7 + 3^4 \cdot 59^6$	=	$7^2 \cdot 17 \cdot 251 + 13 \cdot 31^{10}$
164.	1.21757	$5 \cdot 13^{12}$	=	$3^5 \cdot 11 \cdot 859 + 19^{11} + 7^4 \cdot 31 \cdot 47^2$
165.	1.21737	$5^{11} + 7^{10} + 23^6$	=	$13^5 \cdot 1291$
166.	1.21734	$3^{22} \cdot 241^2 + 13 \cdot 61^9$	=	$5^3 \cdot 40237 + 17^{11} \cdot 67^2$
167.	1.217	59^5	=	$3^{14} \cdot 149 + 5^6 \cdot 11^2 + 13^5$
168.	1.21697	$31^9 \cdot 463$	=	$3^{30} \cdot 59 + 5^3 \cdot 19^7 \cdot 29^2 + 23 \cdot 103^2$
169.	1.21694	$7^{14} \cdot 103 \cdot 1063 + 17^8$	=	$29^9 + 3^2 \cdot 71^7 \cdot 907$
170.	1.21684	$3^{18} \cdot 7 \cdot 23^3 + 11^{13}$	=	$13^9 \cdot 6367 + 17^4 \cdot 31^2$
171.	1.21648	$3^{17} \cdot 47^2 + 7^6 \cdot 349$	=	$11^{11} + 13^2 \cdot 53$
172.	1.21625	$3^{13} \cdot 53 \cdot 1973 + 7^{18} \cdot 157 + 11^{16}$	=	$17^{10} \cdot 41^2 \cdot 89$
173.	1.21599	$7^8 \cdot 79 + 43^2$	=	$5 + 3^5 \cdot 37^4$
174.	1.21595	$17^{11} + 3^4 \cdot 13^2 \cdot 67^2$	=	$19^2 \cdot 37^7 + 61 \cdot 71^4$
175.	1.21579	$13 \cdot 29^5 + 53^7$	=	$3^{18} \cdot 5 \cdot 601 + 47^6$
176.	1.21528	$5 \cdot 13^{12} + 3^5 \cdot 37^2$	=	$19^{11} + 61^2 \cdot 44893$
177.	1.21494	$13^5 \cdot 607 + 53^8 \cdot 4651$	=	$3^5 \cdot 7^{15} \cdot 251 + 23^9$
178.	1.21464	$3^{17} \cdot 67 + 5^8 \cdot 53$	=	$7^{10} + 11^2 \cdot 37^5$
179.	1.21458	$3 \cdot 5^{21} + 19 \cdot 47^8 \cdot 53$	=	$13^3 \cdot 107 \cdot 1049 + 31^{11}$
180.	1.21444	$3^6 \cdot 17 \cdot 2131 + 7 \cdot 11^4 \cdot 37^2 + 19^{11}$	=	$5 \cdot 13^{12}$
181.	1.21414	$5 \cdot 13^{12}$	=	$3^7 \cdot 31 \cdot 2459 + 19^{11} + 29 \cdot 47$
182.	1.21323	$3^{29} \cdot 127 + 67 \cdot 1499$	=	$5^{16} \cdot 239^2 + 7^{11} \cdot 43$
183.	1.21305	$7^{12} + 19^{10}$	=	$3^5 \cdot 13^8 \cdot 31 + 37^2 \cdot 5861$
184.	1.21281	$3^{14} \cdot 7 \cdot 11 + 23^5 + 5^2 \cdot 29^5$	=	31^6
185.	1.21256	$11^6 \cdot 13^3 + 5^2 \cdot 59^7 \cdot 107$	=	$3^{29} \cdot 97 + 7^3 \cdot 23^2 \cdot 53$
186.	1.21248	$3^{20} \cdot 431 + 5^{18} \cdot 17 + 7^{10} \cdot 53$	=	$47^7 \cdot 131$
187.	1.21178	$3^{21} + 19^3 \cdot 37^4$	=	$5^{11} \cdot 7^3 + 11^7 \cdot 337$
188.	1.21075	$41^9 \cdot 307$	=	$11^2 \cdot 31^{10} + 37^8 \cdot 379 + 3^4 \cdot 13^2 \cdot 47^3$
189.	1.20985	$3^3 + 5^5 \cdot 11^2$	=	$13^5 + 19^3$
190.	1.20974	$17^2 \cdot 19^5$	=	$3^7 + 5^3 \cdot 73^2 + 59^5$
191.	1.2096	$5^{20} \cdot 1031 + 7^8 \cdot 137$	=	$3^2 \cdot 11^{12} \cdot 59^2 + 13 \cdot 19^3 \cdot 47^2$
192.	1.20896	$5 \cdot 13^{12} + 7 \cdot 29^4 \cdot 173$	=	$19^{11} + 3 \cdot 31^3 \cdot 107^2$
193.	1.20894	$5^{15} \cdot 83 + 7 \cdot 13^5 \cdot 23 \cdot 149$	=	$3^{26} + 47^3$
194.	1.20891	$7^2 \cdot 17^{12} + 23^2 \cdot 43^5$	=	$13^9 \cdot 2657 + 3 \cdot 5^2 \cdot 11^2 \cdot 61^7$
195.	1.20884	$5 \cdot 13^{12}$	=	$3^{12} \cdot 313 + 11 \cdot 33923 + 19^{11}$
196.	1.20872	$3^{21} \cdot 31 + 13^8 \cdot 284783 + 11^3 \cdot 29^2$	=	7^{17}
197.	1.20829	$7^{11} \cdot 103919 + 13 \cdot 17^{13} + 3^3 \cdot 43^4$	=	$5^{17} \cdot 11^3 \cdot 127$
198.	1.20756	$19^{11} + 3^2 \cdot 43^3 \cdot 233$	=	$5 \cdot 13^{12} + 67 \cdot 179$
199.	1.20751	$11^4 \cdot 59 \cdot 193 + 19^{11}$	=	$5 \cdot 13^{12} + 43 \cdot 67$
200.	1.20749	$11^9 + 17^5 \cdot 103 + 3 \cdot 29^5$	=	37^6

For all 25857 examples see website `http://www.win.tue.nl/~bdeweger/abc/`.

C.2 *abcd*-examples without gcd condition

C.3 *abcde*-examples with gcd condition

Table 5: *abcde*-examples with $\gcd(a, b) = \gcd(a, c) = \ldots = \gcd(d, e) = 1$

1.	1.95577	$2^{25} + 31^6 + 37^6$	=	$3^{20} + 11^2$
2.	1.77341	$2^{28} + 5^{13}$	=	$11^5 + 13^8 + 3^6 \cdot 31^4$
3.	1.65729	$2^{19} \cdot 13 + 5^9 + 7^{10} \cdot 37 + 11^2$	=	3^{21}
4.	1.62733	$2^3 + 5^8 + 7^3$	=	$3^9 + 13^5$
5.	1.62729	$3^{16} + 5^{10}$	=	$7^8 + 2^7 \cdot 13 + 19^6$
6.	1.60251	$5^{11} + 13^4$	=	$2^4 \cdot 3^{12} + 7^9 + 23$
7.	1.57729	29^7	=	$3^2 + 2^8 \cdot 5^5 + 11 \cdot 17^6 + 19^8$
8.	1.57485	$3^{19} + 7^3 \cdot 11^2 + 5^2 \cdot 13^2$	=	$2^{28} + 19^7$
9.	1.57474	$2^{36} + 11 + 3 \cdot 13^9 + 19$	=	$5 \cdot 7^2 \cdot 17^7$
10.	1.56807	$2 + 3^{11} + 5^9 \cdot 89 + 7^9$	=	11^8
11.	1.56674	$2^{20} + 5^2 + 13^2 \cdot 17^7$	=	$3^{18} \cdot 179 + 7^5$
12.	1.56105	$2^{16} \cdot 7^2 + 3^{14} + 11^6$	=	$5^{10} + 13^2$
13.	1.56033	3^{11}	=	$2^7 + 5^6 + 7^3 + 11^5$
14.	1.55651	$7 \cdot 19^7 + 29$	=	$2^7 + 3^{12} \cdot 17^2 + 5^{14}$
15.	1.55109	$2^{23} \cdot 5 + 17^3$	=	$3^{13} + 7^9 + 23$
16.	1.54966	$2^{28} \cdot 13 + 7 \cdot 17$	=	$3^{20} + 5^9 + 31^4$
17.	1.54052	$2^5 + 3^{20} + 11^8 + 13^4$	=	$5^{10} \cdot 379$
18.	1.53685	2^{27}	=	$3^{17} + 5^7 + 11^5 \cdot 31 + 19^3$
19.	1.52944	$2^{29} + 3^9 + 7^5 \cdot 13 + 17^2$	=	$5^{11} \cdot 11$
20.	1.52863	$2^7 + 7^4 + 3^2 \cdot 29^4 + 31^6$	=	19^7
21.	1.52677	$2^{27} + 5^5 \cdot 71 + 23^6$	=	$3^5 + 7^{10}$
22.	1.52605	$5^4 + 19^6$	=	$2^{17} \cdot 17 + 3^{16} + 11^6$
23.	1.52225	$2^{30} + 7^6$	=	$3^{18} + 19^5 \cdot 37 + 29^6$
24.	1.51688	$2^{26} + 5 + 7^6 \cdot 19$	=	$3^5 + 37^5$
25.	1.51512	$7^2 + 29^5$	=	$2^{13} \cdot 5^3 + 3^3 + 11^7$
26.	1.51494	$2^{25} + 13^3 + 17^7$	=	$3^2 \cdot 5^4 + 7^9 \cdot 11$
27.	1.51401	$3^{18} \cdot 71 + 7^8$	=	$2^{10} \cdot 5 + 17^2 + 31^7$
28.	1.51072	$3^9 + 7 + 11^7 + 29^3$	=	$2 \cdot 5^{10}$
29.	1.51051	$2^3 + 5^8 + 3^3 \cdot 29^4$	=	$7^2 + 11^7$
30.	1.51043	$7^{12} + 23^6 \cdot 29$	=	$2^3 + 3^8 + 13^7 \cdot 17^2$
31.	1.50985	$5^2 \cdot 7^9 + 2^3 \cdot 11^7 + 23$	=	$3^{19} + 19^5$
32.	1.50559	$2^{11} \cdot 17 + 3^{13} \cdot 7^2 + 13^8$	=	$5^6 + 19^7$
33.	1.50306	$2^{32} + 5^4 + 17^5$	=	$3^{20} + 11^6 \cdot 457$
34.	1.50286	$2^8 \cdot 101 + 3^{21} + 5^2 \cdot 7^8 + 17^2$	=	13^9
35.	1.50021	$3^{18} + 5^3 + 19 \cdot 31^6$	=	$2^2 \cdot 13^4 + 29^7$
36.	1.50018	17^7	=	$2^5 + 3^{18} + 5^{10} + 23^4 \cdot 47$
37.	1.50018	$5^{11} \cdot 17 + 7 \cdot 19^2$	=	$2^{10} + 3^{15} + 13^8$
38.	1.49781	$7^6 + 2 \cdot 13^6$	=	$3^6 + 5^{10} + 17^3$
39.	1.49775	$3^9 \cdot 29 + 7^{13} + 11 \cdot 17^5$	=	$2^{28} \cdot 19^2 + 5^2$
40.	1.49479	$7 \cdot 19^8$	=	$2^{28} + 3 \cdot 61 + 17^9 + 31^5$
41.	1.49348	$3^{21} + 5^9 \cdot 61 + 7^7 \cdot 31$	=	$2^{19} + 13^9$
42.	1.49294	$3^{17} + 5^8 \cdot 13$	=	$2^{27} + 23^2 + 31$
43.	1.49086	$5^3 \cdot 7^8 + 11^4 + 3^3 \cdot 23^5$	=	$2^{19} + 19^7$
44.	1.49086	23^8	=	$2^{26} \cdot 13^2 + 3^{20} \cdot 19 + 5^3 \cdot 7^8 + 17^4$
45.	1.48497	$7^2 \cdot 17^6 + 19^9 + 29^5$	=	$3^{21} \cdot 11 + 2^8 \cdot 13^8$
46.	1.48061	$11^2 \cdot 29^4$	=	$2^{21} + 3^{16} + 7^9 + 17^4$
47.	1.47952	$2^{19} + 5^3 + 13^2 + 19^3$	=	3^{12}
48.	1.47834	$2^{15} + 17^3 + 31^5$	=	$5^7 + 3^5 \cdot 7^6$
49.	1.47344	$2^{10} + 5^8 + 3^2 \cdot 13^7$	=	$7^3 + 11^7 \cdot 29$
50.	1.47108	$5^{16} + 13^7$	=	$3^4 + 7^2 \cdot 11^2 + 2^2 \cdot 31^6 \cdot 43$
51.	1.47001	$3^{20} + 7^{10} \cdot 17$	=	$2^{31} + 5^{14} + 31^4 \cdot 41$
52.	1.46988	$2^{31} + 5^8 \cdot 7^3 + 11^3 + 19^3$	=	$3^{16} \cdot 53$
53.	1.46914	$2^{19} + 5^{11} + 19^7 \cdot 241$	=	$3^5 \cdot 13^8 + 29^7$
54.	1.46861	$2^{31} + 17^7 + 31^4$	=	$3^{19} + 5^{10} \cdot 11 \cdot 13$
55.	1.46849	$5 + 7 + 17^5$	=	$2^{20} + 13^5$
56.	1.46805	$2^{32} + 11 \cdot 23^7$	=	$3^2 \cdot 5^{12} \cdot 19 + 7^2 + 17^2$
57.	1.46783	$2^{26} + 13^4$	=	$3^{12} \cdot 5^3 + 19 + 29^4$
58.	1.46702	$2^{18} + 3^6 + 19^6$	=	$5^6 + 13^2 \cdot 23^4$
59.	1.46474	$2 \cdot 5^5 + 31^7$	=	$3^{18} \cdot 71 + 7^8 + 29^2$
60.	1.46466	$3^{21} + 5^2 \cdot 7^8 + 11^2 \cdot 233$	=	$2^{11} + 13^9$
61.	1.46141	$3^{19} \cdot 37 + 11^8 + 13^2 \cdot 17^4$	=	$2^{33} \cdot 5 + 7^{10}$
62.	1.46069	$2^3 \cdot 5^{14} + 7^{12} + 23^7$	=	$11^3 \cdot 37 + 3^4 \cdot 13^8$
63.	1.45955	11^8	=	$2^{19} \cdot 17^2 + 3^6 \cdot 5^3 + 7 + 13^7$
64.	1.45876	$2^{25} + 5 \cdot 19^2 + 23^4$	=	$3^{15} + 11^7$
65.	1.45747	$2^3 \cdot 5^{10} + 11^4 + 13^8 + 3^4 \cdot 17$	=	19^7
66.	1.45746	$3^{14} \cdot 7^3 + 5^5 + 31^4$	=	$2^2 \cdot 17^7 + 19^4$
67.	1.45659	3^{16}	=	$2^{23} + 11^3 + 13 + 7^2 \cdot 29^4$
68.	1.45576	$2^{27} + 5^9 + 3^4 \cdot 7^9 + 17^2 \cdot 43$	=	23^7
69.	1.45501	$2^{18} \cdot 7 + 5^3 \cdot 29^5$	=	$3^3 + 13^3 + 37^6$
70.	1.45488	$3^{17} \cdot 29 + 11^8$	=	$2^{26} \cdot 59 + 5^4 + 7$
71.	1.45475	$2^7 \cdot 5^9 + 7^{12}$	=	$3^{20} + 13^9 + 23 \cdot 149$
72.	1.45439	$2^{25} + 5^{14} + 29^2$	=	$3^{17} \cdot 43 + 11^2 \cdot 13^6$
73.	1.45362	$2^{37} + 3^{18} + 7 \cdot 31^4$	=	$5^7 + 11^8 \cdot 643$
74.	1.45353	$2^{29} \cdot 13$	=	$3^{13} + 5^9 + 17^8 + 19^2 \cdot 47$
75.	1.45262	$2^{22} \cdot 3^2 + 7^6 + 17^7$	=	$5^3 + 11^7 \cdot 23$
76.	1.45076	$19^6 + 31^2$	=	$3^{16} + 2^8 \cdot 5^6 + 11^2$
77.	1.44884	$2^{23} \cdot 7^2 + 19$	=	$3^2 \cdot 5^7 + 13 + 17^7$
78.	1.44839	$5^{11} \cdot 17 + 7^2 \cdot 31$	=	$2^4 + 3^{15} + 13^8$
79.	1.44716	$3^4 \cdot 5^{10} + 7^7 + 17^6$	=	$13^8 + 2^8 \cdot 31^2$
80.	1.44716	$2^{26} + 5^3 \cdot 7^6 + 17^6 \cdot 31$	=	$3^{15} + 13^8$

81.	1.44509	$11^7 \cdot 29$	=	$3^{11} + 5^2 + 2 \cdot 7^{10} + 17^2$
82.	1.44497	$11^7 + 17 + 5^4 \cdot 23^6$	=	$2^{31} \cdot 43 + 3^5 \cdot 7^7$
83.	1.44396	$5^{11} \cdot 89$	=	$2^{14} + 3^9 \cdot 23^2 + 7^{11} + 11^9$
84.	1.44307	$2^{23} + 3 \cdot 7^3$	=	$5^9 + 13^2 + 23^5$
85.	1.44142	$2^{21} \cdot 7 + 13^6$	=	$3^9 + 11^7 + 19$
86.	1.44035	$2^{28} + 19^7 + 17 \cdot 29^2$	=	$3^{19} + 5^2 \cdot 7^4$
87.	1.43816	$3^{10} \cdot 19^2 + 7^3 + 11^9 + 13^4$	=	$2^2 \cdot 29^6$
88.	1.43809	$5^{12} + 11^6 + 31^4 \cdot 71$	=	$2^{28} + 3^{16}$
89.	1.4379	$2^{34} + 3^8 \cdot 5^4$	=	$7^2 \cdot 11 \cdot 13^5 + 19^8 + 23^4$
90.	1.4372	19^8	=	$2^{26} \cdot 11 \cdot 23 + 3^{12} + 5 \cdot 7^7 + 13^5$
91.	1.43715	$2^{34} + 3^{12} \cdot 11^2 + 5^7 \cdot 73$	=	$19^2 + 29^7$
92.	1.43627	3^{12}	=	$2^{19} + 5 + 17^2 + 19^3$
93.	1.43517	$2^{13} \cdot 5^3 + 11^7$	=	$3^2 + 13 + 29^5$
94.	1.43493	$2^{23} \cdot 7^3 + 13^9 + 19^2$	=	$3^{15} \cdot 5^4 + 11 \cdot 17^7$
95.	1.43466	$2^{23} + 7^6$	=	$3^{12} \cdot 11 + 5^9 + 29^4$
96.	1.43108	$2^{25} \cdot 59 + 23$	=	$3^9 \cdot 11^2 + 5^5 + 7^{11}$
97.	1.43081	$2^7 + 5^8 + 3^3 \cdot 29^4$	=	$11^7 + 13^2$
98.	1.4305	$11^9 \cdot 13 + 31^5$	=	$5^{15} + 2^2 \cdot 7^9 + 3^5 \cdot 23^3$
99.	1.4301	$2^{34} + 5^5 \cdot 139 + 7^2 \cdot 17^5$	=	$3^5 + 29^7$
100.	1.42934	$2^{20} \cdot 19 + 3^4 + 17^7 \cdot 67$	=	$5 + 31^7$
101.	1.42835	$2^{30} \cdot 17 + 5^{15} + 7^2$	=	$3 \cdot 11^8 + 13^8 \cdot 59$
102.	1.42811	23^7	=	$2^{23} \cdot 13 \cdot 31 + 3^6 + 5^7 + 17^6$
103.	1.42701	$7^4 + 29^7$	=	$2^{34} + 5^{11} + 3^2 \cdot 31^3 \cdot 79$
104.	1.42668	$11^8 \cdot 19 + 29^6$	=	$3^7 + 2^4 \cdot 7^{10} + 23^6$
105.	1.42615	$2^{23} + 7^2 \cdot 29^4 + 37^2$	=	$3^{16} + 5^2$
106.	1.42612	$2^{24} + 11^7$	=	$5^4 + 7 \cdot 13^6 + 19^5$
107.	1.42465	$2^{12} \cdot 3^3 + 5^9 \cdot 37 + 7^8 + 13^8$	=	19^7
108.	1.42434	17^6	=	$2^{20} \cdot 23 + 3^9 + 5^4 + 13$
109.	1.42427	$2^{20} \cdot 23 + 13^3$	=	$3^{15} + 5^{10} + 17^3$
110.	1.4236	$2^{26} \cdot 13 + 3^{11} \cdot 11^2 + 7^5 + 17^3$	=	19^7
111.	1.42336	$2 + 3^{13} + 23^4$	=	$5 + 37^4$
112.	1.42336	$3^{13} + 5 + 23^4$	=	$2^3 + 37^4$
113.	1.42336	$3^{13} + 5^3 + 23^4$	=	$2^7 + 37^4$
114.	1.4215	$2^2 \cdot 3^{18} + 11^7 + 17^8$	=	$5^{13} \cdot 7 + 13 \cdot 19^2$
115.	1.42115	$3^3 + 5^5 + 19^4$	=	$2^{17} + 7^4$
116.	1.42063	13^8	=	$3^{17} + 5^3 + 2^4 \cdot 11^6 \cdot 19 + 23^6$
117.	1.41895	$5^{12} + 3^3 \cdot 17^5$	=	$7^{10} + 11^3 + 2^3 \cdot 23$
118.	1.41824	$11^4 \cdot 47 + 3^6 \cdot 19^6$	=	$2^{33} + 5^9 + 7^{11} \cdot 13$
119.	1.41815	$3^{13} + 2^6 \cdot 23^4$	=	$7^5 + 11^7 + 13^2$
120.	1.41756	$2^7 \cdot 3^7 + 5^9 + 7^6 + 19^5$	=	13^6
121.	1.41727	$2^3 + 3^3 + 5^{10}$	=	$7^2 \cdot 11^5 + 37^4$
122.	1.41727	$2^5 + 3 + 5^{10}$	=	$7^2 \cdot 11^5 + 37^4$
123.	1.41668	$2^{13} + 31^7$	=	$3^{18} \cdot 71 + 7^8 + 11^2 \cdot 23$
124.	1.41522	$2^9 \cdot 5^{12} + 37^4$	=	$3^{20} \cdot 23 + 11^9 \cdot 19 + 13^6$
125.	1.41428	$2^{31} + 17^5$	=	$3^{16} \cdot 7^2 + 5^8 \cdot 101 + 11^5$
126.	1.41319	$3^{11} \cdot 41 + 2^3 \cdot 5^{10} \cdot 19 + 7^{12} + 13^9$	=	11^{10}
127.	1.41188	$3^{17} + 2^3 \cdot 5^8 + 7^2 + 11^6$	=	$13^5 \cdot 19^2$
128.	1.41079	$7^7 \cdot 17^2 + 13^7$	=	$2 \cdot 3^{14} + 5^{12} + 19^6$
129.	1.41036	$2^5 \cdot 3^{15} + 11 \cdot 19^2$	=	$5^{11} + 13^3 + 17^7$
130.	1.40997	$2^{18} + 7^{10} \cdot 11$	=	$3^{19} + 13^7 \cdot 31 + 29^3$
131.	1.40949	$2^8 \cdot 11^6 + 19$	=	$3^{18} + 5^8 \cdot 13^2 + 17^4$
132.	1.40942	$2^{27} + 17^8$	=	$3^8 \cdot 5^2 + 7^8 \cdot 13 + 11^7 \cdot 19^2$
133.	1.40924	$3^2 + 11^8 + 5 \cdot 13^3 + 29^5$	=	$2^{25} \cdot 7$
134.	1.40908	$2^{26} + 3^4 + 17^3$	=	$5^{11} + 19^2 \cdot 37^3$
135.	1.40886	$2^{23} + 3 \cdot 17^2$	=	$5^9 + 7 + 23^5$
136.	1.40886	$5^9 + 23^5$	=	$2^{23} + 3^3 + 7^2 \cdot 17$
137.	1.40882	$17^6 + 5 \cdot 31^3$	=	$2^{14} + 3^{14} + 11^7$
138.	1.4082	$3 \cdot 7^{10} \cdot 23$	=	$2^{32} + 5^5 + 13^3 + 17 \cdot 19^7$
139.	1.40754	$2^{27} + 5^4 \cdot 19^2 + 23^6$	=	$7^{10} + 3 \cdot 11^3$
140.	1.40639	$2^{14} \cdot 11^2$	=	$3^6 + 5^9 + 7^2 + 13^4$
141.	1.40551	$2^3 + 3^{19} + 5^{10} \cdot 13^2 + 31^3$	=	$19 \cdot 23^6$
142.	1.40531	$3^{18} + 2^6 \cdot 13^5$	=	$7^7 + 17^7 + 5^2 \cdot 29^2$
143.	1.40514	$2^{33} + 11^4 + 13^4$	=	$3^{17} \cdot 5^2 + 7^2 \cdot 17 \cdot 23^5$
144.	1.40504	$11^3 + 19^5$	=	$2^{19} + 5^9 + 17$
145.	1.40364	$3^{22} + 7$	=	$2^{29} \cdot 47 + 5^{14} + 11^5 \cdot 277$
146.	1.40309	$17^2 + 29^5$	=	$2^{13} + 3^8 \cdot 5^5 + 11^2$
147.	1.40287	$2^3 \cdot 13^3 + 31^7$	=	$3^{18} \cdot 71 + 7^8 + 23^3$
148.	1.40189	$3^8 + 2 \cdot 5^{10} + 13$	=	$11^7 + 37^3$
149.	1.40159	$7^7 + 5 \cdot 13^5 + 2^2 \cdot 23^6$	=	$3^5 + 29^6$
150.	1.4013	$2^{20} \cdot 19 + 3^3 + 7^2 + 17^7 \cdot 67$	=	31^7
151.	1.40094	$5^5 + 7^8$	=	$2^{19} \cdot 11 + 3^6 + 29$
152.	1.40042	$7^{10} + 19$	=	$2^8 + 3^{14} \cdot 59 + 23^4$
153.	1.40036	3^{15}	=	$2^{18} \cdot 11 + 5^{10} + 17^5 + 23^4$
154.	1.40021	2^{18}	=	$3^{11} + 5^7 + 13 + 19^3$
155.	1.39977	$3^{15} \cdot 7^2 \cdot 17 + 11^8 + 31^2$	=	$2^5 \cdot 5^{11} + 13^9$
156.	1.39745	$3^{19} \cdot 59 + 23^6$	=	$2^{36} + 5^6 \cdot 127 + 11^3$
157.	1.3972	$3^{15} + 5^{10} + 29^3$	=	$2^3 \cdot 13^2 + 17^6$
158.	1.39701	$3^9 + 5^7 + 7^6 \cdot 31 + 11^8$	=	$2^{24} \cdot 13$
159.	1.39689	$7^7 + 13^3 + 2^3 \cdot 23^7$	=	$5^{12} \cdot 83 + 17^8$
160.	1.39628	$3^{17} \cdot 5^2 + 7^8 + 29^5$	=	$2^{25} \cdot 97 + 11^2$

161.	1.39525	$3^{19} \cdot 19 + 17^9$	=	$2^{30} \cdot 131 + 5^4 \cdot 7^5 + 11^5$
162.	1.39411	$3^{19} + 7^7 \cdot 71 + 13^2 \cdot 17$	=	$2^{15} + 5^{13}$
163.	1.39329	$11^7 \cdot 257 + 31^7$	=	$2^{30} + 3^{22} + 5^8 \cdot 13^2$
164.	1.39318	$2^{24} + 3^9$	=	$5^8 \cdot 43 + 7 + 17$
165.	1.39316	$2^6 \cdot 3^8 + 11^5 + 13^8$	=	$5^{10} \cdot 83 + 7^8$
166.	1.39297	$2^{31} + 3^{20} + 7 \cdot 67$	=	$11^7 \cdot 17^2 + 19^5$
167.	1.39182	$2^{25} \cdot 773$	=	$3^{10} + 5^7 + 11^{10} + 7^2 \cdot 17^2$
168.	1.3902	$3 + 19 + 29^5$	=	$2^{13} \cdot 5^3 + 11^7$
169.	1.3891	$5^{11} + 13^9 + 19^2 + 29^3$	=	$2^3 \cdot 3 \cdot 7^9 \cdot 11$
170.	1.38884	$2^{28} \cdot 19^2 + 31^5$	=	$3^{11} \cdot 11 \cdot 23 + 7^{13} + 13^2$
171.	1.38692	$2^{22} + 3^2 \cdot 5^9 \cdot 7^2 + 31^5$	=	$19^7 + 23^4$
172.	1.38638	$2^6 + 17^9$	=	$3^{18} \cdot 5 + 11 \cdot 13^9 + 31^3 \cdot 43$
173.	1.3859	$2^{28} + 3^{16} + 5^3 + 11^4 \cdot 19^2$	=	$7^3 \cdot 31^4$
174.	1.38511	$2^{26} \cdot 7 + 3^2 + 19^6$	=	$11^6 \cdot 17^2 + 13^6$
175.	1.38498	$2^{18} \cdot 5^3 + 13^9$	=	$3^{15} \cdot 577 + 7^3 + 11^9$
176.	1.38477	$5^{11} \cdot 7 + 29^6$	=	$2^{25} + 3^{13} + 13 \cdot 37^5$
177.	1.38399	$5^4 \cdot 11 + 13^2 + 2^6 \cdot 23^5$	=	$3^{13} + 17^7$
178.	1.3835	29^7	=	$2^{34} + 3^4 + 5^8 \cdot 179 + 7 \cdot 23^3$
179.	1.38298	$2^3 + 11^6 \cdot 19 + 13^6$	=	$3^{15} + 17^6$
180.	1.38253	$2^{26} + 23^6 + 7 \cdot 29^5$	=	$3^{15} \cdot 5^2 + 11^2$
181.	1.38253	$7^{12} + 19^2$	=	$2^{24} \cdot 3 \cdot 5^2 \cdot 11 + 17^4 + 29^2$
182.	1.38239	$2^6 \cdot 61 + 5^{11} \cdot 17$	=	$3^{15} + 7^4 + 13^8$
183.	1.38145	$13^9 + 37 \cdot 179$	=	$2^{15} + 3^{21} + 5^2 \cdot 7^8$
184.	1.38106	$3^{16} + 5^{11} + 13^4 + 17^5$	=	$2^{20} \cdot 89$
185.	1.3807	$3^{15} + 11^6 + 13^5 \cdot 47$	=	$2^{25} + 7^5$
186.	1.38014	$3^{17} + 7^5 \cdot 677 + 11^2 + 13^{10}$	=	$2^3 \cdot 29^7$
187.	1.37903	$2^{12} \cdot 13^2 + 5^{10} \cdot 29$	=	$3^5 + 7^{10} + 17^5$
188.	1.37885	$7^{10} + 2^7 \cdot 29$	=	$5^{12} + 13^3 + 3^3 \cdot 17^5$
189.	1.37808	$5^{11} \cdot 229 + 7^2$	=	$2^{18} + 3^{21} + 11^7 \cdot 37$
190.	1.37777	$2^{30} + 5^3$	=	$7^3 \cdot 19^5 + 29^3 + 3^5 \cdot 31^4$
191.	1.37768	$7 + 37^4$	=	$2^2 + 3^{13} + 23^4$
192.	1.37715	$7^7 + 11^3 + 2^5 \cdot 3 \cdot 13^8 + 19^2 \cdot 31$	=	23^8
193.	1.37708	$2^{32} + 23^2 + 37^3 \cdot 149$	=	$3^{20} + 13^8$
194.	1.37663	$3^{14} + 11^9$	=	$2^{25} + 19^7 + 23^5 \cdot 223$
195.	1.37659	$2^{21} + 11^4$	=	$3^3 \cdot 5^7 + 7^4 + 17$
196.	1.3765	$2^9 \cdot 13^7 + 31$	=	$3^5 \cdot 5^2 + 11^3 \cdot 17^6 + 19^4$
197.	1.37542	$3^{16} + 17^6 + 23$	=	$2^{26} + 11 \cdot 19^3$
198.	1.37489	$2^7 + 3^3 + 5^9 \cdot 11^2 + 13^4$	=	$7^8 \cdot 41$
199.	1.37362	$3^4 + 7^8 \cdot 43 + 13^3$	=	$5^{12} + 2^8 \cdot 11^4$
200.	1.3736	$2^{28} + 5^5 + 19^7$	=	$3^{19} + 7^2 \cdot 997$

For all 65609 examples see website http://www.win.tue.nl/~bdeweger/abc/. These examples were generated with the Mathematica notebook given in Appendix B.2 with as primebase all subsets of length 5 of the first 12 primes, with powers up to $\frac{30\log(2)}{\log(p)}$ and linear combinations of the reduced lattice base vectors with coefficients up to 2.

C.4 *abcde*-examples without gcd condition

Table 6: *abcde*-examples with $\gcd(a,b,c,d,e) = 1$

1.	6.96675	$2^{24} \cdot 5^3 + 3^{11} + 3^{20} \cdot 5$	=	$2^7 \cdot 3^2 + 2^4 \cdot 5^{13}$
2.	6.11452	$2^{15} \cdot 3 \cdot 5^2 + 2^{30}$	=	$2^{10} + 3^{16} \cdot 5^2 + 3^5 \cdot 5^3$
3.	5.91006	$2^{29} + 3^2 \cdot 5^7$	=	$2^2 \cdot 3 + 2^{22} \cdot 5^3 + 3^{12} \cdot 5^2$
4.	5.8601	$2^4 \cdot 5^3 + 3^{15} + 3^2 \cdot 5^{11}$	=	$2^{15} \cdot 5^2 + 2^{24} \cdot 3^3$
5.	5.81656	$3^3 \cdot 5 + 2^3 \cdot 5^{11}$	=	$2^7 \cdot 5^3 + 2 \cdot 3^{13} + 3^{18}$
6.	5.79151	$2^2 \cdot 3^4 + 3^{15} \cdot 5^2$	=	$2^{22} + 2^{26} \cdot 5 + 3^5 \cdot 5^7$
7.	5.79151	$2^2 + 2^6 \cdot 5 + 3^{15} \cdot 5^2$	=	$2^{22} \cdot 3^4 + 3^5 \cdot 5^7$
8.	5.701	$2^7 \cdot 3^4 \cdot 5 + 2^{18} + 2^{27} + 3^{17}$	=	$3^3 \cdot 5^{10}$
9.	5.61869	$3^3 + 2 \cdot 3^{10} + 3^{13} \cdot 5^3$	=	$2^{15} \cdot 5^3 + 2^2 \cdot 5^{11}$
10.	5.50247	$2^{27} + 3^4 \cdot 5^7$	=	$2^2 \cdot 3^3 + 2^{22} \cdot 5 + 3^{14} \cdot 5^2$
11.	5.50247	$2^2 \cdot 5 + 2^{22} \cdot 3^3 + 3^{11} \cdot 5^3$	=	$2^{27} + 3 \cdot 5^8$
12.	5.49113	$2^5 \cdot 3^{13} + 2^3 \cdot 5^{10}$	=	$2^{11} + 3^2 \cdot 5^3 + 3^{17}$
13.	5.49113	$3^3 + 2^5 \cdot 3^{13} + 2^3 \cdot 5^{10}$	=	$2^7 \cdot 5^2 + 3^{17}$
14.	5.4685	$2^7 + 3^{14} \cdot 5^2$	=	$2^2 \cdot 5 + 2^{22} \cdot 3^3 + 3^4 \cdot 5^7$
15.	5.4685	$2^{19} \cdot 3^3 \cdot 5 + 5^{11}$	=	$2^2 \cdot 3 + 2^{15} + 3^{14} \cdot 5^2$
16.	5.4685	$2^2 \cdot 3^3 + 3^{14} \cdot 5^2$	=	$2^{23} + 2^{22} \cdot 5^2 + 3^4 \cdot 5^7$
17.	5.42989	$2^{22} \cdot 5^2 + 3^3 \cdot 5^7$	=	$2^2 \cdot 3^2 + 2^{26} + 3^{13} \cdot 5^2$
18.	5.37192	$2^{11} + 3^2 \cdot 5^3 + 2 \cdot 3^{16}$	=	$3^{13} \cdot 5 + 2^3 \cdot 5^{10}$
19.	5.37192	$3^3 + 3^{13} \cdot 5 + 2^3 \cdot 5^{10}$	=	$2^7 \cdot 5^2 + 2 \cdot 3^{16}$
20.	5.37192	$2^7 \cdot 3^{10} + 2 \cdot 3^{16}$	=	$2^{24} \cdot 5 + 3^2 + 5^{10}$
21.	5.34929	$3^3 + 3^{13} + 2^3 \cdot 5^{10}$	=	$2^7 \cdot 5^2 + 2 \cdot 3^{13} \cdot 5^2$
22.	5.34929	$3^{13} + 2^3 \cdot 5^{10}$	=	$2^{11} + 3^2 \cdot 5^3 + 2 \cdot 3^{13} \cdot 5^2$
23.	5.34336	$2^{11} + 3^2 \cdot 5^3 + 3^{13} + 2^4 \cdot 3^{14}$	=	$2^3 \cdot 5^{10}$
24.	5.34336	$2^3 \cdot 5^{10}$	=	$2^{11} + 3^2 \cdot 5^3 + 3^{13} \cdot 5^2 + 2^3 \cdot 3^{14}$
25.	5.34336	$2^{11} + 3^2 \cdot 5^3 + 2^2 \cdot 3^{13} + 3^{15} \cdot 5$	=	$2^3 \cdot 5^{10}$
26.	5.34336	$3^3 + 2^3 \cdot 5^{10}$	=	$2^7 \cdot 5^2 + 3^{13} + 2^4 \cdot 3^{14}$
27.	5.34336	$3^3 + 2^3 \cdot 5^{10}$	=	$2^7 \cdot 5^2 + 2^2 \cdot 3^{13} + 3^{15} \cdot 5$
28.	5.34336	$2^7 \cdot 5^2 + 2^3 \cdot 3^{14} + 3^{13} \cdot 5^2$	=	$3^3 + 2^3 \cdot 5^{10}$

29.	5.29867	$2 \cdot 3^2 + 2^{26}$	$=$	$2^{10} \cdot 5^2 + 3^{15} + 3^3 \cdot 5^9$
30.	5.29867	$2^2 \cdot 3 \cdot 5 + 2^{22} + 3^{12} \cdot 5^3$	$=$	$2^{26} + 3^2 \cdot 5^8$
31.	5.29867	$2^2 + 2^{22} \cdot 3 \cdot 5 + 3^{11} \cdot 5^2$	$=$	$2^{26} + 3 \cdot 5^7$
32.	5.29867	2^{26}	$=$	$2^5 \cdot 3^3 + 2^{11} \cdot 5^3 + 3^{12} \cdot 5^3 + 3^3 \cdot 5^6$
33.	5.29568	$2^2 + 2^{22} \cdot 3 \cdot 5 + 3^2 \cdot 5^8$	$=$	$2^6 + 3^{12} \cdot 5^3$
34.	5.20517	$3^7 \cdot 5^2 + 2^4 \cdot 3^{10} + 2 \cdot 3^{14} \cdot 5$	$=$	$2^{10} + 5^{11}$
35.	5.20517	$2^7 \cdot 3^3 + 2^{23} \cdot 3 + 3^{14} \cdot 5$	$=$	$2^{11} \cdot 5^3 + 5^{11}$
36.	5.20517	$2^{18} + 5^{11}$	$=$	$2^7 \cdot 3 \cdot 5^2 + 2^{23} \cdot 3 + 3^{14} \cdot 5$
37.	5.16812	$2^{18} \cdot 5^3 + 2^7 \cdot 3^7 + 2^7 \cdot 5^7$	$=$	$3^5 \cdot 5 + 3^{16}$
38.	5.16812	$2^{13} \cdot 3^3 \cdot 5 + 2^{23} \cdot 5$	$=$	$2^{10} + 3^{16} + 3^5 \cdot 5$
39.	5.16414	$2^8 \cdot 3 + 2^{17} \cdot 5 + 3^{13} \cdot 5^2 + 5^9$	$=$	$2^{19} \cdot 3^4$
40.	5.16048	$2^2 \cdot 3^2 + 2^{22} + 3^{13} \cdot 5^2$	$=$	$2^{23} \cdot 5 + 3^3 \cdot 5^7$
41.	5.16048	$2^8 \cdot 3 + 2^{17} + 3^{13} \cdot 5^2 + 5^9$	$=$	$2^{23} \cdot 5$
42.	5.14549	$2^4 + 2^2 \cdot 5 + 3^{13} \cdot 5^2$	$=$	$2^{22} \cdot 3^2 + 3^3 \cdot 5^7$
43.	5.14549	$2^2 + 2^5 + 3^{13} \cdot 5^2$	$=$	$2^{22} \cdot 3^2 + 3^3 \cdot 5^7$
44.	5.14549	$2^2 + 2^{22} \cdot 3^2 + 3^3 \cdot 5^7$	$=$	$2^3 \cdot 5 + 3^{13} \cdot 5^2$
45.	5.14549	$2^2 \cdot 3^2 + 3^{13} \cdot 5^2$	$=$	$2^{22} + 2^{25} + 3^3 \cdot 5^7$
46.	5.14549	$2^{22} \cdot 5 + 2^{24} + 3^3 \cdot 5^7$	$=$	$2^2 \cdot 3^2 + 3^{13} \cdot 5^2$
47.	5.12951	$2^2 + 2^{25} + 3^{11} \cdot 5^2$	$=$	$2^{22} \cdot 3^2 + 3 \cdot 5^7$
48.	5.09488	$2^2 \cdot 5 + 2^{22} \cdot 3 + 3^{11} \cdot 5^3$	$=$	$2^{25} + 3 \cdot 5^8$
49.	5.09488	$2^2 \cdot 3 + 2^{22} \cdot 5 + 3^{12} \cdot 5^2$	$=$	$2^{25} + 3^2 \cdot 5^7$
50.	5.05498	$2^7 \cdot 3^2 \cdot 5 + 2^{18} \cdot 5^2 + 2^{23} + 3^{15}$	$=$	$3 \cdot 5^{10}$
51.	5.04891	$2^7 \cdot 3^2 + 2 \cdot 3^{15}$	$=$	$2^{20} \cdot 5^2 + 3^{12} + 5^9$
52.	5.0223	$2^7 \cdot 3^2 + 3^{12} \cdot 5 + 2^4 \cdot 3^{13}$	$=$	$2^{20} \cdot 5^2 + 5^9$
53.	5.0223	$2^{20} \cdot 5^2 + 5^9$	$=$	$2^7 \cdot 3^2 + 2^3 \cdot 3^{12} + 3^{14} \cdot 5$
54.	5.01029	$2^2 \cdot 5 + 2^{22} + 3^{11} \cdot 5^3$	$=$	$2^{23} \cdot 3 + 3 \cdot 5^8$
55.	5.01029	$2^{23} \cdot 3 + 5^7$	$=$	$2^4 \cdot 5^3 + 2^{14} \cdot 3^4 + 3^{14} \cdot 5$

These examples were generated with the Mathematica notebook given in Appendix B.2 with as primebases $\{2,2,2,3,5\}$, $\{2,3,3,3,5\}$, $\{2,3,3,5,7\}$, $\{2,3,5,5,7\}$, $\{2,3,5,5,11\}$, $\{2,5,5,5,7\}$, $\{2,5,5,5,11\}$, $\{2,5,5,7,11\}$, $\{2,5,7,7,11\}$, $\{2,7,7,7,11\}$, with powers up to $\frac{30\log(2)}{\log(p)}$ and linear combinations of the reduced lattice base vectors with coefficients up to 1.

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