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A Practical Analysis of Kernelization Techniques for the Maximum Cut Problem

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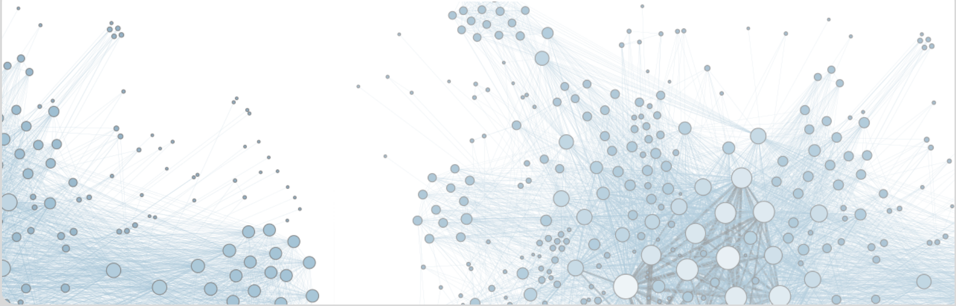
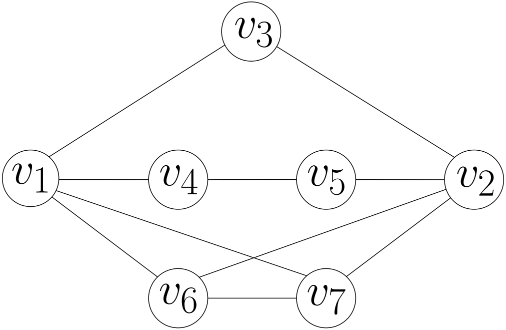


Max-Cut: Definition and Example

- Given $G = (V, E)$, find $S \subseteq V$ such that $|E(S, V \setminus S)|$ is maximum
- Notation: $mc(G) := \max_{S \subseteq V} |E(S, V \setminus S)|$



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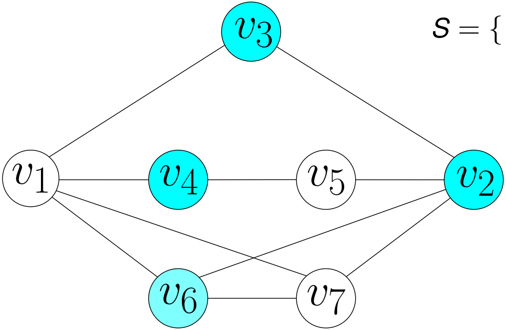

$S = \{v_2, v_3, v_4, v_6\}$

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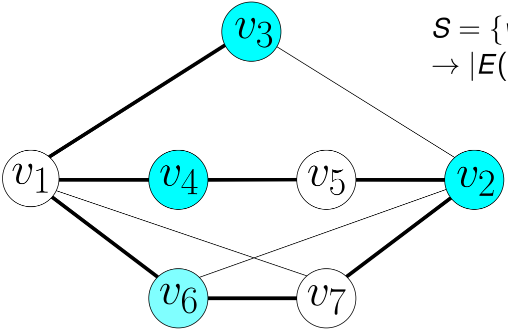

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$\rightarrow |E(S, V \setminus S)| = 7$

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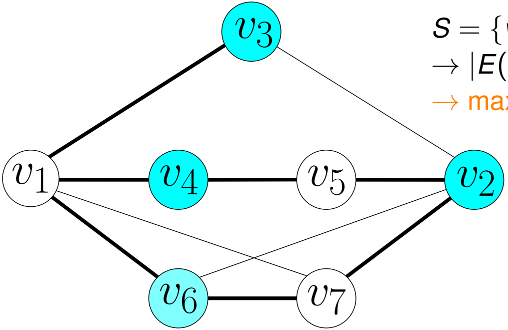

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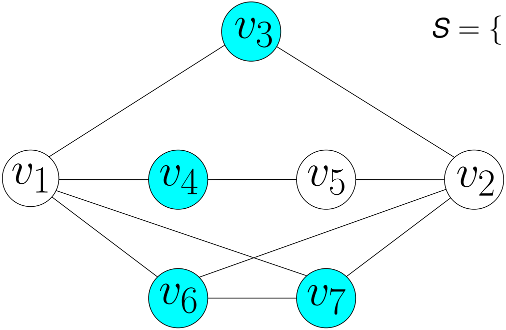

$S = \{v_3, v_4, v_6, v_7\}$

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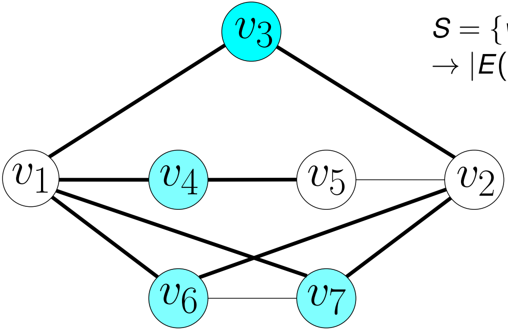

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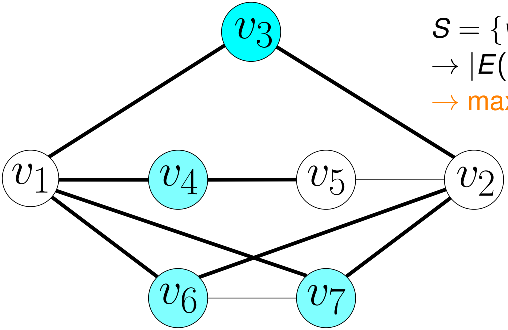
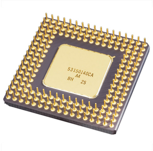


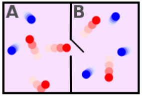
$S = \{v_3, v_4, v_6, v_7\}$
$\rightarrow |E(S, V \setminus S)| = 8$
$\rightarrow$ maximum?

Max-Cut: Importance of Studying it

- Member of Karp's 21 **NP-complete** problems
- Used in...



Circuit design



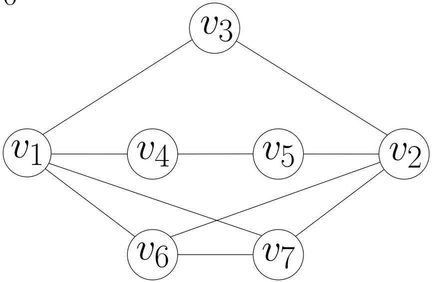
Statistical physics

Social networks

Kernelization: Definition and Example

- Kernelization: Compress graph while preserving optimality

$G_0 = G$:



Kernelization: Definition and Example

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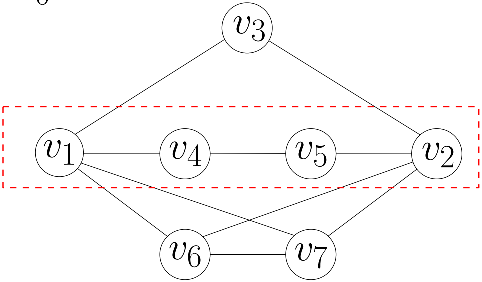

Kernelization: Definition and Example

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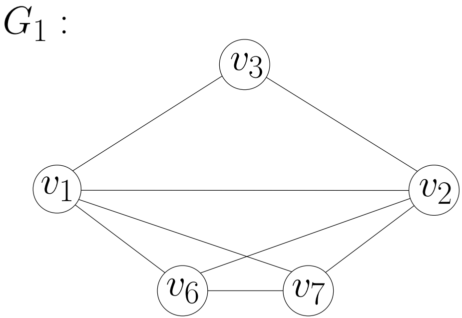
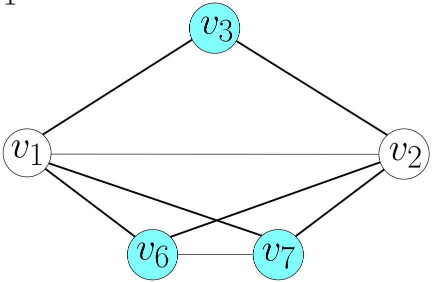


$$mc(G_0) = mc(G_1) + 2$$

Kernelization: Definition and Example

- Kernelization: Compress graph while preserving optimality

G_1 :



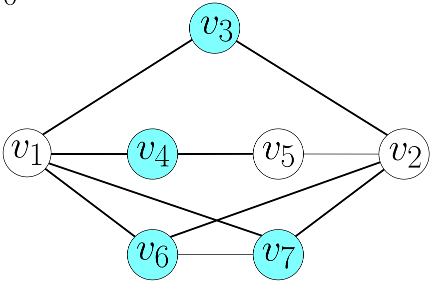
$mc(G_0) = mc(G_1) + 2$

$mc(G_1) = 6$

Kernelization: Definition and Example

- Kernelization: Compress graph while preserving optimality

$G_0 = G:$

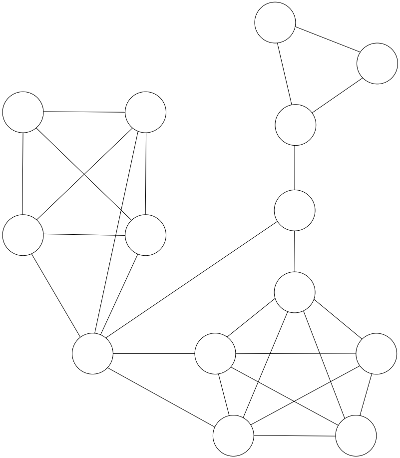


$mc(G_0) = 6 + 2 = 8$

Max-Cut: Current Research on Kernelization

- Previous work mostly of theoretical nature
  - Analyze problem $k + \textit{mc-lowerbound}(G)$
  - Different reformulations (**Etscheid and Mnich 2018**, Madathil, Saurabh, and Zehavi 2018, Prieto 2005)
- Research on practicality missing
  - Present for other problems
  - INDEPENDENT SET, VERTEX COVER (Hespe, Schulz, and Strash 2018, Akiba and Iwata 2016)

Max-Cut: Reduction Rule 8 in Etscheid and Mnich 2018



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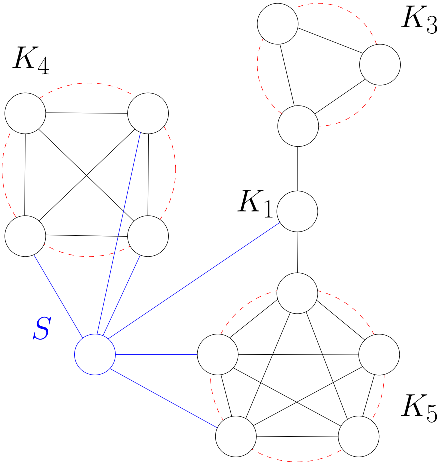

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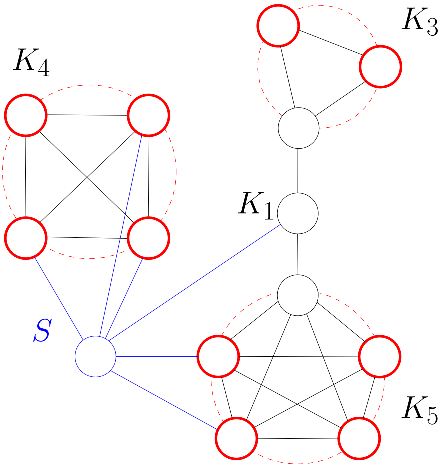
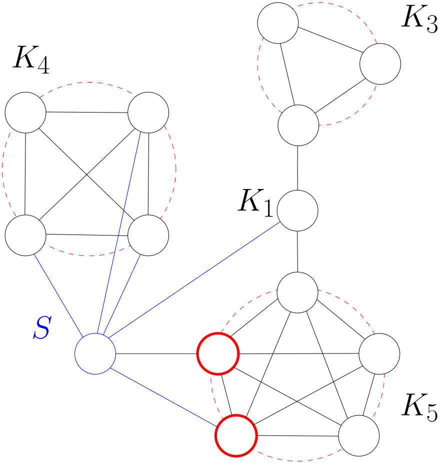


1. $N_G(x) \cap S = N_G(X) \cap S$
2. $|X| > \frac{|K|+|N_G(X)\cap S|}{2} \geq 1$

Max-Cut: Reduction Rule 8 in Etscheid and Mnich 2018



1. $N_G(x) \cap S = N_G(X) \cap S$ ✓
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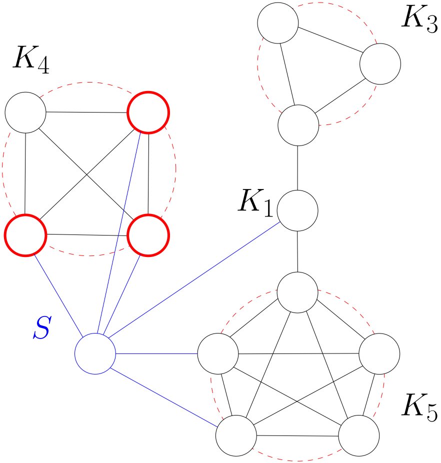

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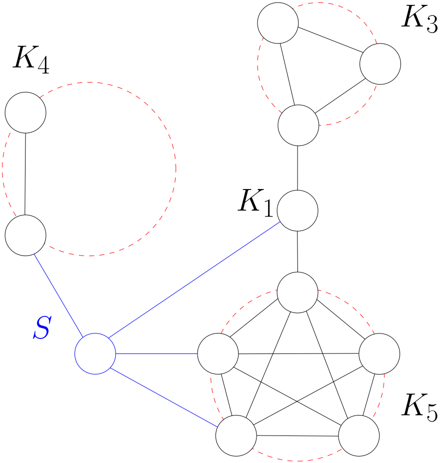

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Performance of Work from Etscheid and Mnich 2018

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- Kernelization mostly **driven by rule 8**
- Weak-points in practice
  - Reliance on clique-forest
  - Parameter k large in practice
    - **Kernel size $O(k)$ too large**
    - $O(k \cdot |E(G)|)$ time too slow

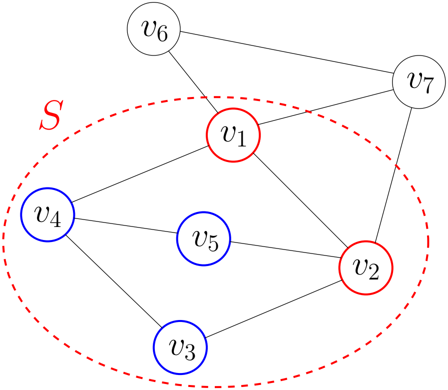
Overview of Our Contributions

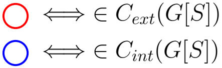
- Implemented and evaluated work of Etscheid and Mnich 2018
- **Generalized existing reduction rules**
  - Rules not dependent on a subgraph anymore
- **Developed new reduction rules**
  - Simplistic but significant improvement in practice
  - Identified inclusions
- **Efficient implementation**
  - Timestamping system
- Benchmark over a variety of instances

Overview of Our Contributions

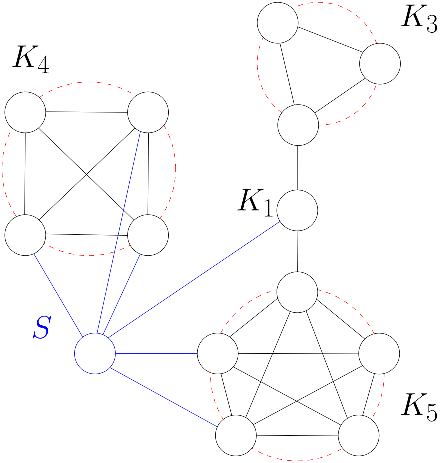
- Five new unweighted reduction rules
  - Rule to compress induced 3-paths
  - Two rules reducing cliques **(R8, S2)**
  - Two antagonizing rules – merge and divide of cliques
- Briefly investigated: Weighted path compression

External/Internal Vertices





Rule Generalization: R8 – “Sharing Adjacencies”



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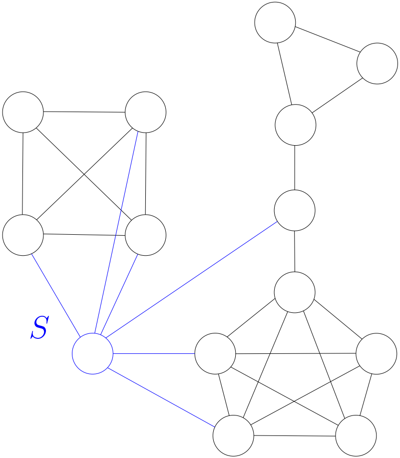

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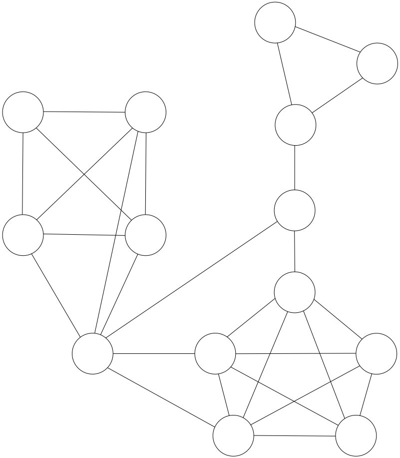

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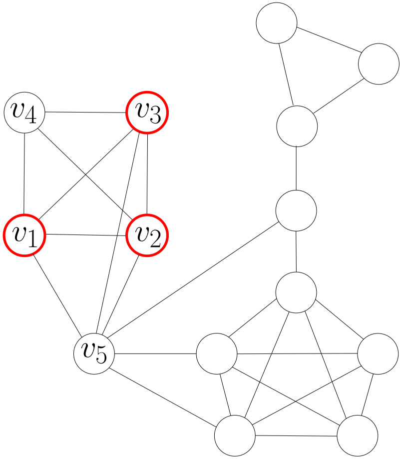

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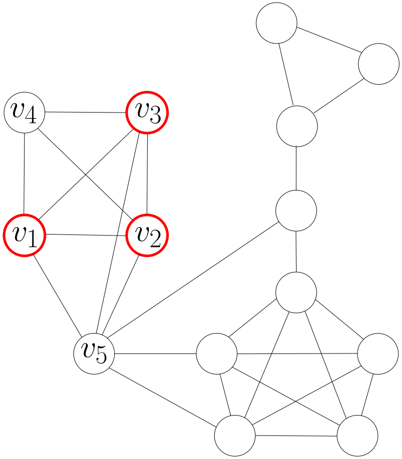

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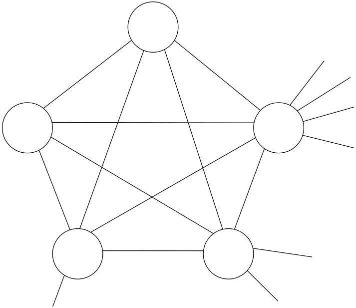
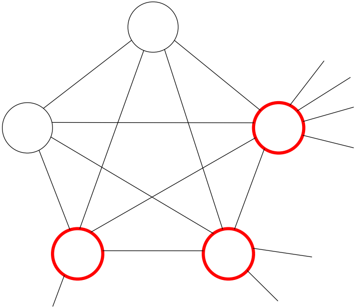
New Reduction Rule: S2 – "Semi-Isolated Cliques"



- Clique $G[S]$ with $|C_{ext}(G[S])| \leq \left\lceil \frac{|S|}{2} \right\rceil$

New Reduction Rule: S2 – "Semi-Isolated Cliques"



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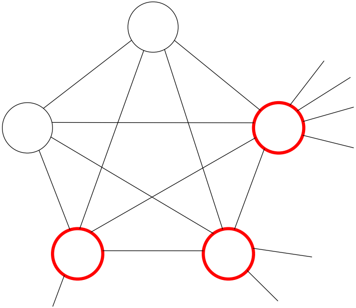

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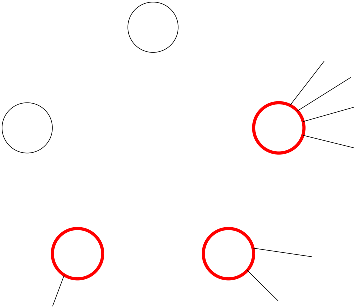

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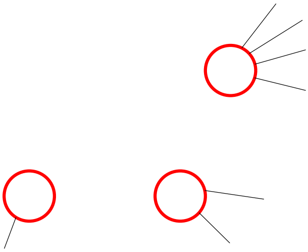

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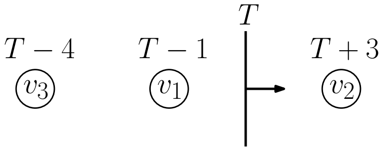
Techniques Used for Performance

- **Avoid time-intensive checks**
  - Vertex v internal in clique: $\forall w \in N_G(v) : Deg(v) \leq Deg(w)$
- **Avoid checking the same reduction rules**
  - Timestamp of most recent change in neighborhood for each vertex
  - Keep timestamp T for each rule:
    All vertices with timestamp $\leq T$ already processed
  - Update vertex on change



Experiments on KaGen Graphs

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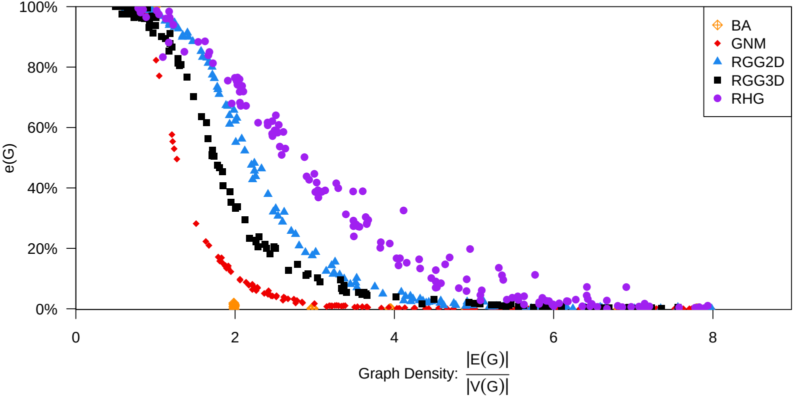
- Random graphs by KaGen, 150 per each graph type. $|V| = 2048$
- **Total runtime: 16 sec.** (68 min. by Etscheid and Mnich 2018!)

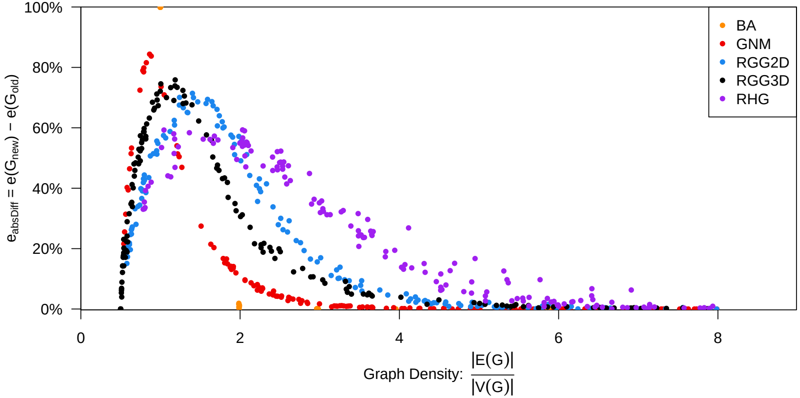
Kernelization efficiency for KaGen graphs; metric: $e(G) = 1 - \frac{|V(G_{\text{ker}})|}{|V(G)|}$



Experiments on KaGen Graphs

- Improvement over Etscheid and Mnich 2018. $|V| = 2048$
- Discrepancy between theory and practice

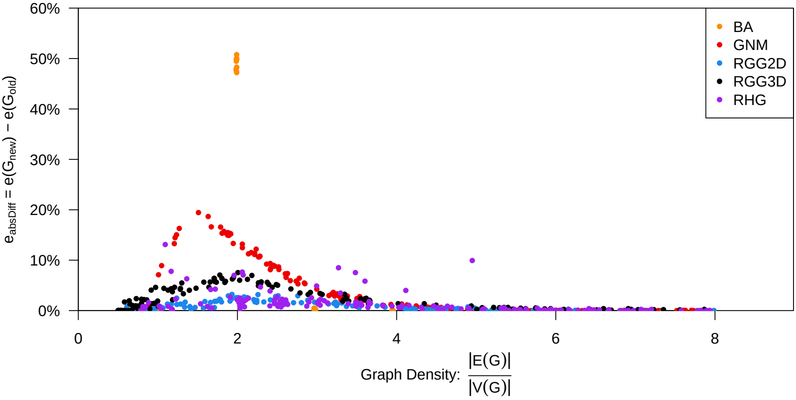
Absolute difference in efficiency: $e_{\text{absDiff}} = e(G_{\text{new}}) - e(G_{\text{old}})$



Experiments on KaGen Graphs

- Improvement on our results with weighted path compression

Absolute difference in efficiency: $e_{\text{absDiff}} = e(G_{\text{newWeighted}}) - e(G_{\text{new}})$



Experiments – BiqMac Solver

Name	$\lvert V(G)\rvert$	deg_{avg}	$e(G)$	$T_{\text{BM}}(G)$	$T_{\text{BM}}(G_{\text{ker}})$
ego-facebook	2888	1.03	1.00	-	0.01 $[\infty]$
road-euroroad	1174	1.21	0.79	-	- -
rt-twitter-copen	761	1.35	0.85	-	1.77 $[\infty]$
bio-diseasome	516	2.30	0.93	-	0.07 $[\infty]$
ca-netscience	379	2.41	0.77	-	0.67 $[\infty]$
g000302	317	1.50	0.21	1.88	0.74 [2.53]
g001918	777	1.59	0.12	31.11	17.45 [1.78]
g000981	110	1.71	0.28	531.47	21.53 [24.68]
imgseg_105019	3548	1.22	0.93	f	13748.62 $[\infty]$
imgseg_35058	1274	1.42	0.37	-	- -
imgseg_374020	5735	1.52	0.82	f	- -

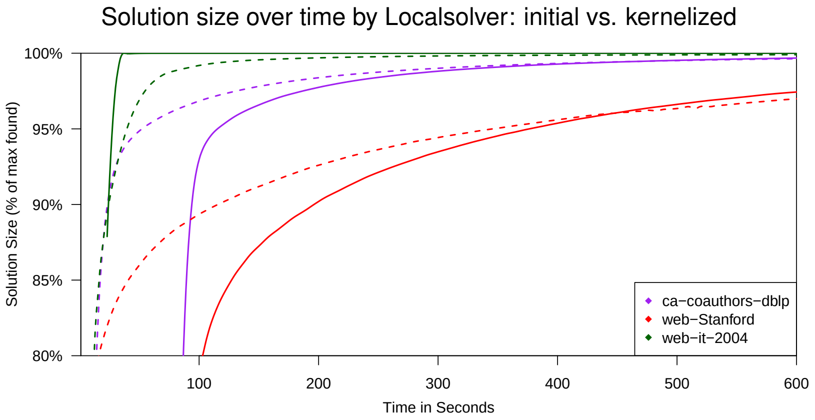
Times in seconds. 10 hour time limit with 5 iterations.

Experiments – Localsolver

Name	$\lvert V(G) \rvert$	deg_{avg}	$e(G)$	$T_{LS}(G)$	$T_{LS}(G_{ker})$
ego-facebook	2888	1.03	1.00	20.09	0.09 [228.91]
road-euroroad	1174	1.21	0.79	-	- -
rt-twitter-copen	761	1.35	0.85	-	834.71 [∞]
bio-diseasome	516	2.30	0.93	-	4.91 [∞]
ca-netscience	379	2.41	0.77	-	956.03 [∞]
g000302	317	1.50	0.21	0.58	0.49 [1.17]
g001918	777	1.59	0.12	1.47	1.41 [1.04]
g000981	110	1.71	0.28	10.73	4.73 [2.27]
imgseg_105019	3548	1.22	0.93	234.01	22.68 [10.32]
imgseg_35058	1274	1.42	0.37	34.93	24.71 [1.41]
imgseg_374020	5735	1.52	0.82	1739.11	72.23 [24.08]

Times in seconds. 10 hour time limit with 5 iterations.

Experiments – Localsolver



Conclusion

Summary

- Previous work: Good in theory, bad in practice
- **Set of new (unweighted) reduction rules**
- **Sparse graphs highly reducible**
- **Significant benefits for existing solvers**

Future Work

- Add parallelism?
- New (weighted) reduction rules?
- Hybrid approach: Use solver for reductions?