Bisimulation for probabilistic transition systems: a coalgebraic approach

E.P. de Vink[1] and J.J.M.M. Rutten[2]

[1] Faculty of Mathematics and Computer Science, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands, e-mail: `vink@cs.vu.nl`
[2] Department of Software Technology, CWI, P.O.Box 94079, 1090 GB Amsterdam, The Netherlands, e-mail: `janr@cwi.nl`

Abstract. The notion of bisimulation as proposed by Larsen and Skou for discrete probabilistic transition systems is shown to coincide with a coalgebraic definition in the sense of Aczel and Mendler in terms of a set functor. This coalgebraic formulation makes it possible to generalize the concepts to a continuous setting involving Borel probability measures. Under reasonable conditions, generalized probabilistic bisimilarity can be characterized categorically. Application of the final coalgebra paradigm then yields an internally fully abstract semantical domain with respect to probabilistic bisimulation.

Keywords. Bisimulation, probabilistic transition system, coalgebra, ultrametric space, Borel measure, final coalgebra.

1 Introduction

For discrete probabilistic transition systems the notion of probabilistic bisimilarity of Larsen and Skou [LS91] is regarded as the basic process equivalence. The definition was given for reactive systems. However, Van Glabbeek, Smolka and Steffen showed in joint work with Tofts [GSS95], that for a concrete process language the usual notion of strong bisimilarity and the probabilistic concepts of reactive, generative and so-called stratified bisimulation constitute a hierarchy of observational congruences. Several other probabilistic equivalences are proposed as well in the literature. However, in all papers, *discrete* probability distributions are used, and hence the transition systems that are treated are in essence of a finitely branching or image-finite nature. The recent work of Blute et al. [BDEP97] is the single execption that we know of.

For the exploration of probabilistic transition systems and stochastic equivalences in the setting of modeling continuous systems, such as real-time or hybrid systems, one usually wants to allow more general probability measures than the more limited discrete probability distributions. [BDEP97] use stochastic kernels and spans of zigzags to underpin their notion of process equivalence. They prove that their notion of bisimulation agrees in the discrete case with the Larsen-Skou definition, but do not provide a characterization of bisimilarity in terms of transition steps, i.e., they do not give a continuous analogue for the Larsen-Skou bisimulation.

Here we attack the problem of continuous probabilistic transition systems and bisimulation by exploiting the transition-systems-as-coalgebras paradigm. Using a minimal amount of category theory, it can be summarized as follows: Let $\mathcal{F}: \mathcal{C} \to \mathcal{C}$ be any functor on a category $\mathcal{C}$. A *coalgebra* of $\mathcal{F}$ is an object S in $\mathcal{C}$ together with an arrow $\alpha: S \to \mathcal{F}(S)$. For many categories and functors, such a pair (S, α) represents a transition system, the type of which is determined by the functor $\mathcal{F}$. Vice versa, many types of transition systems can be captured by a functor this way. For instance, consider the familiar *labeled transition systems* $(S, A, \to)$, consisting of a set S of states, a set A of actions, and a transition relation $\to \subseteq S \times A \times S$. Put $\mathcal{L}(X) = \mathcal{P}(A \times X)$, the collection of all subsets of $A \times X$, for any set X, and, for $f: X \to Y$, $\mathcal{L}(f): \mathcal{L}(X) \to \mathcal{L}(Y)$, by $\mathcal{L}(f)(\{(a_i, x_i) \mid i \in I\}) = \{(a_i, f(x_i)) \mid i \in I\}$. It can be easily shown that $\mathcal{L}$ is a functor on the category of sets and functions. A labeled transition system $(S, A, \to)$ can now be represented as an $\mathcal{L}$-coalgebra by defining

$$\alpha: S \to \mathcal{L}(S), \quad s \mapsto \{(a, s') \mid (s, a, s') \in \to\}.$$

Conversely, any $\mathcal{L}$-coalgebra corresponds to a transition system: If (S, α) is a coalgebra for $\mathcal{L}$, then $(S, A, \to)$, with $\to \subseteq S \times A \times S$ given by $(s, a, s') \in \to$ *iff* $(a, s') \in \alpha(s)$, is clearly a transition system. (See [Rut96] for more details.)

One of the advantages of the coalgebraic view on transition systems is the existence of a general definition of $\mathcal{F}$-bisimulation, for any functor $\mathcal{F}$ (cf. [AM89]). For instance, applying that definition to the functor $\mathcal{L}$ above yields the standard notion of strong bisimulation. In general, the coalgebraic theory gives a generic approach to the definition and description of bisimulation: First define or characterize the transition systems one is interested in as coalgebras of a suitably chosen functor $\mathcal{F}$. Then obtain a definition of bisimulation for those systems by applying the categorical definition of $\mathcal{F}$-bisimulation.

The coalgebraic approach is applicable to many kinds of transition systems—see [Rut96] for many examples. In the present paper, this scheme is used to describe discrete and continuous probabilistic transition systems and bisimulations. The functor $\mathcal{M}_1$ assigns to a metric space its collection of Borel probability measures. It is shown that the corresponding notion of $\mathcal{M}_1$-bisimulation coincides, under mild conditions, with the continuous analogue of Larsen-Skou bisimulation. This extends a similar result for the discrete case, which is in fact given first: the functor $\mathcal{D}$, which assigns to a set the collection of its simple probability distributions, is shown to yield a categorical characterization of Larsen-Skou bisimulation. Hence, in agreement with general opinion, also from the coalgebraic point of view the latter equivalence is suggested as the canonical one.

Another appealing aspect of the coalgebraic approach is a canonical way of finding internally fully abstract domains of bisimulation, where two elements are equal if and only if they are bisimilar. It follows from a simple but very general argument that *final* coalgebras are fully abstract (see Aczel's final coalgebra model for nonwellfounded sets [Acz88], and also [RT93]). We shall show that it follows from general coalgebraic considerations [AR89,Bar93,RT93] that both our functors $\mathcal{D}$ and $\mathcal{M}_1$ have a final coalgebra, which consequently are internally fully abstract with respect to (discrete and continuous) probabilistic bisimulation. Therefore these final coalgebras can be exploited as semantic domains for probabilistic bisimulation (an important direction for future research).

As mentioned above, the functor $\mathcal{M}_1$ is defined on ultrametric spaces, and the Borel σ-algebras and associated measures are taken with respect to the metric topology. Our reasons for considering metric spaces rather than the, in semantical contexts, more standard use of ordered structures, as studied, e.g., by Jones and Plotkin [JP89] and by Edalat [Eda94] are twofold. Firstly, one can resort to the rich literature for standard measure theory on metric spaces. Secondly, we can apply the recently developed theory on coalgebraic bisimulation and final coalgebras in the metric setting [AM89,RT94]. Notably, we shall see that $\mathcal{M}_1$ is *locally contractive*, from which it follows that it has a final coalgebra. Because of the coalgebraic definition of bisimulation, we thus obtain an internally fully abstract domain. Such a full abstractness result has been lacking so far in the literature.

In conclusion, $\mathcal{D}$-bisimilarity and Larsen-Skou bisimilarity coincide for discrete probabilistic transition systems. For the continuous case, the functor $\mathcal{M}_1$ captures the generalization of probabilistic transition systems, and, under conditions, characterizes the associated notion of probabilistic bisimulation. For both functors a final coalgebra and hence, internally fully abstract domain exists, which can be exploited in the construction of domains for probabilistic bisimulation semantics.

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Note A technical report version of this paper is available by anonymous ftp from `ftp.cs.vu.nl` as `/pub/papers/theory/IR-423.ps.Z`.

2 Mathematical Preliminaries

Basic measure theoretic definitions (See, e.g., the standard textbook [Rud66].) A σ-algebra Σ on a set X is a collection of subsets which contains X and is closed under complement and countable union. Elements E of Σ are called measurable subsets of X. Trivially, the powerset $\mathcal{P}(X)$ is a σ-algebra for X. If X is a topological space, the Borel σ-algebra $\mathcal{B}(X)$ is defined as the least σ-algebra containing all open sets.

A function $\mu: \Sigma \to [0,1]$, where Σ is a σ-algebra on a set X, is called a Σ-probability measure if $\mu(X) = 1$ and μ is σ-additive, i.e., $\mu(\bigcup_{i \in I} E_k) = \sum_{i \in I} \mu(E_i)$ for any countable disjoint collection of measurable sets $\{E_i \mid i \in I\}$. For X a topological space, a Borel probability measure is a probability measure on X taken with respect to the Borel σ-algebra $\mathcal{B}(X)$. For $x \in X$, the Dirac-measure δ_x is given by $\delta_x(E) = 1$ if $x \in E$, and $\delta_x(E) = 0$ otherwise. A function $\mu: X \to [0,1]$ is called a simple probability distribution if there exist n distinct points $x_1, \ldots, x_n$, $n > 0$, such that $\mu(x_1) + \cdots + \mu(x_n) = 1$ and $\mu(x) = 0$ for $x \notin \{x_1, \ldots, x_n\}$. $\mathcal{D}(X)$ denotes the collection of all simple probability distributions on X. For $E \subseteq X$, $\mu[E]$ is short for $\sum_{x \in E} \mu(x)$. This way, a simple probability distribution corresponds to a convex linear combination of Dirac-measures.

Metric spaces (See, e.g., the monograph [BV96].) A pair (M, d) with M a nonempty set and $d\colon M^2 \to [0,1]$ is called an ultrametric space if, for all $x, y, z \in M$: $d(x,y) = d(y,x)$, $d(x,y) = 0 \Leftrightarrow x = y$, and $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. The last expression is referred to as the strong triangle inequality. For metric spaces M_1, M_2, a function $f\colon M_1 \to M_2$ is called nonexpansive if $d_2(f(x), f(y)) \leq d_1(x,y)$, for all $x, y \in M$. In case $d_2(f(x), f(y)) \leq \kappa \cdot d_1(x,y)$, for all $x, y \in M$, the function f is called κ-contractive, where κ is a constant with $0 \leq \kappa < 1$. The collection of all nonexpansive mappings from M_1 to M_2 is denoted by $M_1 \to_1 M_2$. We use the notation $\mathcal{O}$, or more explicit $\mathcal{O}(M)$, for the collection of all open subsets of M. For $\varepsilon > 0$ we put $\mathcal{O}_\varepsilon = \{\, O \in \mathcal{O} \mid \forall x \in O\colon B_\varepsilon(x) \subseteq O \,\}$.

Binary relations For a binary relation $R \subseteq S \times T$ we use π_1 and π_2 for the projections of R on S and T, respectively. R is called total if the two projections π_1 and π_2 are surjective. We say that R is z-closed if, for all $s, s' \in S$, $t, t' \in T$, $R(s,t) \wedge R(s',t) \wedge R(s',t') \Rightarrow R(s,t')$. If we put, for $n \in \mathbb{N}$, $R_0 = R$, $R_{n+1} = \{(s,t') \in S \times T \mid \exists s' \in S, t' \in T{:}R(s,t) \wedge R_n(s',t) \wedge R(s',t')\}$, and $R^* = \bigcup_{n \in \mathbb{N}} R_n$, we have that R^* is the least z-closed binary relation on $S \times T$ containing R. Below we will employ, for $s \in S$, the notation $F(s) = \{t \in T \mid R(s,t)\}$ and, for $U \subseteq S$, $F[U] = \bigcup_{s \in U} F(s)$, and, likewise, for $t \in T$, $E(t) = \{s \in S \mid R(s,t)\}$, and, for $V \subseteq T$, $E[V] = \bigcup_{t \in V} E(t)$.

Coalgebras (See, e.g., [Rut96].) Let $\mathcal{C}$ be either the category of sets and functions, or the category of ultrametric spaces and nonexpansive mappings. (These are the only categories playing a role in this paper.) Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a functor. An $\mathcal{F}$-coalgebra is a pair (S, α) consisting of an object S in $\mathcal{C}$ together with an arrow $\alpha\colon S \to \mathcal{F}(S)$ in $\mathcal{C}$ called a coalgebra structure on S. A homomorphism between two $\mathcal{F}$-coalgebras (S, α) and (T, β) is an arrow $f\colon S \to T$ in $\mathcal{C}$ such that $\mathcal{F}(f) \circ \alpha = \beta \circ f$.

An $\mathcal{F}$*-bisimulation* between two $\mathcal{F}$-coalgebras (S, α) and (T, β) is a relation $R \subseteq S \times T$ for which there exists a coalgebra structure $\gamma\colon R \to \mathcal{F}(R)$ such that the projections $\pi_1\colon R \to S$ and $\pi_2\colon R \to T$ are homomorphisms: $\mathcal{F}(\pi_1) \circ \alpha = \gamma \circ \pi_1$ and $\mathcal{F}(\pi_2) \circ \beta = \gamma \circ \pi_2$. We then say that R is an $\mathcal{F}$-bisimulation for α and β. The arrow γ is called mediating for α and β. We write $x \sim y$ ('x and y are $\mathcal{F}$-bisimilar') whenever there exists an $\mathcal{F}$-bisimulation R with $(x,y) \in R$.
An $\mathcal{F}$-coalgebra (D, δ) is called *final* if there exists for any $\mathcal{F}$-coalgebra (S, α) a unique homomorphism from (S, α) to (D, δ). We have the following result.

Theorem 1. *(Internal full abstractness) For a final $\mathcal{F}$-coalgebra (D, δ) and $x, y \in D$, $x = y$ if and only if $x \sim y$.*

The proof is easy, see, e.g., [Rut96], Theorem 9.2. The main difficulty in obtaining full abstractness lies in the *construction* of a final coalgebra, which in general is nontrivial.

3 A coalgebraic interpretation of Larsen-Skou bisimulation

Starting from the definitions of a discrete probabilistic transition system and probabilistic bisimulation as proposed in the literature, we will consider generalizations of (discrete) probabilistic transition systems as coalgebras of a functor $\mathcal{D}$ on **Set**. We argue that $\mathcal{D}$-bisimilarity implies probabilistic bisimilarity, and, using the notion of z-closure, that probabilistic bisimulation and totality imply $\mathcal{D}$-bisimilarity. Then it is shown how this leads to the existence of a fully abstract domain.

Definition 2. [LS91,GSS95] A discrete probabilistic transition system is a tuple (Pr, Act, μ) where Pr is a given set of processes, Act is a given set actions, and $\mu\colon Pr \times Act \times Pr \to [0,1]$ is a so-called transition probability function, i.e., for all $P \in Pr$, $a \in Act$, $\mu(P, a, \cdot)$ is either the zero-map or a simple probability distribution.

A probabilistic bisimulation for a discrete probabilistic transition system is an equivalence '$\equiv$' on Pr such that

$$P \equiv Q \;\Rightarrow\; \sum_{P' \in E} \mu(P, a, P') = \sum_{P' \in E} \mu(Q, a, P')$$

for all $P, Q \in Pr$, $a \in Act$, and equivalence classes $E \in Pr/{\equiv}$. (Using the conventions of Section 2, the implication can also be written as $P \equiv Q \Rightarrow \mu[P, a, E] = \mu[Q, a, E]$.) Two processes P and Q are said to be probabilistic bisimilar if some probabilistic bisimulation contains the pair (P, Q).

Above we introduced the notation $\mathcal{D}(S)$ for the collection of all simple probability distributions over a set S. In fact, $\mathcal{D}$ can be extended to a **Set**-functor by defining for a mapping $f\colon S \to T$ a function $\mathcal{D}(f)\colon \mathcal{D}(S) \to \mathcal{D}(T)$ which maps a simple distribution μ on S to a simple distribution $\mathcal{D}(f)(\mu)$ on T such that $\mathcal{D}(f)(\mu)(t) = \mu[f^{-1}(\{t\})]$.

Let $\mathbf{0}$ represent termination. Note that a probabilistic transition system is just a mapping $\mu\colon Pr \times Act \to \mathcal{D}(Pr) + \{\mathbf{0}\}$ or, equivalently, a function $\mu\colon Pr \to (Act \to (\mathcal{D}(Pr) + \{\mathbf{0}\}))$. In other words, a probabilistic transition system is precisely a coalgebra of the functor $Act \to (\mathcal{D}(\cdot) + \{\mathbf{0}\})$. Applying the category theoretical machinery as described in Section 2 now gives us the coalgebraic notion of bisimulation. We will show that it corresponds to (actually generalizes) the notion of probabilistic bisimulation of Definition 2, thus providing categorical evidence for the Larsen-Skou bisimulation as the canonical process equivalence for discrete probabilistic transition systems.

For clarity of presentation we suppress, for the moment, the action component of a probabilistic transition system, and also do not bother about termination. Thus we consider coalgebras of the functor $\mathcal{D}$ itself. As it turns out, the presence of labels and termination does not make any essential difference for the technical content of what follows. Before we relate probabilistic bisimulation with $\mathcal{D}$-bisimulation, we first give a generalization of Definition 2, by allowing bisimulations between different transition systems, which are not necessarily equivalence relations.

Definition 3. Let $\alpha: S \to \mathcal{D}(S)$, $\beta: T \to \mathcal{D}(T)$ be two (stripped) discrete probabilistic transition systems. A binary relation $R \subseteq S \times T$ is called a probabilistic bisimulation for α, β iff $R(s,t) \Rightarrow \alpha(s)[U] = \beta(t)[V]$, for all $s \in S, t \in T$ and $U \subseteq S, V \subseteq T$ such that $\pi_1^{-1}(U) = \pi_2^{-1}(V)$. Two elements $s \in S$, $t \in T$ are said to be probabilistic bisimilar if some probabilistic bisimulation contains the pair (s,t).

Note that if R is an equivalence relation, then $\pi_1^{-1}(U) = \pi_2^{-1}(V)$ if and only if $U = \bigcup_{i \in I} E_i = V$, for some collection of equivalence classes $\{E_i | i \in I\}$ of R. Thus in this case, the condition on U and V in Definition 3 amounts to the assumption of E being an equivalence class in Definition 2, or, following the terminology of [Hen95], U and V are the same '$\equiv$'-block. This shows that Definition 2 is a special instance of Definition 3 ('modulo' the presence of labels and termination).

By exploitation of the various definitions one straightforwardly verifies that $\mathcal{D}$-bisimulation implies probabilistic bisimulation.

Lemma 4. *Let $\alpha: S \to \mathcal{D}(S)$ and $\beta: T \to \mathcal{D}(T)$ be two discrete probabilistic transition systems. Let R be a $\mathcal{D}$-bisimulation for α, β. Then R is a probabilistic bisimulation for α, β.*

The reverse of the above lemma is more intricate. We will first use the concept of z-closure and associated properties as developed in Section 2.

Lemma 5. *If $R \subseteq S \times T$ is a probabilistic bisimulation for $\alpha: S \to \mathcal{D}(S)$, $\beta: T \to \mathcal{D}(T)$, then so is R^*, the z-closure of R.*

So, if $s \in S$ and $t \in T$ are probabilistic bisimilar, we can assume —without loss of generality— that there exists a z-closed probabilistic bisimulation containing (s,t). We will need, for technical reasons, that R is total. This is equivalent with the common assumption of transition systems to have a distinguished initial state and considering reachable states only.

Theorem 6. *Let $R \subseteq S \times T$ be a probabilistic bisimulation for $\alpha: S \to \mathcal{D}(S)$ and $\beta: T \to \mathcal{D}(T)$. Moreover, assume R to be z-closed and total. Then R is a $\mathcal{D}$-bisimulation.*

Proof. The mapping $\gamma: R \to \mathcal{D}(R)$ given by

$$\gamma(s,t)(s',t') = \begin{cases} 0 & \text{if } \beta(t)[F(s')] = 0 \\ \dfrac{\alpha(s)(s') \cdot \beta(t)(t')}{\beta(t)[F(s')]} & \text{otherwise,} \end{cases}$$

for $(s,t) \in R$, is mediating for α and β. □

The format of the definition of $\gamma(s,t)$ is reminiscent of the discrete probability distributions of [JL91]. It is however not clear how their notion of probabilistic specification extends to the continuous setting of Section 4.

It is straightforward to adapt the above line of reasoning to a functor $\mathcal{D}'$ given by $\mathcal{D}' = Act \to (\mathcal{D}(\cdot) + \{\mathbf{0}\})$. The discrete probabilistic transition systems of Definition 2 are in 1–1 correspondence with the coalgebras of this functor, and the notion of $\mathcal{D}'$-bisimulation coincides with that of probabilistic bisimulation of Definition 2 (for total relations R).

We can now benefit from some general insights in the theory of coalgebras, by applying (a minor variation on) a result from [Bar93] involving boundedness of a set functor.

Theorem 7. *The functor $\mathcal{D}'$ (and also $\mathcal{D}$) has a final coalgebra.*

The final coalgebra for $\mathcal{D}'$ is nontrivial. The final coalgebra for $\mathcal{D}$, though, is degenerate: it equals the one element set. This is equivalent to the fact that, due to the absence of labels and a concept of termination as present for $\mathcal{D}'$, all elements in any two $\mathcal{D}$-coalgebras are probablisitically bisimilar.

Let $\mathbb{P}$ be the final $\mathcal{D}'$-coalgebra, so $\mathbb{P} \cong Act \to (\mathcal{D}(\mathbb{P}) + \{\mathbf{0}\})$. (Note that final coalgebras are always fixed points. See, e.g., [Rut96], Theorem 9.1.) The following is immediate by Theorem 1.

Corollary 8. *The system $\mathbb{P}$ is internally fully abstract with respect to the original notion of probabilistic bisimulation of Definition 2.*

4 $\mathcal{M}_1$-Bisimilarity for Probabilistic Transition Systems

The previous section illustrates that in a discrete probabilistic setting, a coalgebraic interpretation of probabilistic transition systems and bisimulation can be given, which is equivalent with the usual 'direct' approach. One of the advantages of the abstract coalgebraic approach is that it can fairly easily be generalized to the continuous setting of stochastic systems. We will now, in fact, allow probability measures to play the role of the simple distributions in the definition of a probabilistic transition system.

Probability measures only make sense in the context of a σ-algebra. When the collection of processes comes equipped with a topology —as is the case if the set of processes is endowed with an order or a metric structure— the obvious choice for this σ-algebra is the Borel σ-algebra, i.e. the least σ-algebra containing all the open sets. As mentioned in the introduction, we prefer the use of ultrametric (cf. [BV96]) above order, because of a combination of the following two reasons: (1) the technical advantage of a close relationship between standard measure theory and metric topology, and (2) the availability of a final coalgebra theorem in the metric setting, leading to a fully abstract domain for general probabilistic bisimulation.

The generalization of the notion of a discrete probabilistic transition system and the associated concept of bisimulation as proposed by Larsen and Skou is as follows.

Definition 9. A (general) probabilistic transition system is a tuple (Pr, Act, μ) where Pr is a given ultrametric space of processes, Act is a given set of actions, and $\mu: Pr \times Act \times \mathcal{B}(Pr) \to [0,1]$ is a so-called (general) transition probability function, i.e., $\mu(P, a, \cdot)$ is either the zero-map, or a Borel probability measure, for all $P \in Pr$, $a \in Act$. (Here $\mathcal{B}(Pr)$ denotes the collection of Borel measurable subsets of Pr.)

A probabilistic bisimulation for a probabilistic transition system (Pr, Act, μ) is an equivalence '$\equiv$' on Pr such that every equivalence class $E \subseteq Pr$ of '$\equiv$' is measurable, and

$$P \equiv Q \Rightarrow \mu(P, a, E) = \mu(Q, a, E)$$

for all $P, Q \in Pr$, $a \in Act$, and $E \in Pr/\equiv$. Two processes P and Q in Pr are said to be probabilistic bisimilar if there exists a probabilistic bisimulation containing the pair (P, Q).

Note that the equivalence classes E of '$\equiv$' must be measurable, since only then the values $\mu(P, a, E), \mu(Q, a, E)$ are well-defined.

For reasons of presentation, we dispense with the actions and with the treatment of termination. They can be added again later. In this way, a probabilistic transition system becomes a function $\alpha: S \to \mathcal{M}_1(S)$ where $\mathcal{M}_1(S)$ denotes the collection of all Borel probability measures. In the reformulation of the related notion of probabilistic bisimulation we give, as before, first a slightly more general definition of bisimilarity of systems with different carriers.

Definition 10. Let $\alpha: S \to \mathcal{M}_1(S)$ and $\beta: T \to \mathcal{M}_1(T)$ be two probabilistic transition systems. A relation $R \subseteq S \times T$ is called a probabilistic bisimulation for α, β iff $R(s,t) \Rightarrow \alpha(s)(U) = \beta(t)(V)$ for all $s \in S$, $t \in T$ and $U \in \mathcal{B}(S)$, $V \in \mathcal{B}(T)$ such that $\pi_1^{-1}(U) = \pi_2^{-1}(V)$. Two elements $s \in S$, $t \in T$ are said to be probabilistic bisimilar iff some probabilistic bisimulation contains the pair (s,t).

As for $\mathcal{D}$ in the previous section, $\mathcal{M}_1$ can be regarded as a functor, viz. a functor on the category **UMS** of ultrametric spaces and nonexpansive mappings.

Definition 11. The functor $\mathcal{M}_1: \mathbf{UMS} \to \mathbf{UMS}$ is given as follows: $\mathcal{M}_1(M)$ is the collection of all Borel probability measures endowed with the metric d such that $d(\mu, \nu) \leq \varepsilon \iff \forall O \in \mathcal{O}_\varepsilon: \mu(O) = \nu(O)$, for all $\mu, \nu \in \mathcal{M}_1(M)$, $\varepsilon > 0$. For nonexpansive $f: M \to N$ the mapping $\mathcal{M}_1(f): \mathcal{M}_1(M) \to \mathcal{M}_1(N)$ is defined by $\mathcal{M}_1(f)(\mu)(V) = \mu(f^{-1}(V))$, for all $V \in \mathcal{B}(N)$.

Elementary considerations concerning Borel-σ-algebras and nonexpansive maps show that $\mathcal{M}_1$ is a well-defined functor on **UMS**. Following the coalgebraic paradigm, $\mathcal{M}_1$ induces a notion of $\mathcal{M}_1$-bisimulation. One half of the relationship of $\mathcal{M}_1$-bisimulation and probabilistic bisimulation can be shown directly.

Lemma 12. *Let $\alpha: S \to \mathcal{M}_1(S)$, $\beta: T \to \mathcal{M}_1(T)$ be two probabilistic transition systems. Any $\mathcal{M}_1$-bisimulation R for α and β is also a probabilistic bisimulation for α, β.*

Below we show that the reverse also holds under reasonable conditions. The technicality to be dealt with concerns the proper generalization of the measurability condition of the equivalence classes E.

For a probabilistic bisimulation '$\equiv$' in the sense of Definition 9 we have, by an elementary set-theoretic argument, a partitioning into squares of subsets. Moreover, these subsets are measurable by assumption. So, we have $\equiv\, = \bigcup_{i \in I} E_i \times E_i$. Similarly, for the general set-up, we want a decomposition $R = \bigcup_{k \in K} E_k \times F_k$ where the E_k and F_k are Borel sets in S and T, respectively. Additionally, for measure theoretical considerations, we will assume the number of rectangles $E_k \times F_k$ that constitute R to be countable.

Definition 13. A binary relation $R \subseteq S \times T$ on two ultrametric spaces S and T is said to have a Borel decomposition iff $R = \bigcup_{k \in K} E_k \times F_k$ where $\{ E_k \mid k \in K \}$, $\{ F_k \mid k \in K \}$ are countable partitions of Borel sets of S and T, respectively.

In the construction of a mediating probabilistic transition system $\gamma\colon R \to \mathcal{M}_1(R)$, for a given probabilistic bisimulation R, we can again assume that R is z-closed. Since no measure theoretical considerations are involved, the proof of this is literally as for Lemma 5. The property is used in the next result.

Theorem 14. *Let $\alpha\colon S \to \mathcal{M}_1(S)$, $\beta\colon T \to \mathcal{M}_1(T)$ be two probabilistic transition systems. Let R be a probabilistic bisimulation for α, β in the sense of Definition 10. Assume that R is z-closed. If R has a Borel decomposition, then R is an $\mathcal{M}_1$-bisimulation for α, β.*

Proof. Let $\{ E_k \times F_k \mid k \in K \}$ be a Borel decomposition of R. Suppose $R(s,t)$ holds. The mapping $\gamma(s,t)\colon \mathcal{B}(R) \to [0,1]$ is then given by

$$\gamma(s,t)((U \times V) \cap R) \;=\; \textstyle\sum_{k \in K} \dfrac{\alpha(s)(U \cap E_k) \cdot \beta(t)(V \cap F_k)}{\beta(t)(F_k)} \tag{4.1}$$

for $U \in \mathcal{B}(S)$, $V \in \mathcal{B}(T)$. The verification that $\gamma(s,t)$ is well-defined and mediating for $\alpha(s)$, $\beta(t)$ is nontrivial but omitted for reasons of space. $\square$

In the remainder of this section, we shall again use some general insights from the theory of coalgebras, this time by applying a result from [AR89,RT93].

In turns out, that we are only able to show the existence of a final coalgebra when we consider an adaptation of $\mathcal{M}_1$, say $\mathcal{M}_1'$, which delivers Borel probability measures with so-called compact support, i.e., measures that vanish outside a compact set. More precisely, for a metric space M, $\mu\colon \mathcal{B}(M) \to [0,1]$ is said to have a compact support if, for some compact subset $K \subseteq M$, we have that $U \cap K = \emptyset \Rightarrow \mu(U) = 0$, for all $U \in \mathcal{B}(M)$. Let $\mathcal{M}_1'(M)$ denote the collection of all Borel probability measures of an ultrametric space M. Similarly as for $\mathcal{M}_1$, the new $\mathcal{M}_1'$ extends to a functor on **UMS**.

Additionally, to ensure the property of local contractivity (see, e.g., [RT93]), we put in a scaling functor $\cdot/2$. This operation is harmless from a semantical point of view. The usage of $\mathcal{M}'_1$, though, does narrow the type of transition systems falling within the framework. However, we stress that the established relationship of coalgebraic and probabilistic bisimulation, still carry through for the modified setting. Additionally, for the class of transition systems, now captured by the functor $Act \to (\mathcal{M}'_1(\cdot)/2 + \{\mathbf{0}\})$, the existence of a final coalgebra is guaranteed.

Theorem 15. *Let the functor* $\mathcal{F}\colon \mathbf{UMS} \to \mathbf{UMS}$ *be given by* $\mathcal{F} = Act \to (\mathcal{M}'_1(\cdot)/2 + \{\mathbf{0}\})$. *Then the following holds:*

(a) $\mathcal{F}$ *is locally contractive, i.e., for some* κ, $0 \leq \kappa < 1$, *and all ultrametric spaces* M *and* N, *the function* $\mathcal{F}_{M,N}\colon (M \to_1 N) \to (\mathcal{F}(M) \to_1 \mathcal{F}(N))$ *given by* $\mathcal{F}_{M,N}(f) = \mathcal{F}(f)$ *is* κ*-contractive.*
(b) If M *is complete, then* $\mathcal{F}(M)$ *is complete.*
(c) The functor $\mathcal{F}$ *has a final coalgebra.*

The presence of '$\cdot/2$' in the definition of $\mathcal{F}$ results in (a). (The other constituent functors are locally nonexpansive.) Only for part (b) the assumption of measures having a compact support is necessary. Its proof is non-trivial. Finally, part (c) follows from (a), (b), and (a minor variation of) [RT93], Theorem 4.8.

Let $\mathbb{Q}$ be the final $\mathcal{F}$-coalgebra: $\mathbb{Q} \cong Act \to (\mathcal{M}'_1(\mathbb{Q})/2 + \{\mathbf{0}\})$. From Theorem 1 and 15 we then immediately obtain the following result.

Corollary 16. *The system* $\mathbb{Q}$ *is internally fully abstract with respect to probabilistic bisimulation.*

5 Conclusion and future research

In this paper, a framework is proposed for probabilistic transition systems, involving general probability measures, and an associated notion of probabilistic bisimulation. Most research reported in the literature so far deals with *discrete* probabilistic transition systems, employing simple probability distributions only. The use of Borel measures allows for an extension of this to a continuous setting, which is necessary for the further development of models for dynamical, real-time, and in particular hybrid systems, for which discreteness and image-finiteness are often too restrictive.

Following the transition-systems-as-coalgebras paradigm, the categorical set-up provides a characterization of the Larsen-Skou bisimulation in terms of a set functor. For the continuous case, a similar result is shown for a functor on the category of ultrametric spaces. Moreover, exploiting parts of the theory of coalgebras, both for the discrete case and for the continuous case, internally fully abstract domains are constructed.

Further investigations of the proposed notion of Borel decomposition should clarify how the latter relates to the use of Polish spaces as in [BDEP97]. We expect that the technical result obtained there, on the existence of weak pullbacks, applies also to our setting. Also, once a suitable continuous process language is identified (such as PCCS [GJS90] for the discrete case), the process equivalences and fully abstract domains presented in this paper may be fruitfully applied in the semantical study of dynamical and hybrid systems.

References

[Acz88] P. Aczel. *Non-Well-Founded Sets.* CSLI Lecture Notes 14. Center for the Study of Languages and Information, Stanford, 1988.

[AM89] P. Aczel and N. Mendler. A final coalgebra theorem. In D.H. Pitt et al., editor, *Proc. Category Theory and Computer Science*, pages 357–365. LNCS 389, 1989.

[AR89] P. America and J.J.M.M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *Journal of Computer Systems and Sciences*, 39:343–375, 1989.

[Bar93] M. Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science*, 114:299–315, 1993. See also the addendum, Theoretical Computer Science, 124:189–192, 1994.

[BDEP97] R. Blute, J. Desharnais, A. Edalat, and P. Panangaden. Bisimulation for labelled Markov processes. In *Proc. LICS'97.* Warzaw, 1997.

[BV96] J.W. de Bakker and E.P. de Vink. *Control Flow Semantics.* The MIT Press, 1996.

[Eda94] A. Edalat. Domain theory and integration. In *Proc. LICS'94*, pages 115–124. Paris, 1994.

[GJS90] A. Giacalone, C. Jou, and S.A. Smolka. Algebraic reasoning for probabilisitic concurrent systems. In *Proc. Working Concerence on Programming Concepts and Methods.* IFIP TC2, Sea of Gallilee, 1990.

[GSS95] R.J. van Glabbeek, S.A. Smolka, and B. Steffen. Reactive, generative and stratified models of probabilistic processes. *Information and Computation*, 121:59–80, 1995.

[Hen95] T.A. Henzinger. Hybrid automata with finite bisimulations. In Z. Fülöp and F. Gécseg, editors, *Proc. ICALP'95*, pages 324–335. LNCS 944, 1995.

[JL91] B. Jonsson and K.G. Larsen. Specification and refinement of probabilistic processes. In *Proc. LICS'91*, pages 266–277. Amsterdam, 1991.

[JP89] C. Jones and G. Plotkin. A probabilistic powerdomain of evaluations. In *Proc. LICS'89*, pages 186–195. Asilomar, 1989.

[LS91] K.G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94:1–28, 1991.

[RT93] J.J.M.M. Rutten and D. Turi. On the foundations of final semantics: non-standard sets, metric spaces, partial orders. In J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors, *Proc. REX Workshop on Semantics: Foundations and Applications*, pages 477–530. LNCS 666, 1993.

[RT94] J.J.M.M. Rutten and D. Turi. Initial algebra and final coalgebra semantics for concurrency. In J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors, *Proc. REX School/Symposium 'A Decade of Concurrency'*, pages 530–582. LNCS 803, 1994.

[Rud66] W. Rudin. *Real and Complex Analysis.* McGraw-Hill, 1966.

[Rut96] J.J.M.M. Rutten. Universal coalgebra: a theory of systems. Report CS-R9652, CWI, 1996. Ftp-available at `ftp.cwi.nl` as pub/CWIreports/-AP/CS-R9652.ps.Z.