Exact Algorithms for Finding Well-Connected 2-Clubs in Sparse Real-World Graphs: Theory and Experiments

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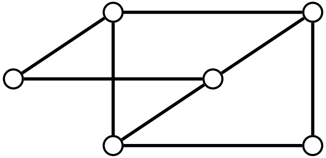
Based on joint work with Christian Komusiewicz, Rolf Niedermeier, and Marten Picker appearing in European Journal of Operational Research, 2019.

2-Club Problem

Input: An undirected graph $G = (V, E)$.
Task: Find the maximum size 2-club ($=$ diameter-two subgraph) in G.



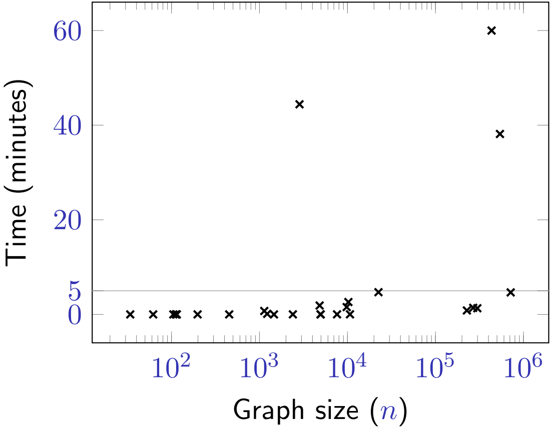
- proposed as clique relaxation in social network analysis [Mokken; Quality and Quantity, 1979]
- NP-hard [Balasundaram, Butenko, Trukhanov; Journal of Combinatorial Optimization, 2005]
- NP-hard to approximate within a factor $|V|^{1/2-\epsilon}$ [Asahiro, Doi, Miyano, Samizo, Shimizu; Algorithmica, 2018]

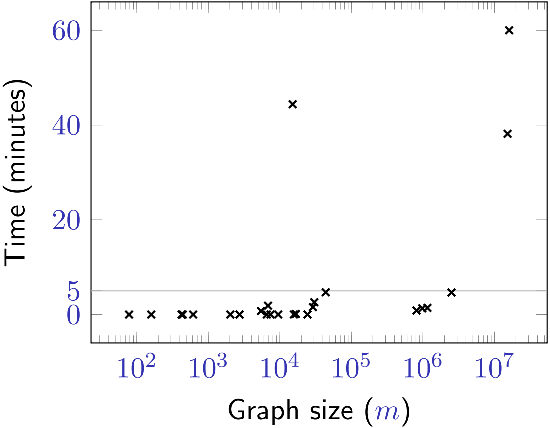
2-Club is hard? Not in practice!

Existing implementation: [Hartung, Komusiewicz, N.; Journal of Graph Algorithms and Applications, 2015]

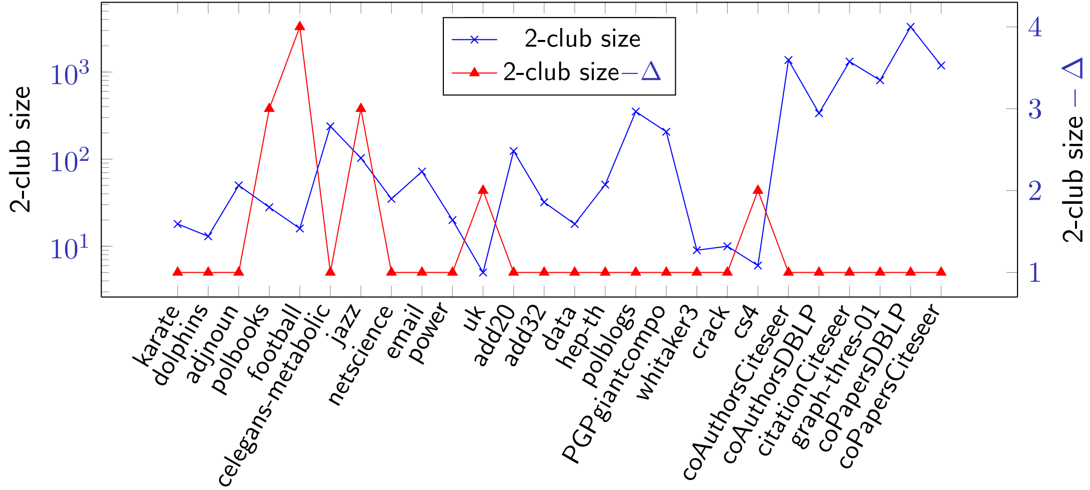
Data set: Graphs from **clustering** and **coauthor** category of the 10th DIMACS challenge

Implementation: Written in Java Machine: CPU 3.60 GHz (Xeon); 64 GB main memory





Analysis: 2-Club size



⇝ **Almost optimal algorithm:**

Return a maximum degree vertex with its neighbors

2-Club extensions

Definition ([Veremyev, Boginski; Eur J Oper Res, 2012])

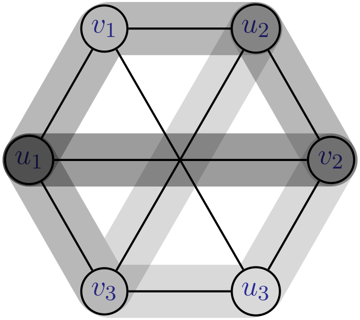
t-robust 2-club G:
Any pair of vertices is connected by t internally vertex-disjoint paths of length at most two.

Definition ([Pattillo, Youssef, Butenko; Eur J Oper Res, 2013])

t-hereditary 2-club G:
$G - U$ is a 2-club for all $U \subset V(G)$ where $|U| \leq t$.
$\iff$ any pair of nonadjacent vertices has $t+1$ common neighbors.

Definition ([Pattillo, Youssef, Butenko; Eur J Oper Res, 2013])

t-connected 2-club G:
G is a 2-club and t-vertex-connected.



Example: A $K_{3,3}$ is a

- 1-robust 2-club
- 2-hereditary 2-club
- 3-connected 2-club

Our results

Goal:

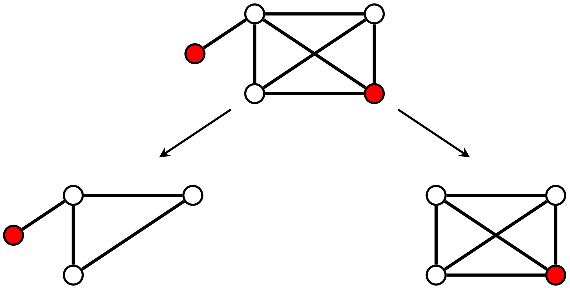
transfer algorithmic work (theoretical & practical) from **2-Club** to
t-robust / t-hereditary / t-connected **2-Club**

Results:

- "unifying view" on all three considered models
- FPT algorithms
- competitive implementation

Simple search tree

Example: Find largest 2-**hereditary 2-club** (deleting any 2 vertices yields a **2-club**)



Observation: At most one red vertex in a solution.

Generic Search tree:

FindSolution(G)

1. **If** G is a solution **then return** G
2. $u, v \leftarrow$ two "**incompatible**" vertices
3. **Return** $\max\{$**FindSolution**$(G - v)$,**FindSolution**$(G - u)\}$

$\rightsquigarrow$ running time $O(2^{\ell} nm)$ $\qquad$ $\ell \ldots$ number of vertices not in a solution

Note: no $2^{(1-\varepsilon)\ell} n^{O(1)}$ algorithm for any $\varepsilon > 0$, unless SETH fails

[Hartung, Komusiewicz, N.; Journal of Graph Algorithms and Applications, 2015]

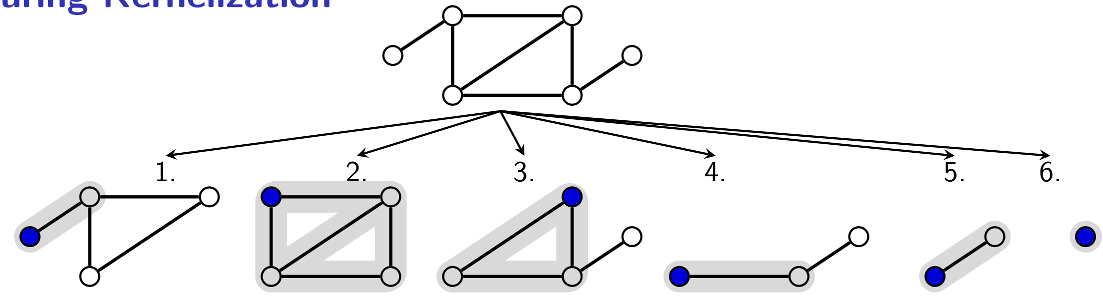
Compatible vertices — unifying view

Definition

Two vertices v and w in a graph are called **compatible**

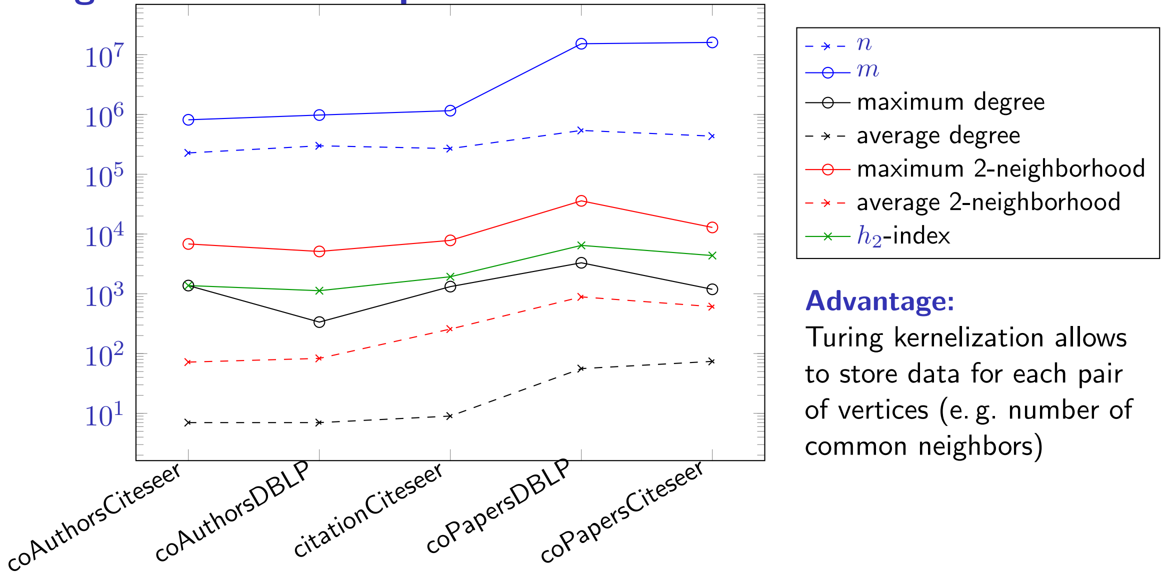
- for t-**robust 2-clubs** if they are adjacent and have at least $t-1$ common neighbors, or if they have at least t common neighbors,
- for t-**hereditary 2-clubs** if they are adjacent or if they have at least $t+1$ common neighbors,
- for t-**connected 2-clubs** if they are at distance at most two and are connected by at least t internally vertex-disjoint paths.

Turing Kernelization



1. sol $\leftarrow \emptyset$
2. **foreach** $v \in V$ **do**
3. $\quad$ $T \leftarrow$ all vertices at distance ≥ 2 from v
4. $\quad$ $S \leftarrow$ largest solution in T that contains v
5. $\quad$ **if** S is larger than sol **then** sol $\leftarrow S$
6. $\quad$ delete v
7. **return** sol

Turing kernelization — practical effect



Advantage:

Turing kernelization allows to store data for each pair of vertices (e. g. number of common neighbors)

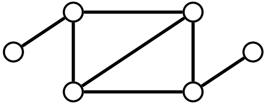
Data reduction & lower bounds

Reduction Rule

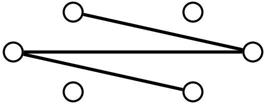
Remove vertices whose degree is too low.

Incompatibility graph:

Two vertices are adjacent in the the **incompatibility graph** iff they are not **compatible**.



input graph



incompatibility graph

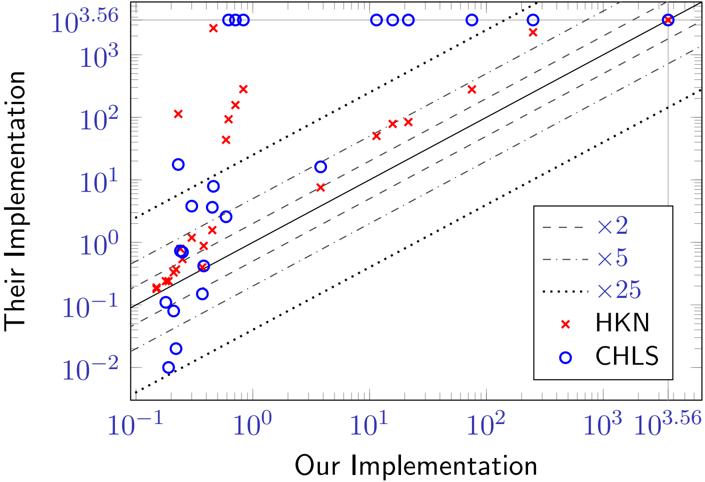
Observation:

The size of a maximum **independent set** in the **incompatibility graph** is an upper bound on the **solution size** in the input graph.

$\rightsquigarrow$ upper bound worse than best previously found solution $\Rightarrow$ discard current Turing kernel

Experiments I

Data set: Graphs from **clustering** and **coauthor** category of the 10th DIMACS challenge
Implementation: Written in Java   Machine: CPU 3.60 GHz (Xeon); 64 GB main memory



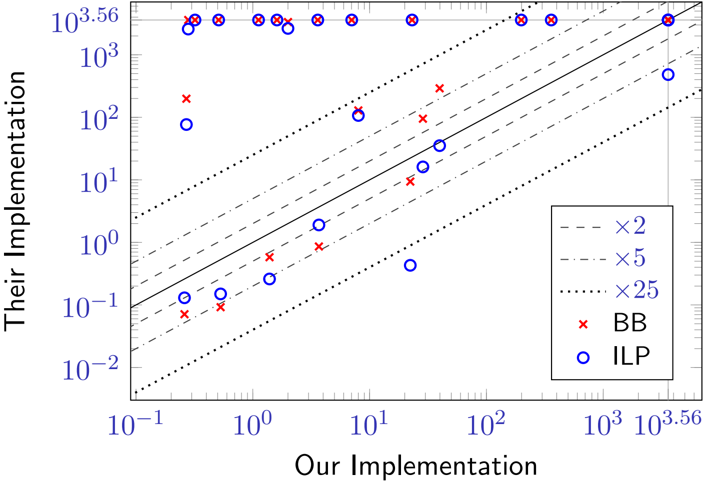
2-Club implementations:

- **HKN** (Java)
  [Hartung, Komusiewicz, N.; Journal of Graph Algorithms and Applications, 2015]
- **CHLS** (c++)
  [Chang, Hung, Lin, Su; Computing, 2013.]

Experiments II

Data set: Graphs from **clustering** and **coauthor** category of the 10th DIMACS challenge

Implementation: Written in Java   Machine: CPU 3.60 GHz (Xeon); 64 GB main memory



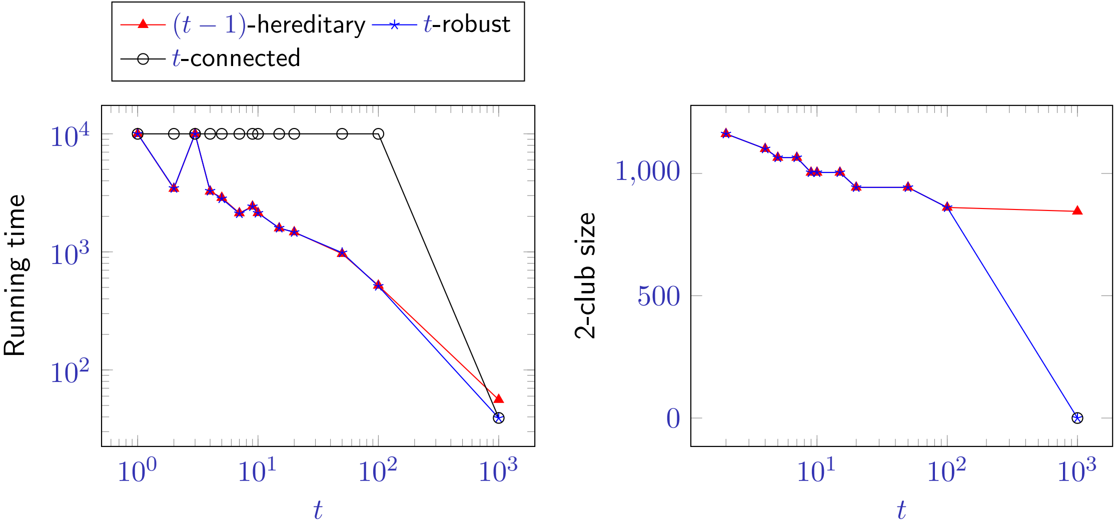
Implementations:

2-connected 2-Club

- **BB** - Branch & Bound (c++)
- **ILP** (c++ & Gurobi)

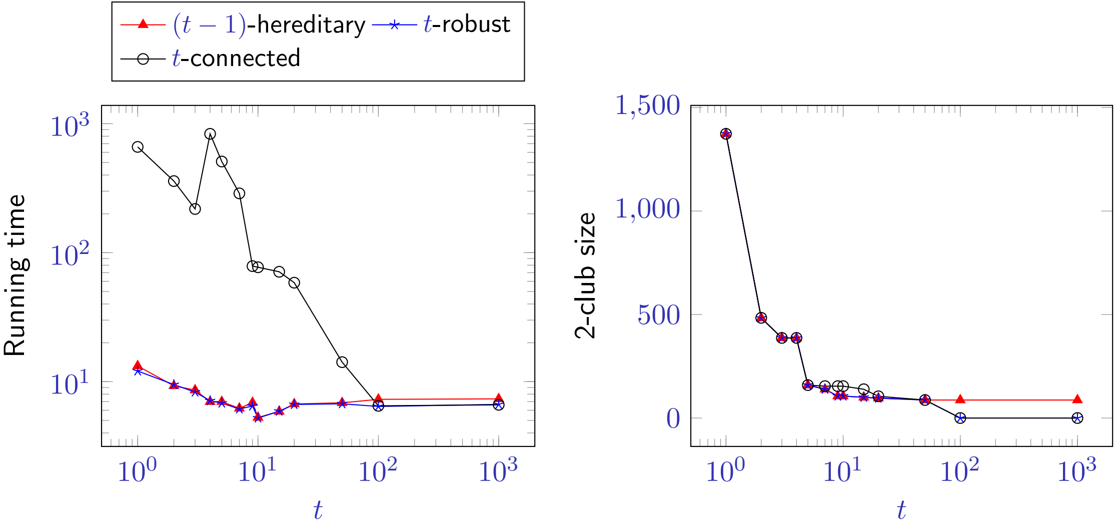
[Yezerska, Pajouh, Butenko; European Journal of Operational Research, 2017.]

Experiments III



Graph: coPapersCiteseer

Experiments IV



Graph: coAuthorsCiteseer

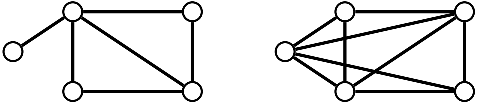
Summary & Outlook

Key results:

- Unifying approach for several **2-club** variants.
- Efficient implementation (= data reduction + Turing kernelization + search tree).

Work in progress: γ-**relative robust 2-club** S:
$0 < \gamma \leq 1$: Any pair of vertices connected by at least $\gamma \cdot |S|$ paths of length at most two.

Example: $\gamma = 0.5$



input graph incompatibility graph

input graph incompatibility graph

Open Question: Is t-**robust** / t-**hereditary** / t-**connected 2-club** fixed parameter tractable with respect to the solution size?

Thank you!