Computing sparsity stuff in real world graphs

Marcin Pilipczuk

a lot of slides by Wojciech Nadara and Michał Pilipczuk

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Shonan Village Center

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Mandatory slide

Current research (and my stay here) is a part of projects that have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 714704.

European Research Council

Established by the European Commission

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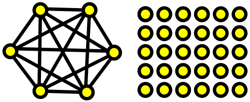

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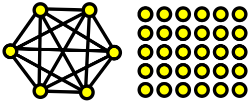

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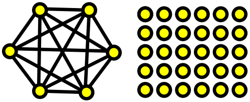

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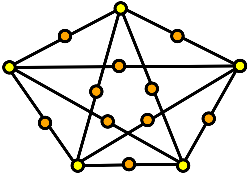

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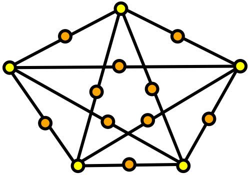

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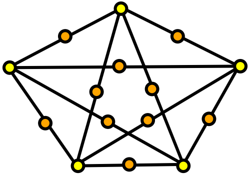

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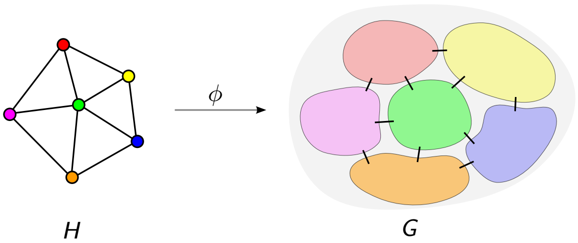

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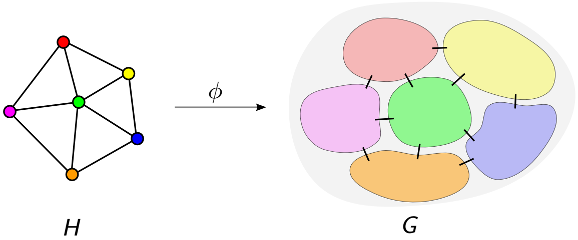

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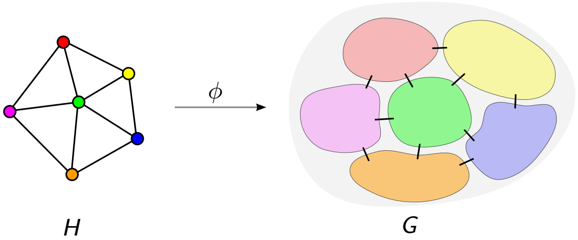

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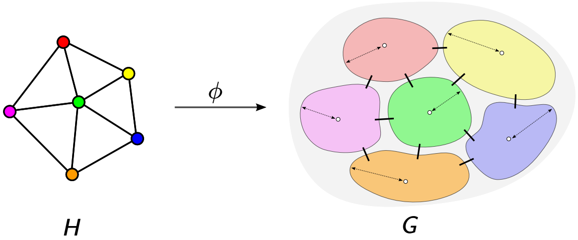

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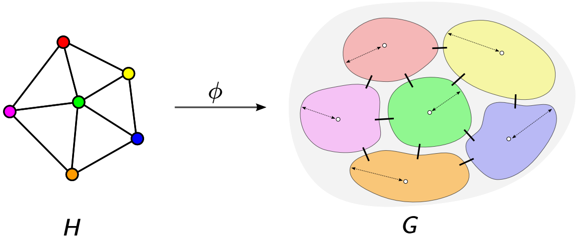
- **Idea**: Replace *subgraphs* with *shallow minors* in the def. of sparsity.



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  - If $H \in \mathcal{C}\nabla d$, then H has $\mathcal{O}_{\varepsilon,d}(n^{1+\varepsilon})$ edges, for any $\varepsilon > 0$.

Hierarchy of sparsity

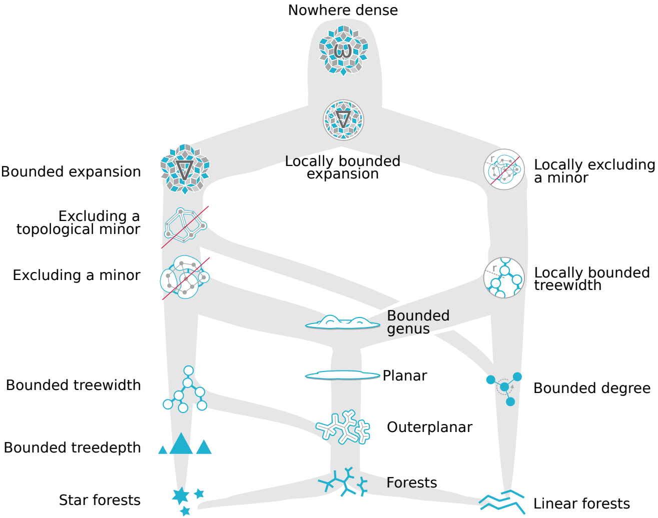

Figure by Felix Reidl

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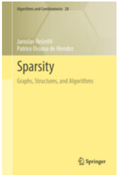

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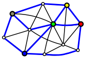
  - Toolbox seems much more suitable than using decomposition theorems for classes excluding a fixed (topological) minor.

Sparsity of shallow minors

Characterizations

Sparsity of shallow topological minors



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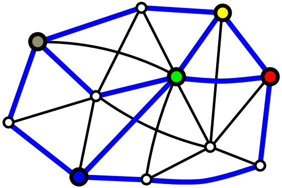
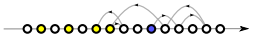


Characterizations

Generalized coloring numbers



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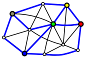

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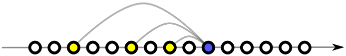
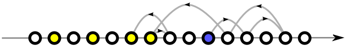


Degeneracy



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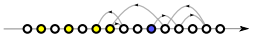

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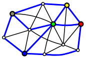

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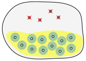

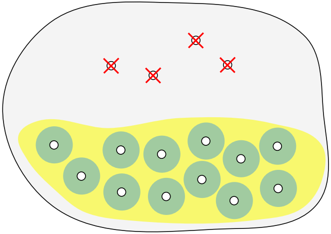

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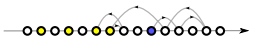

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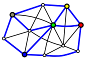

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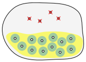
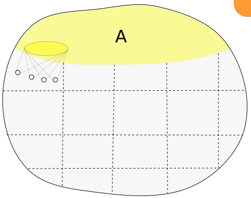
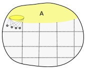




Neighborhood complexity



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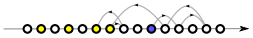
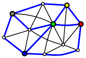


Sparsity of shallow topological minors



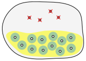
Fraternal augmentations

Sparsity of shallow minors

Neighborhood covers

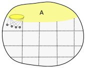
Low treedepth colorings

Uniform quasi-wideness



k-Helly property

Neighborhood complexity



Splitter game

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- FPT algorithms for bounded degree, planar, H-minor-free, ...

FO model-checking dichotomy

Theorem [Grohe et al., Dvořák et al.]

Let $\mathcal{C}$ be a monotone graph class (closed under taking subgraphs). Then:

- If $\mathcal{C}$ is nowhere dense, then FO model-checking can be done in time $f(\varphi) \cdot n^{1+\varepsilon}$ on graphs from $\mathcal{C}$, for any $\varepsilon > 0$.
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- Provides a natural barrier for locality-based methods.

Gaifman normal form

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Every FO sentence on graphs is equivalent to a boolean combination of **basic local sentences**, each having the following form:

There exist $u_1, u_2, \ldots, u_k$ that are pairwise at distance $> 2r$, and $\psi^r(u_i)$ holds for each $i = 1, \ldots, k$.

where r is some integer and $\psi^r(x)$ is an r-**local formula**, i.e. satisfaction of $\psi^r(u)$ depends only on the r-neighborhood of u.

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- Can be lifted to bounded expansion and nowhere dense classes.

Scattered sets and dominating sets

r-Scattered Set

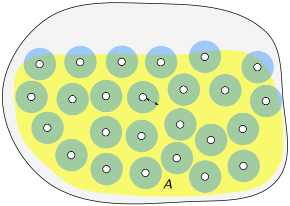
I Graph G, vertex subset $A \subseteq V(G)$, integer k

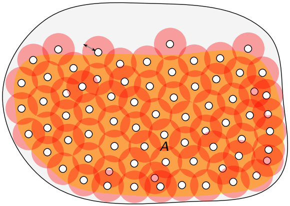
Q Is there $I \subseteq A$ with $|I| = k$ s.t. r-balls around vrts of I are disjoint?

r-Dominating Set

I Graph G, vertex subset $A \subseteq V(G)$, integer k

Q Is there $D \subseteq V(G)$ with $|D| = k$ s.t. every vertex of A is at distance $\leqslant r$ from some vertex of D?





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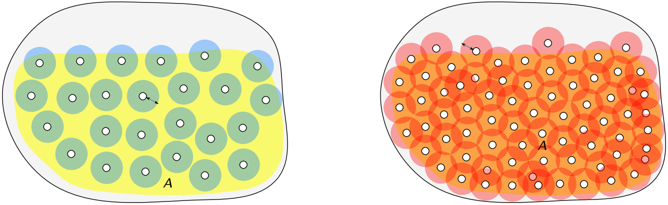
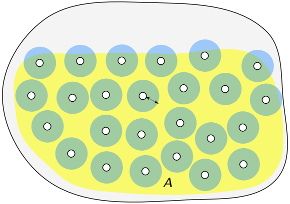
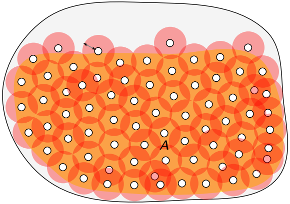


- **Note**: $\mathrm{sca}_r(G, A) \leqslant \mathrm{dom}_r(G, A)$





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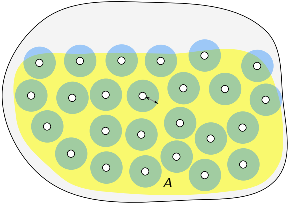

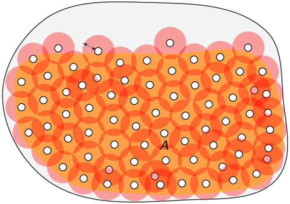

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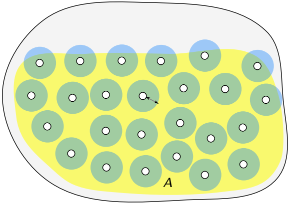

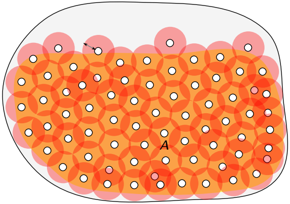

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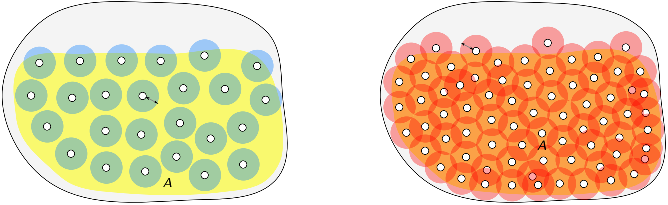

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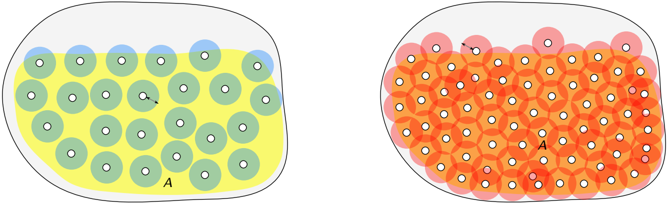

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- **Now**: runtime $f(k) \cdot |G|^2$ for r-ScaSet on any nowhere dense $\mathcal{C}$.

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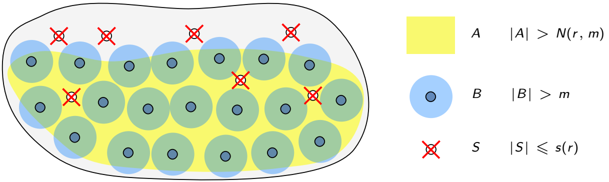

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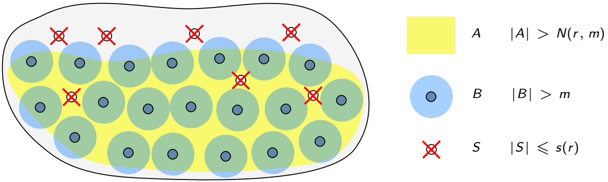

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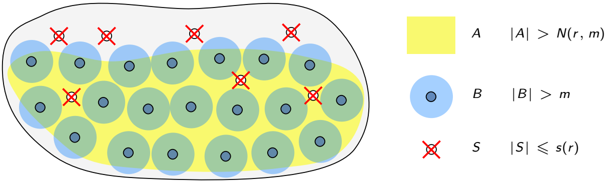

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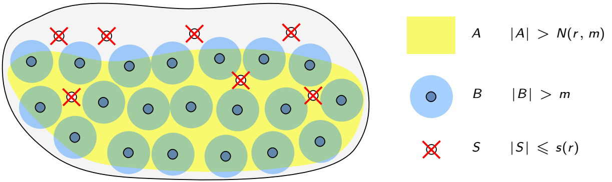

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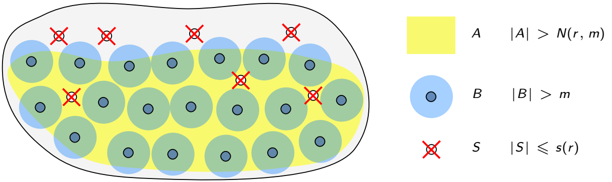

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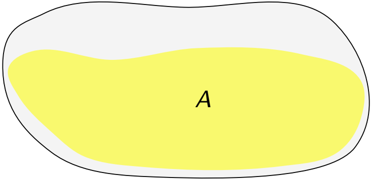

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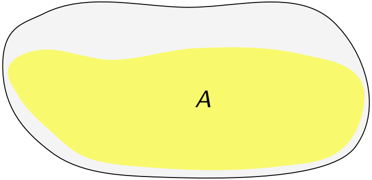

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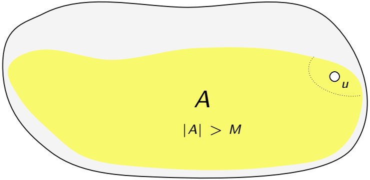

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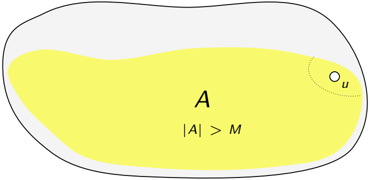

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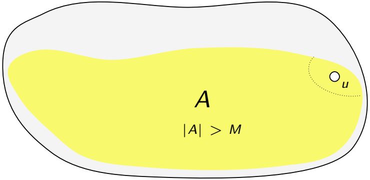

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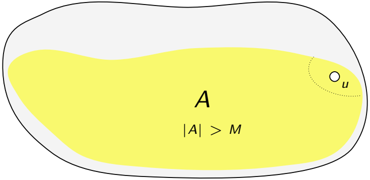

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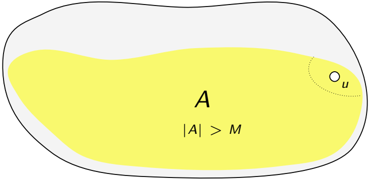

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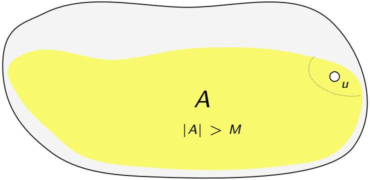

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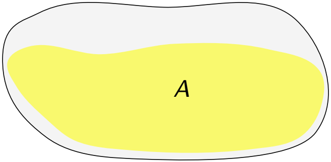

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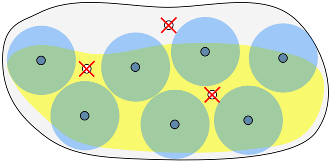

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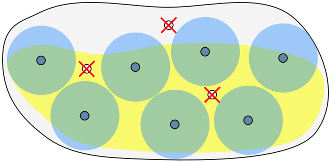

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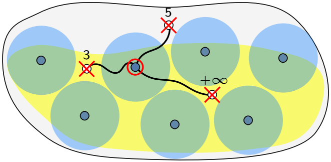

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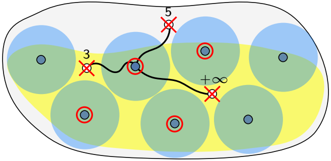
  $\Rightarrow$ Set $B' \subseteq B$ of more than k vertices with same profile.



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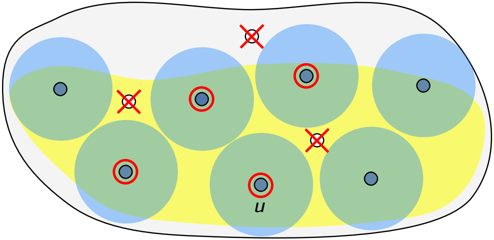

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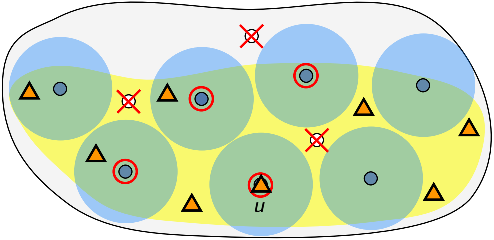

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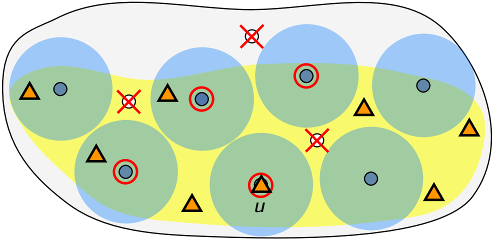

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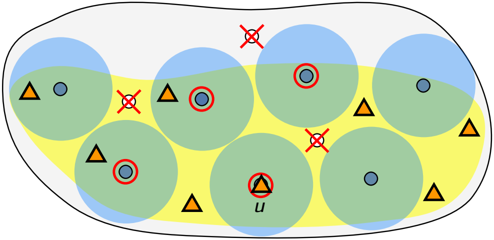

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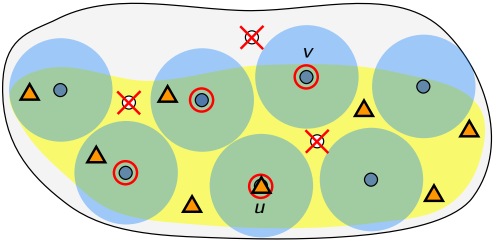

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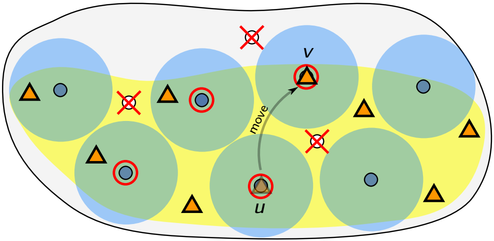

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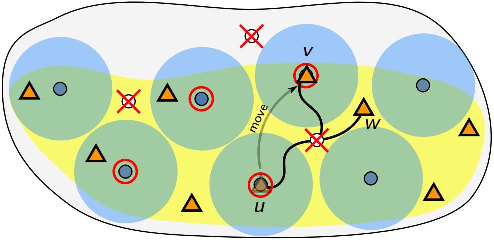
- **Pf**: If some $w \in I - u$ conflicted v, then it would already conflict u.



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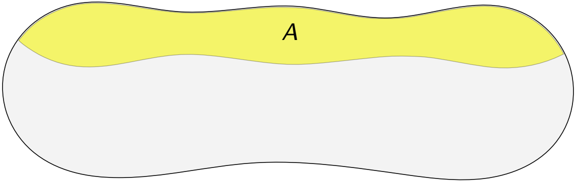

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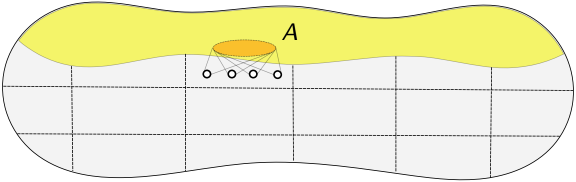

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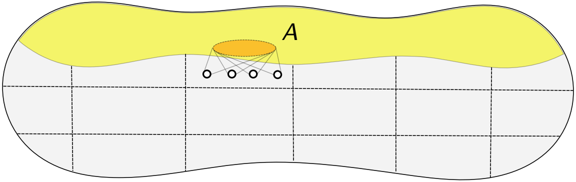

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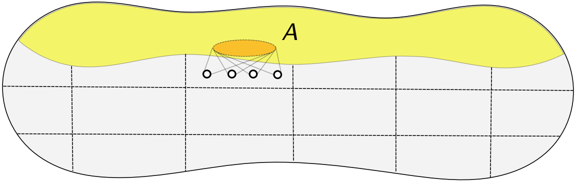

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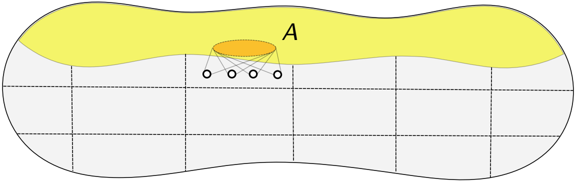

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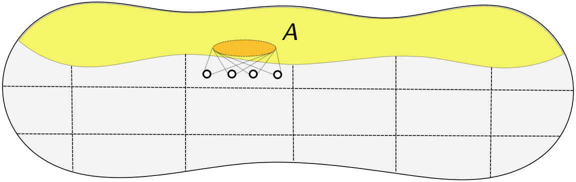
- Again, a very useful tool for algorithmic analysis of the instance.



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- Very useful for finding or counting constant-sized subgraphs.
  - Say we want to count the number of P_4 subgraphs in G.
  - Take 4-centered coloring c.
  - For every $X \subseteq [k]$ of size at most 4 count the number of P_4s that use exactly colors of X.
  - Complexity $\mathcal{O}(k^4 n)$ (without finding the coloring).

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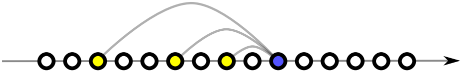

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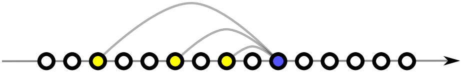

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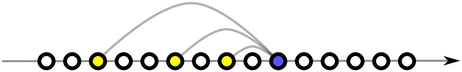

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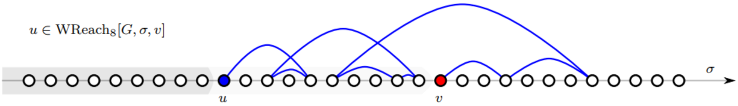

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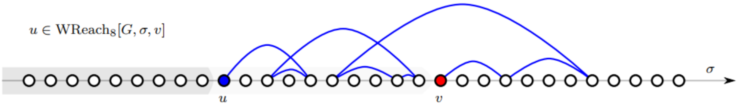

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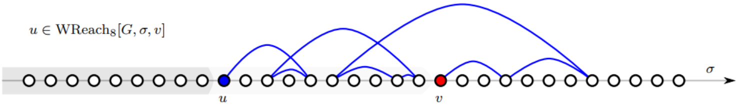

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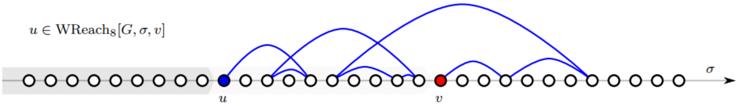

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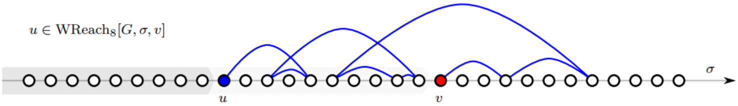

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- **Candidate**: **Stability**

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- If interested, see **lecture notes**: `https://www.mimuw.edu.pl/~mp248287/sparsity/`

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- Analysis of low treedepth coloring numbers for a number of real-world graphs.

DRRSSS results

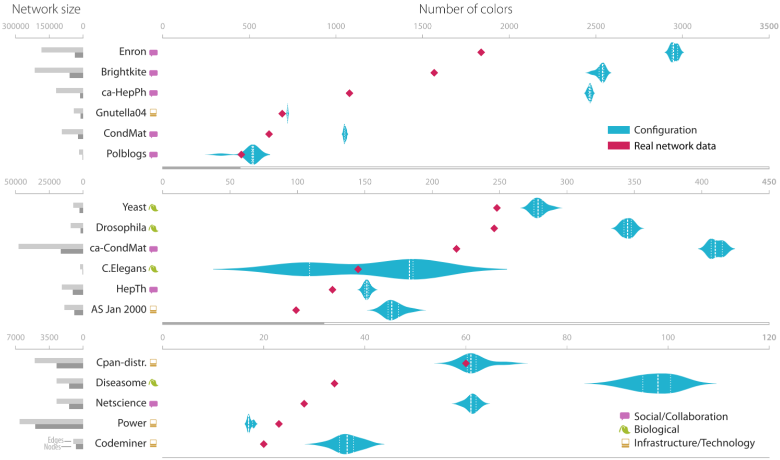

Figure 2: Comparison of 4-centered coloring numbers on real-world networks (red diamonds) compared to synthetic graphs (blue violins) with the same degree distribution. Each violin represents 10 random instances generated with the configuration model, with median and quartiles marked with dashed and dotted lines. Networks are partitioned into three groups by size (indicated on the left) to enable rescaling axes. See Table 2 for data sources.

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- The NCSU group is working on new implementation of pattern counting with a new algorithm based on weak coloring numbers.

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3. Nadara, P., Rabinovich, Reidl, Siebertz, SEA 2018.

Tested approaches

1. Distance-constrained transitive fraternal augmentations
2. Flat decompositions
3. Treedepth heuristic
4. Treewidth heuristic
5. Two greedy approaches
   1. Left-to-right, take vertex maximizing current size of the *WReach* set
   2. Right-to-left, take vertex minimizing current size of the *SReach* set
6. Other simple heuristics
   1. Sorting by descending degree
   2. Degeneracy ordering
   3. Doing these on graph G^r (where $V(G) = V(G^r), uv \in E(G^r) \Leftrightarrow dist_G(u, v) \leq r$)
7. Local search applied on top of any produced result

Test data

1. Selection of tests from KONECT base (modelling various real-world data: graphs of collaborations, airlines connections, political blogosphere, neural networks etc.)
2. Random planar graphs generated by LEDA library
3. Tests from Parameterized Algorithms and Computational Experiments Challenge 2016 competition — Feedback Vertex Set track
4. Graphs generated by random models producing graphs of bounded expansion: stochastic blocks model, Chung-Lu model and Chung-Lu model with households

We partitioned all tests into *small, medium, big* and *huge* based on number of edges (respectively $[0, 10^3), [10^3, 10^4), [10^4, 4,8 \cdot 10^4), [4,8 \cdot 10^4, \infty))$.

Comparison of all approaches

tests	r	dtf		flat		treedepth		treewidth		degree sort		WReach		SReach	
small	2	1.275	0:04.74	1.289	0:00.02	1.514	0:09.57	1.202	0:00.35	1.267	0:00.32	1.083	0:00.04	1.155	0:00.06
	3	1.513	0:04.18	1.307		1.516		1.186		1.276		1.100	0:00.05	1.107	0:00.02
	4	1.627	0:04.70	1.346		1.447		1.184		1.269		1.177	0:00.07	1.075	0:00.03
	5	1.749	0:05.61	1.382		1.440		1.187		1.290		1.226	0:00.06	1.084	0:00.07
medium	2	1.326	0:20.41	1.541	0:01.13	2.474	—	1.751	0:18.36	1.285	0:00.85	1.085	0:00.60	1.191	0:00.98
	3	1.440	0:44.33	1.655		2.240		1.513		1.271		1.116	0:00.71	1.104	0:00.96
	4	1.698	1:11.08	1.672		1.974		1.343		1.285		1.089	0:01.11	1.058	0:01.14
	5	1.777	1:37.55	1.660		1.816		1.232		1.294		1.163	0:01.52	1.040	0:01.34
big	2	1.304	—	1.706	0:17.54	—	—	2.773	—	1.400	0:02.28	1.075	0:03.73	1.202	0:11.32
	3	1.528	—	1.796		—		2.452		1.356		1.084	0:06.61	1.185	0:12.40
	4	—	—	1.827		—		1.862		1.382		1.097	0:14.57	1.117	0:16.02
	5	—	—	1.777		—		1.495		1.329		1.345	0:25.35	1.042	0:24.80
huge	2	—	—	2.124	4:14.11	—	—	—	—	1.432	0:16.91	1.086	1:07.98	—	—
	3	—	—	2.618		—		—		1.342		1.152	—	—	—
	4	—	—	2.506		—		—		1.293		—	—	—	—
	5	—	—	2.389		—		—		1.234		—	—	—	—

Table 3. *Gray columns*: Comparison of the main approaches and their average approximation ratio to the best found coloring number. Some of the approaches did not finish in time on larger graphs or ran out of memory. *White columns*: Total running time of the main approaches. Note that for some approaches the ordering (and thus running time) is independent of the radius.

Local search

tests	radius	dtf		flat		treedepth		treewidth		degree sort		WReach		SReach	
small	2	1.155	16.7%	1.060	16.9%	1.172	15.2%	1.087	7.0%	1.053	16.2%	1.069	6.7%	1.063	7.3%
	3	1.256		1.100		1.263		1.122		1.065		1.053		1.041	
	4	1.343		1.105		1.299		1.145		1.066		1.096		1.032	
	5	1.480		1.148		1.325		1.165		1.100		1.136		1.056	
medium	2	1.207	13.9%	1.151	21.4%	1.224	30.9%	1.149	15.3%	1.024	17.1%	1.070	2.8%	1.012	9.9%
	3	1.249		1.159		1.354		1.167		1.062		1.110		1.011	
	4	1.530		1.359		1.440		1.216		1.087		1.108		1.006	
	5	1.582		1.424		1.505		1.226		1.118		1.161		1.021	
big	2	1.172	—	1.196	24.4%	—	—	1.268	24.3%	1.091	18.5%	1.087	1.6%	1.023	11.5%
	3	1.321		1.239		—		1.415		1.097		1.105		1.019	
	4	—		1.390		—		1.434		1.145		1.123		1.020	
	5	—		1.438		—		1.387		1.164		1.177		1.010	

Table 4. *Gray columns*: Comparison of average approximation ratio after local search. *White columns*: Relative improvement of local search for ordering output by the studied approaches.

UQW: Tested approaches

1. Distance trees (variants denoted by `tree1, tree2, ld_it`)
2. From weak coloring numbers to uniform quasi-wideness (variants `mfcs, new1, new2, new_ld`)
3. Naive approach of removing vertices with biggest degrees and greedily computing independent set in r-th power of remaining graph (denoted as `ld`)

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2. We want best tradeoff between number of deleted vertices and size of r-independent set.
3. Biggest class of r-independent vertices when grouped by r-distance profiles!

Results

r	algorithm	start with whole $V(G)$				start with 20% of $V(G)$			
		deleted	independent	score	time	deleted	independent	score	time
3	mfcs	5076	11471	2153	0:01.25	1922	3459	1135	0:00.48
	new1	78	2345	2211	0:37.53	49	1192	1159	0:29.96
	new2	84	3820	3673	0:34.34	49	2132	2096	0:23.36
	new_ld	—	—	—	—	5	2926	2873	11:10.63
	tree1	7	6072	5686	0:02.77	4	2652	2598	0:00.48
	tree2	5	5645	5645	0:01.00	4	2603	2603	0:00.38
	ld_it	7	6136	5748	0:01.71	4	2741	2688	0:00.39
	ld	5	6471	6296	0:08.13	6	2972	2871	0:02.01
5	mfcs	7946	15773	1164	0:01.93	4057	4396	594	0:00.67
	new1	115	1623	1445	4:38.57	84	709	676	3:20.15
	new2	122	2079	1888	4:19.50	103	1036	982	3:07.82
	new_ld	—	—	—	—	—	—	—	—
	tree1	11	2988	2643	0:02.85	4	1325	1282	0:00.53
	tree2	5	2603	2603	0:01.05	4	1284	1284	0:00.45
	ld_it	12	3102	2752	0:01.84	5	1380	1336	0:00.64
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- Nadara: applied UQW to DOMINATING SET kernelization, compared to old experiments in planar graphs (Alber, Fellows, Niedermeier, J.ACM 2004 and Alber, Betzler, Niedermeier, Annals OR 2006).
  - The sparsity-based reduction does not add any power over the known simple neighborhood rules.
- **Thank you for your attention!**