Building Metric Structures with the *Meas*-Functor

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Abstract We introduce the functor *Meas* in the category of complete ultra metric spaces and nonexpansive mapping. The main result of this paper is that *Meas* is a welldefined and locally nonexpansive endofunctor. Therefore the functor fits naturally in the metric approach to programming language semantics. The use of *Meas* in the construction of probabilistic powerdomains, either directly or through the use of domain equations, is illustrated with two examples.

1 Introduction

Recently there has been a growing interest in probabilistic systems and the application of formal methods to probabilistic systems. Probabilistic models are designed to describe systems that are inherently probabilistic, such as a faulty communication channel, or programs where probability is explicitly introduced, i.e. random algorithms. These random algorithms have shown to be successful at solving many problems that deterministic algorithms cannot solve or cannot solve efficiently.

Often the most difficult problems to find and diagnose are intermittent problems which occur only in a fraction of the executions or which occur only under specific circumstances. Many properties of probabilistic programs are intermittent by nature. These probabilistic properties cannot be tested by a single or a couple of executions of a program. Instead one needs to consider the average behavior over many program runs. The difficulty in testing for errors in probabilistic systems, which may be very subtle, makes the use of formal methods even more pressing.

An important part of a formal approach is the semantical modeling of programs to give a precise description of a program and to create a setting in which one can reason about a program. Even when using a syntactical reasoning approach, such as e.g. process algebra [5, 3] or Hoare logic [20] one still needs a semantical model to check the correctness of reasoning rules. A metric approach to defining semantics uses metric spaces as semantical domains. In reasoning about the processes in these domains, the properties of metric spaces can be used. Banach's theorem, for example, which guarantees existence of a unique fixed points for so-called 'contractive functions' on a complete metric space, is a useful tool for definitions and in showing equality. After initial work by

Arnold and Nivat [4, 24], a vast amount of work has been done in this area by the Amsterdam Concurrency Group led by Jaco de Bakker, see e.g. [8]. An overview of the use of metric techniques and of many of the results obtained can be found in [9, 12].

Domain equations play an important role in the specification of semantical domains. Domain equations over metric spaces were pioneered by De Bakker and Zucker [10] in the early eighties. Several types of domain equations can be solved by purely metric means. See e.g. [9]. A categorical approach to solving domain equations was developed by America and Rutten [1], while [27] provides a coalgebraic reformulation of these results. The approach of America and Rutten, translating results of [31] to the metric context, can be used to solve domain equations based on so-called locally contractive functors. Locally nonexpansive functors also play an important role in this setting as the composition of a locally nonexpansive functor and a locally contractive functor is again locally contractive.

Several methods are used to describe probabilistic choices within a semantical domain. For a probabilistic choice between finitely many options, a set or a multiset of options labeled with probabilities can be used, see e.g. [17, 15], or a discrete probability distribution over options [29]. The number of options for a probabilistic choice, however, is not always finite. When modeling the performance of a system, the amount of time needed or the achieved throughput may be probabilistic. As such an interval of real values may be possible as outcomes, at least conceptually. Programs which run indefinitely can also result in a probabilistic choice between infinitely many options even if only finite choices are possible in each step. For probabilistic choices with infinitely many options, the probability of each of the single options is insufficient to find the probability of all events. More information has to be included in the domain. A solution is the use of measures or valuations.

In this paper we will deal with a metric version of measures: The functor *Meas*, which yields an ultrametric space of measures over a given ultrametric space, is used to build metric domains. The functor *Meas* was introduced in [33] and also studied in [32]. An issue, not dealt with in these papers is the completeness of the resulting space which is required for the application of Banach's theorem to obtain unique fixed points. We will show that the functor *Meas* preserves completeness, i.e. $Meas(M)$ is complete for a given complete ultrametric space M. We will also show that the functor *Meas* is locally nonexpansive on the category of complement ultra metric spaces and thus fits in the America-Rutten approach to solving domain equations. This is illustrated in section 4 by building a simple probabilistic branching domain using domain equations.

Van Breugel and Worrell, inspired by work done by Desharnais, Gupta, Jagadeesan and Panangaden [16], introduce in [14, 13] a metric on a domain of measures. A key difference with their approach is that they use the Hutchinson metric to provide quantitative information about the probabilities and in this way a pseudometric rather than an ultrametric is obtained. The processes in their distance have distance zero when they are probabilistically bisimilar and close when they exhibit approximately the same probabilities. In the metric obtained by using *Meas*, processes that are close have exactly the same probabilities for approximately the same processes. Probabilistic bisimilar programs are assigned the same process. An advantage of the Van Breugel and Worrell approach is the intuitive way of dealing with distances between probabilities; small differences in probabilities lead to a small distance between processes. A disadvantage is that a pseudometric is obtained rather than an ultrametric. Only having a pseudometric instead of a metric means that the uniqueness of fixed points is not guaranteed. Additionally, ultrametricity is often useful in reasoning about processes in a metric domain. A property of ultrametric spaces that is exploited in this setting, for example, is the fact that two open balls are either disjoint or one is contained in the other.

In [14] a real-valued logic is introduced which can be used to characterize the distance between processes. In this paper a comparison is also given with the pseudo metric introduced by Desharnais, Gupta, Jagadeesan and Panangaden in [16], the metric on compact support measures from [33] also used here and with the pseudometric introduced by Norman in his thesis [25, section 6.1].

In [6] Baier and Kwiatkowska present a metric space of valuations and compare it with spaces defined using a set-theoretic and a complete partial order approach. The valuations introduced in [6] are similar to the compact support measures used here.

Slightly different are the valuations used in the thesis work of Jones [21] to introduce probabilistic semantical domains. These valuations are a slight simplification of measures. Not every measure necessarily has a corresponding valuation. In the setting of metric spaces each valuation does have unique extensions to a measure, though not necessarily a compact support measure.

The remainder of this paper is organized as follows: Section 2 collects some metric and measure theoretic definitions and results. Section 3 introduces the functor *Meas*. It is shown that this functor is well-defined and locally nonexpansive on the category of complete ultrametric spaces with nonexpansive mappings. Finally, section 4 illustrates the use of the functor for the construction of a probabilistic semantical domain.

2 Mathematical preliminaries

We assume the reader to be familiar with the basic notions from metric topology, measure theory and category theory. (See, e.g., [9, 30], [18, 26], and [22, 11], respectively.)

Definition 2.1 *A function $d\colon M \times M \to [0,1]$ is a 1-bounded ultrametric on the nonempty set M if the following conditions are met:*

(i) $d(x,y) = d(y,x)$

(ii) $d(x,y) = 0 \iff x = y$

(iii) $d(x,z) \leq \max\{\, d(x,y), d(y,z)\,\}$

for all $x, y, z \in M$.

Condition (iii) of Definition 2.1 is referred to as the strong triangle-inequality. For a function d to be a metric on M, only the weaker triangle-inequality

$$d(x,z) \leq d(x,y) + d(y,z)$$

is required. An immediate consequence of the strong triangle-inequality is that two ε-balls $B_\varepsilon(x)$ and $B_\varepsilon(z)$ are either disjoint or coincide. For, suppose $y \in B_\varepsilon(x) \cap B_\varepsilon(z)$, then $d(x,z) \leq \max\{\, d(x,y), d(y,z) \,\} < \varepsilon$. Now, if $w \in B_\varepsilon(x)$, then $d(w,z) < \max\{\, d(w,x), d(x,z) \,\} < \varepsilon$. Hence, $w \in B_\varepsilon(z)$. So $B_\varepsilon(x) \subseteq B_\varepsilon(z)$. By symmetry, $B_\varepsilon(x) = B_\varepsilon(z)$ follows. Similarly one can show that $B_\delta(x) \subseteq B_\varepsilon(z)$ if $\delta \leq \varepsilon$ and $x \in B_\varepsilon(z)$.

Clearly, an ultrametric on a set is also an metric on that set. So, an ultrametric induces a topology, viz. the topology $\mathcal{O}$ generated by the open balls. The notation $\mathcal{O}_\varepsilon$ denotes the collection $\{\, O \in \mathcal{O} \mid x \in O \Rightarrow B_\varepsilon(x) \subseteq O \,\}$. It follows that, in an ultrametric space, each $O \in \mathcal{O}_\varepsilon$ is the union of ε-balls, viz. $O = \bigcup\{\, B_\varepsilon(x) \mid x \in O \,\}$, and that the complement $M \setminus O$ is also in $\mathcal{O}_\varepsilon$, hence open, since $M \setminus O = \bigcup\{\, B_\varepsilon(x) \mid x \notin O \,\}$. We use $\mathcal{O}_*$ as shorthand for the set $\bigcup_n \mathcal{O}_{2^{-n}}$.

A metric d on a set M also induces a distance-function on the collection of subsets of M. This is the so-called Hausdorff-distance d_H, given by

$$d_H(X,Y) = \{\, \varepsilon > 0 \mid \forall O \in \mathcal{O}_\varepsilon \colon X \subseteq O \iff Y \subseteq O \,\}.$$

The Hausdorff-metric enjoys various nice properties. For example, if d is an ultrametric on M, so is d_H on $\mathcal{P}_{cl}(M)$ with $\mathcal{P}_{cl}(M)$ the collection of closed subsets of M. A similar results holds for the hyperspace $\mathcal{P}_{co}(M)$ of compact subsets of M. See, e.g., [10, 9].

Lemma 2.2 *If M with distance function d is a complete ultrametric space, then $\mathcal{P}_{co}(M)$ endowed with the Hausdorff-distance induced by d is also a complete metric space.*

The metric d on M also induces a metric on the collection $A \to M$ of mappings from A to M. This distance is called the distance of pointwise convergence and is given by

$$d_F(f,g) = \sup\{\, d(f(a), g(a)) \mid a \in A \,\}$$

for $f, g \colon A \to M$. Again, as can be straightforwardly shown, ultrametricity and completeness of d implies ultrametricity and completeness of d_F.

The notation $\mathcal{B}(M)$ for an ultrametric space M denotes the Borel-σ-algebra on M, i.e. the least σ-algebra containing all open sets of M. Recall that a Borel probability measure is a Borel-measure μ such that $\mu(M) = 1$.

From a technical point of view it is attractive when a Borel probability measure 'lives' on a compact set. This will be instrumental in our proof of the preservation of completeness by the functor *Meas* below.

Definition 2.3 *Let M be an ultrametric space. A Borel probability measure μ is said to have compact support if μ vanishes outside some compact set K, i.e. $\mu(M \setminus K) = 0$ or, equivalently, $\mu(K) = 1$. In this case the support* $\mathrm{spt}(\mu)$ *of μ is defined as the smallest compact set K for which $\mu(M \setminus K) = 0$.*

From the definition of a compact support it follows that for any open O in M we have that

$$\mu(O) = 0 \iff \mathrm{spt}(\mu) \cap O = \emptyset, \tag{1}$$

since the intersection of a closed and a compact set is again compact. The property 1 is used in the proof of the following lemma. which stretches σ-additivity of measures to arbitrary additivity of certain collections of balls.

Lemma 2.4 *If μ is a measure on M and $\{ B_\varepsilon(x_i) \mid i \in I \}$ a collection of pairwise disjoint ε-balls in M, then it holds that $\mu(\bigcup\{ B_\varepsilon(x_i) \mid i \in I \}) = \sum\{ \mu(B_\varepsilon(x_i)) \mid i \in I \}$.*

Proof Let $K = \mathrm{spt}(\mu)$. As $\{ B_\varepsilon(x) \mid x \in K \}$ is an open cover of K, it follows from ultrametricity of d and compactness of K, that $K \cap B_\varepsilon(x_i) \neq \emptyset$ iff $i \in I'$, for some finite subset $I' \subseteq I$. Hence $\mu(B_\varepsilon(x_i)) = 0$ for $i \in I \setminus I'$. Likewise, since $K \cap \bigcup\{ B_\varepsilon(x_i) \mid i \in I \setminus I' \} = \emptyset$, we have that $\mu(\bigcup\{ B_\varepsilon(x_i) \mid i \in I \setminus I' \}) = 0$. Therefore,

$$\begin{aligned}
&\mu(\bigcup\{ B_\varepsilon(x_i) \mid i \in I \}) \\
&\quad = \mu(\bigcup\{ B_\varepsilon(x_i) \mid i \in I' \}) + \mu(\bigcup\{ B_\varepsilon(x_i) \mid i \in I \setminus I' \}) \\
&\quad = \mu(\bigcup\{ B_\varepsilon(x_i) \mid i \in I' \}) \\
&\quad = \sum\{ \mu(B_\varepsilon(x_i)) \mid i \in I' \} \\
&\quad = \sum\{ \mu(B_\varepsilon(x_i)) \mid i \in I \}.
\end{aligned}$$

which was to be proven. $\square$

We have the following extension result for measures defined on $\mathcal{O}_*$. A proof, in a more general setting, can be found in [19, Sec. 13].

Theorem 2.5 *If μ is a Borel probability measure on $\mathcal{O}_*$ then there is a unique extension to a measure $\bar{\mu}$ on $\mathcal{B}$.*

The next lemma states that two measures coincide if they agree on the open balls.

Lemma 2.6 *If μ, ν are two Borel probability measures of compact support on an ultrametric space M such that $\mu(B_\varepsilon(x)) = \nu(B_\varepsilon(x))$ for all $\varepsilon > 0$ and $x \in M$, then $\mu = \nu$.*

Proof It is first shown that the two measures coincide on all sets in $\mathcal{O}_\varepsilon = \{ O \in \mathcal{O} \mid x \in O \Rightarrow B_\varepsilon(x) \subseteq O \}$ (for all $\epsilon > 0$). A basic result from measure theory can then be used to show that the two measures are the same.

For a given $\epsilon > 0$ take a finite collection of points $x_1, \ldots x_n$ such that both μ and ν vanish outside $B_\varepsilon(x_1) \cup \ldots \cup B_\varepsilon(x_n)$. Clearly such points exists as the collection of all ϵ-balls is a cover of $\mathrm{spt}(\mu) \cup \mathrm{spt}(\nu)$. This set is compact and consequently has a finite subcover. By ultrametricity, the balls $B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)$ can be assumed to be pairwise disjoint. Also, for a given set $O \in \mathcal{O}_\varepsilon$, a ball $B_\varepsilon(x)$ is either completely inside O or completely outside O. We now reason as follows:

$$\begin{aligned}
\mu(O) &= \mu(\cup_{i=1}^n B_\varepsilon(x_i) \cap O) \\
&= \mu(\cup \{ B_\varepsilon(x_i) \mid x_i \in O, i = 1, \ldots, n \}) \\
&= \sum \{ \mu(B_\varepsilon(x_i)) \mid x_i \in O, i = 1, \ldots, n \} \\
&= \sum \{ \nu(B_\varepsilon(x_i)) \mid x_i \in O, i = 1, \ldots, n \} \\
&= \nu(\cup \{ B_\varepsilon(x_i) \mid x_i \in O, i = 1, \ldots, n \}) \\
&= \nu(\cup_{i=1}^n B_\varepsilon(x_i) \cap O) \\
&= \nu(O).
\end{aligned}$$

The two sums are equal because the two measures coincide on balls.

We have now shown that the measures coincide on the ring $\mathcal{O}_* = \bigcup_{\epsilon > 0} \mathcal{O}_\varepsilon$. The σ-ring generated by $\mathcal{O}_*$ is the collection of all Borel sets. A basic result from measure theory states that a measure on a ring extends uniquely to a measure on the generated σ-ring (cf., e.g., [18]). Therefore, two measures which coincide on $\mathcal{O}_*$ must also coincide on all Borel sets. Hence $\mu = \nu$. $\square$

The following property claims that if two Borel probability measures of compact support coincide on small balls, the also coincide on all bigger balls.

Lemma 2.7 *Let μ, ν be two Borel-measures of compact support on an ultrametric space M. Let $\delta > 0$ be such that $\mu(B_\delta(x)) = \nu(B_\delta(x))$ for any $x \in M$. Then $\mu(B_\varepsilon(x)) = \nu(B_\varepsilon(x))$ for any $x \in M$ and $\varepsilon \geq \delta$.*

Proof Pick $x \in M$ and $\varepsilon \geq \delta$. We have, by ultrametricity, that

$$B_\varepsilon(x) = \bigcup \{ B_\delta(x') \mid d(x, x') < \varepsilon \}.$$

Moreover, $\{ B_\delta(x') \mid d(x, x') < \varepsilon \}$ is a collection of pairwise disjoint δ-balls (although for some x', x'' the balls $B_\delta(x'), B_\delta(x'')$ may coincide). By applying Lemma 2.4 we obtain

$$\begin{aligned}
\mu(B_\varepsilon(x)) &= \mu(\bigcup \{ B_\delta(x') \mid d(x, x') < \varepsilon \}) \\
&= \mu(\bigcup \{ B_\delta(x') \mid d(x, x') < \varepsilon,\ B_\delta(x') \cap \mathrm{spt}(\mu) \neq \emptyset \}) \\
&= \sum \{ \mu(B_\delta(x')) \mid d(x, x') < \varepsilon \}.
\end{aligned}$$

A similar argument holds for the Borel measure ν. We conclude that

$$\begin{aligned}\mu(B_\varepsilon(x)) &= \sum\{\, \mu(B_\delta(x')) \mid d(x,x') < \varepsilon \,\} \\ &= \sum\{\, \nu(B_\delta(x')) \mid d(x,x') < \varepsilon \,\} \\ &= \nu(B_\varepsilon(x)),\end{aligned}$$

since μ and ν coincide on δ-balls. □

Next we give the definition of a locally contractive and locally nonexpansive functor on the category *CUMS* of complete ultrametric spaces and nonexpansive mappings.

Definition 2.8 *Let $\mathcal{F}\colon CUMS \to CUMS$ be a functor.*

(a) $\mathcal{F}$ is called locally contractive if, for some α, $0 \leq \alpha < 1$, it holds, for all arrows f, g, that $d_F(\mathcal{F}(f), \mathcal{F}(g)) \leq \alpha \cdot d_F(f, g)$.

(b) $\mathcal{F}$ is called locally nonexpansive if it holds that $d_F(\mathcal{F}(f), \mathcal{F}(g)) \leq d_F(f, g)$ for all arrows f, g.

It is readily checked that, e.g., the composition of a locally contractive and a locally nonexpansive functor is itself locally contractive. In the literature various semantical domains are constructed using different combinations of locally contractive and locally nonexpansive functors, usually given by domain equations. We mention

- $\mathbb{P} = \{p_0\} \cup \mathcal{P}_{cl}((A \cup \{\epsilon\}) \times \mathbb{P}/2$ from [10]
- $\mathbb{P} = \{p_0\} \cup \mathcal{P}_{co}((A \cup \{\tau\}) \times \mathbb{P}/2$ from [7]
- $\mathbb{P} = \{p_0\} \cup (\Sigma \to \mathcal{P}_{cl}(\mathit{Step}))/2$ where $\mathit{Step} = (\Sigma \times \mathbb{P}) \cup \mathit{Send} \cup \mathit{Answer}$ from [2].

The result underpinning these equations, stated in the theorem below, is proven in [1].

Theorem 2.9 *Let $\mathcal{F}$ be a locally contractive functor $\mathcal{F} : CUMS \to CUMS$ Then the domain equation $\mathbb{P} \simeq \mathcal{F}(\mathbb{P})$ has a unique solution up to isomorphism.*

$$\begin{aligned}\mu(B_\varepsilon(x)) &= \mu(\bigcup\{\, B_\delta(x') \mid d(x,x') < \varepsilon \,\}) \\ &= \mu(\bigcup\{\, B_\delta(x') \mid d(x,x') < \varepsilon,\ B_\delta(x') \cap \mathrm{spt}(\mu) \neq \emptyset \,\}) \\ &= \sum\{\, \mu(B_\delta(x')) \mid d(x,x') < \varepsilon,\ B_\delta(x') \cap \mathrm{spt}(\mu) \neq \emptyset \,\} \\ &= \sum\{\, \mu(B_\delta(x')) \mid d(x,x') < \varepsilon \,\}\end{aligned}$$

since $\mu(\bigcup\{\, B_\delta(x') \mid d(x,x') < \varepsilon, B_\delta(x') \cap \mathrm{spt}(\mu) = \emptyset \,\}) = 0$ and $\mu(B_\delta(x')) = 0$ for $x' \in M$ such that $d(x,x') < \varepsilon$ and $B_\delta(x') \cap \mathrm{spt}(\mu) = \emptyset$. A similar argument holds for the Borel measure ν. We conclude that

3 The functor *Meas*

In this section we introduce the functor *Meas* on the category of *CUMS* of complete ultrametric spaces and nonexpansive mappings. At the object-level *Meas* yields the metric space of all Borel probability measures of compact support; at the arrow-level *Meas* the construction of a measure-along-a-mapping is used. We will show that *Meas* is a well-defined and locally nonexpansive functor, which makes it a useful building block for the definition of metric domains for probabilistic program constructs.

Definition 3.1 *Let M be an ultrametric space with metric d.*

(a) $\mathit{Meas}(M)$ denotes the space of all Borel probability measures on M of compact support.

(b) The distance d_{Meas} on $\mathit{Meas}(M)$ is defined by

$$d_{\mathit{Meas}}(\mu, \nu) = \inf\{\, \epsilon > 0 \mid \forall x \in M : \mu(B_\varepsilon(x)) = \nu(B_\varepsilon(x)) \,\}.$$

In the remainder of this paper the term measure refers to a Borel probability measure with compact support. We also fix an ultrametric space M with distance d. Lemma 3.3 states that d_{Meas} is an ultrametric on $\mathit{Meas}(M)$. For the proof of the lemma we need a technical property first.

Lemma 3.2 *If $\mu, \nu \in \mathit{Meas}(M)$ are such that $d_{\mathit{Meas}}(\mu, \nu) \leq \varepsilon$ for some $\varepsilon > 0$, then $\mu(O) = \nu(O)$ for any $O \in \mathcal{O}_\varepsilon$.*

Proof By ultrametricity of M, any $O \in \mathcal{O}_\varepsilon$ can be represented as a disjoint union of ε-balls. From Lemma 2.7 we have that μ and ν coincide on ε-balls. Therefore we conclude

$$\mu(O) = \sum\{\, \mu(B_\varepsilon(x_i)) \mid i \in I \,\} = \sum\{\, \nu(B_\varepsilon(x_i)) \mid i \in I \,\} = \nu(O)$$

using Lemma 2.4 twice. □

We are now ready to show that d_{Meas} is a distance function on $\mathit{Meas}(M)$.

Lemma 3.3 *For an ultrametric space (M, d) the pair $(\mathit{Meas}(M), d_{\mathit{Meas}})$ is an ultrametric space as well.*

Proof We check the conditions of Definition 2.1. Symmetry is obvious. Clearly $d_{\mathit{Meas}}(\mu, \mu) = 0$ for any $\mu \in \mathit{Meas}(M)$. Suppose $d_{\mathit{Meas}}(\mu, \nu) = 0$. In order to prove that $\mu = \nu$ it suffices, by Lemma 2.6, to check that $\mu(B_\varepsilon(x)) = \nu(B_\varepsilon(x))$ for any $\varepsilon > 0$ and $x \in M$. This is immediate from the definition of d_{Meas}. Finally, in order to verify the strong triangle-inequality $d_{\mathit{Meas}}(\mu, \rho) \leq \max\{\, d_{\mathit{Meas}}(\mu, \nu), d_{\mathit{Meas}}(\nu, \rho) \,\}$ we reason as follows: Suppose that we have that $\max\{\, d(\mu, \nu), d(\nu, \rho) \,\} < \epsilon$. Then both $d_{\mathit{Meas}}(\mu, \nu) < \varepsilon$ and $d_{\mathit{Meas}}(\nu, \rho) < \varepsilon$. Then, by Lemma 2.7, we have that $\mu(B_\varepsilon(x)) = \nu(B_\varepsilon(x))$ and $\nu(B_\varepsilon(x)) = \rho(B_\varepsilon(x))$ for any $x \in M$. Therefore, $\mu(B_\varepsilon(x)) = \rho(B_\varepsilon(x))$ for any ε-ball, thus $d(\mu, \rho) \leq \varepsilon$. So we have

$$\max\{\, d(\mu, \nu), d(\nu, \rho) \,\} < \varepsilon \Rightarrow d(\mu, \rho) \leq \varepsilon$$

and, subsequently, $d(\mu, \rho) \leq \max\{\, d(\mu, \nu), d(\nu, \rho) \,\}$. □

Before addressing the completeness of $Meas(M)$ we need a property relating the Hausdorff-distance and the distance on measures.

Lemma 3.4 *Let $\mu, \nu \in Meas(M)$ and $\varepsilon > 0$. Then it holds that*

$$d(\mu, \nu) \leq \varepsilon \Rightarrow d_H(\mathrm{spt}(\mu), \mathrm{spt}(\nu)) \leq \varepsilon.$$

Proof Pick $\mu, \nu \in Meas(M)$. Put $K = \mathrm{spt}(\mu)$, $L = \mathrm{spt}(\nu)$. If $d_{Meas}(\mu, \nu) \leq \varepsilon$, then, by Lemma 3.2, $\mu(O) = \nu(O)$ for $O \in \mathcal{O}_\varepsilon$. In particular $\forall O \in \mathcal{O}_\varepsilon$: $\mu(O) = 0 \iff \nu(O) = 0$. This is equivalent to $\forall O \in \mathcal{O}_\varepsilon$: $K \cap O = \emptyset \iff L \cap O = \emptyset$. Since, by ultrametricity, $O \in \mathcal{O}_\varepsilon \iff M \setminus O \in \mathcal{O}_\varepsilon$, it follows that $\forall O \in \mathcal{O}_\varepsilon$: $K \subseteq O \iff L \subseteq O$, i.e. $d_H(K, L) \leq \varepsilon$. □

So far we have that $Meas(M)$ is an ultrametric space if $M \in CUMS$. For $Meas(M) \in CUMS$ we need to show the completeness of $Meas(M)$.

Theorem 3.5 *If the ultrametric space M is complete, so is the ultrametric space $Meas(M)$.*

Proof Suppose that M is an ultra-metric space and let $(\mu_i)_i$ be a Cauchy sequence in $Meas(M)$. Let the index i_n be such that $\forall i, j > i_n$: $d_{Meas}(\mu_i, \mu_j) \leq 2^{-n}$. So, by Lemma 3.2 we have that $\forall i, j > i_n$: $\mu_i(O) = \mu_j(O)$ for $O \in \mathcal{O}_{2^{-n}}$. Additionally, we can assume $i_{n+1} > i_n$. Define the mapping $\mu: \mathcal{O}_* \to [0, 1]$ by $\mu(O) = \lim_i \mu_i(O)$.

Claim 1: μ is finitely additive. Proof of claim 1: Suppose $O_1, \ldots, O_r \in \mathcal{O}_*$ are disjoint. Pick n such that $O_1, \ldots, O_r \in \mathcal{O}_n$. Note $O_1 \cup \ldots \cup O_r \in \mathcal{O}_n$. Pick $i > i_n$. Then $\mu(O_1 \cup \ldots \cup O_r) = \mu_i(O_1 \cup \ldots \cup O_r) = \mu_i(O_1) + \cdots + \mu_i(O_r) = \mu(O_1) + \cdots + \mu(O_r)$ by additivity of μ_i.

Claim 2: μ has a compact support. Proof of claim 2: Put $K_i = \mathrm{spt}(\mu_i)$. By Lemma 3.4 it holds that $(K_i)_i$ is a Cauchy sequence in the hyperspace $\mathcal{P}_{co}(M)$ of compact subsets of M. By completeness of M, the hyperspace $\mathcal{P}_{co}(M)$ is complete. Therefore $K = \lim_i K_i$ exists and is compact. Let $O \in \mathcal{O}_n$. Pick $i > i_n$. Then

$$\begin{aligned} O \cap K = \emptyset \quad &\iff \quad [\text{lemma 3.4a, } d(K, K_i) \leq 2^{-n}] \quad O \cap K_i = \emptyset \\ &\iff \quad [K_i \text{ support of } \mu_i] \quad \mu_i(O) = 0 \\ &\iff \quad [i > i_n] \quad \mu(O) = 0. \end{aligned}$$

So, $K = \mathrm{spt}(\mu)$ and K is compact.

Claim 3: μ is σ-additive. Proof of claim 3: Let $(O_i)_i$ be a disjoint sequence in $\mathcal{O}_*$ such that $\bigcup_i O_i \in \mathcal{O}_*$. Pick n such that $\bigcup_i O_i \in \mathcal{O}_{2^{-n}}$. Put $O' = M \setminus (\bigcup_i O_i)$. Note that $O' \in \mathcal{O}_{2^{-n}}$. In particular, O' is an open set. We have that $\{ O_i \mid i \in \mathbb{N} \} \cup \{O'\}$ is an open cover of K. Since K is compact, we derive that the index-set $I = \{ i \mid O_i \cap K \neq \emptyset \}$ is finite. Note, by disjointness, $(\bigcup_{i \notin I} O_i) \cap K = \emptyset$, hence $\mu(\bigcup_{i \notin I} O_i) = 0$. Also, for $i \notin I$ we have $\mu(O_i) = 0$. Therefore $\mu(\bigcup_{i \notin I} O_i) = \sum_{i \notin I} \mu(O_i)$. We conclude

$$\mu(\bigcup_i O_i) \quad = \quad \mu(\bigcup_{i \in I} O_i \cup \bigcup_{i \notin I} O_i)$$

$$
\begin{aligned}
&= \text{[finite additivity]} \quad \sum_{i \in I} \mu(O_i) + \mu(\bigcup_{i \notin I} O_i) \\
&= \sum_{i \in I} \mu(O_i) \sum_{i \notin I} \mu(O_i) \\
&= \sum_{i} \mu(O_i).
\end{aligned}
$$

Define $\mu: \mathcal{B} \to [0,1]$, using Theorem 2.5, as the unique extension on $\mathcal{B}$ of μ on $\mathcal{O}_*$. Then we have that $\mu \in \mathit{Meas}(M)$ by the above and $d_{\mathit{Meas}}(\mu, \mu_i) \leq 2^{-n}$ for $i > i_n$, so $\mu = \lim_i \mu_i$. □

We are now in a position to define *Meas* as a functor on the category *CUMS* of complete ultrametric spaces with nonexpansive maps.

Definition 3.6 *The functor Meas: CUMS → CUMS is given as follows:*

(a) The functor Meas assigns the complete ultrametric space Meas(M) to any complete ultrametric space $M \in$ CUMS.

(b) The functor Meas assigns to each nonexpansive mapping $f: M \to N$ from a complete ultrametric space M to a complete ultrametric space N the mapping Meas(f): Meas(M) → Meas(N) given by

$$\mathit{Meas}(f)(\mu)(B) = \mu(f^{-1}(B)).$$

From Lemma 3.3 and 3.5 we have that indeed $\mathit{Meas}(M) \in \mathit{CUMS}$ for $M \in \mathit{CUMS}$. It can be straightforwardly verified, for $f: M \to N$ nonexpansive, that $\mathit{Meas}(f)(\mu) \in \mathit{Meas}(N)$ using the fact that $f^{-1}(B) \in \mathcal{B}(M)$ for $B \in \mathcal{B}(N)$ and using basic set-theoretic properties of f^{-1}. In fact, $\mathit{Meas}(f)(\mu)$ is referred to as the measure μ along f. So, *Meas* is a well-defined functor on *CUMS*. The last result of this section states that *Meas* is a locally nonexpansive functor as well.

Theorem 3.7 *The functor Meas on CUMS is locally nonexpansive functor.*

Proof Let $M, N \in \mathit{CUMS}$ and $f, g : M \to N$ be two nonexpansive functions. If $d_H(f,g) < \varepsilon$, then for all $y \in N$ we have that $f^{-1}(B_\varepsilon(y)) = g^{-1}(B_\varepsilon(y))$. For, if $x \in f^{-1}(B_\varepsilon(y))$, then $f(x) \in B_\varepsilon(y)$ and, since $d_H(f,g) < \varepsilon$, $d(f(x), g(x)) < \varepsilon$. So, by ultrametricity, $d(y, g(x)) \leq \max\{\, d(y, f(x)), d(f(x), g(x)) \,\} < \varepsilon$, i.e. $g(x) \in B_\varepsilon(y)$. But then $\mathit{Meas}(f)(\mu)(B_\varepsilon(x)) = \mu(f^{-1}(B_\varepsilon(x))) = \mu(g^{-1}(B_\varepsilon(x))) = \mathit{Meas}(g)(\mu)(B_\varepsilon(x))$ or $d_{\mathit{Meas}}(\mathit{Meas}(f)(\mu), \mathit{Meas}(g)(\mu)) < \varepsilon$ for all $\mu \in \mathit{Meas}(M)$. Thus,

$$d(f,g) < \varepsilon \Rightarrow d(\mathit{Meas}(g), \mathit{Meas}(f)) \leq \varepsilon.$$

Since this holds for all $\varepsilon > 0$ it follows that $d(\mathit{Meas}(g), \mathit{Meas}(f)) \leq d(f,g)$. □

Summarizing this section we have established that the functor *Meas* on the category of complete ultrametric spaces and nonexpansive mappings is well-defined and locally nonexpansive.

4 Examples and Conclusions

Above we have discussed the functor *Meas* first introduced in [33] and further studied in [32]. Using *Meas*, one can construct an ultrametric space of measures over a given ultrametric space. More precisely, the space of all compact support Borel probability measures is obtained. We have shown that the functor *Meas* preserves completeness and is locally nonexpansive on *CUMS*, the category of complete ultrametric spaces and nonexpansive mapping.

In definition 3.1 the ultrametric space $Meas(M)$ is defined for any given ultrametric space M. Banach's fixed point theorem, which is often used in the metric approach to guarantee the existence of unique fixed points, requires that the metric space used is complete. It is therefore important that completeness is preserved by *Meas*, i.e. if a space M is complete than so is $Meas(M)$. That *Meas* does preserves completeness has been shown in theorem 3.5. Thus *Meas* can be used to construct suitable metric spaces.

As an example consider the space of finite or infinite action sequences Act^∞ over some action alphabet Act. Equipped with the *Baire metric* this space is a complete ultrametric space. The metric space of all Borel probability measures with compact support $Meas(Act^\infty)$ can be used to describe trace behavior of probabilistic process languages with alphabet Act.

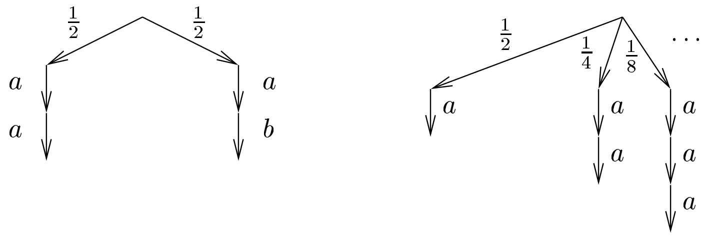
Example 4.1 *Let Act be a given alphabet of action. Let Act^∞ denote the space of finite and infinite sequences of actions with the distance function d_B, the Baire-distance, given by*

$$d_B(w, w') \quad = \quad \inf\{\, \tfrac{1}{2}^n \mid w[n] = w'[n]\,\}$$

for $w, w' \in Act^\infty$. Here $w[n]$ denote the sequence w cut of after n actions.
Let the space $\mathbb{P}_l$ be given by

$$\mathbb{P}_l \quad = \quad Meas(Act^\infty).$$

It contains all compact support Borel probability measures over Act^∞ and can be used as a domain to describe behavior of (linear) probabilistic processes. For example the measure Δ_a for some $a \in Act$ is an element of $Meas(Act^\infty)$ that describes that with probability 1 the sequence consisting only of a single action a is executed. The measures Δ_{aa}, Δ_{aaa} and Δ_{a^ω} ($a^\omega = aaaa\ldots$) respectively describe that the sequence aa, aaa or a^ω is executed with probability 1. More precisely, $\Delta_a(B) = 1$ if $a \in B$, $\Delta_a(B) = 0$ otherwise. A measure like Δ_a is referred to as a Dirac measure. Combinations of such measures are also present in $Meas(Act^\infty)$. Finite combinations such as $\frac{1}{2}\Delta_{aa} + \frac{1}{2}\Delta_{ab}$ which chooses between the sequences aa and ab giving equal probability to both, but also infinite combinations such as $\frac{1}{2}\Delta_a + \frac{1}{4}\Delta_{aa} + \frac{1}{8}\Delta_{aaa} + \ldots$ in which any finite number of actions a can be selected according to a geometric distribution.



There are also measures in Meas which cannot be depicted this easily. Define the operation $\bullet/a$ on measures in $\mathbb{P}_l$ by $\mu/a(B) = \mu(\{\, w \mid aw \in B \,\})$ for all Borel sets B, then $\bullet/a$ is contractive function on $\mathbb{P}_l$. That a solution μ for the equation $\mu = \frac{1}{2}\mu/a + \frac{1}{2}\mu/b$ must be present in Meas(Act^∞) can easily be shown using Banach's fixed point theorem. This measure μ assigns "equal probability" to all infinite sequences of a's and b's. More precisely, with probability 1 some infinite sequence is produced and for any given n and any given sequence of a's and b's of length n the probability of starting with the sequence is $\frac{1}{2}^n$.

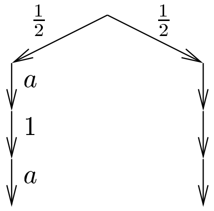
The space $Meas(Act^\infty)$ in the example above is a so-called linear domain; a single probabilistic choice is made to decide the sequence of actions that is produced. One also wants to be able to build branching domains in which choices are made along the way. These branching domains are typically described by domain equations. Definition 3.6 defines *Meas* on nonexpansive functions, the arrows of the category *CUMS*, turning *Meas* into an endofunctor on *CUMS*. The functor *Meas* is shown to be locally nonexpansive in theorem 3.7. As such *Meas* can be used in domain equations in composition with a locally contractive functor. A frequently deployed locally contractive functor is $id_{\frac{1}{2}}(\bullet)$ which simply reduces distances by a half but does not change the space in any other way. The space $Act \times id_{\frac{1}{2}}(\mathbb{P})$, for example can be used to describe processes in $\mathbb{P}$ prefixed with an action in Act.

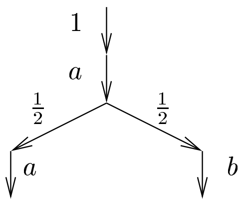
Example 4.2 *Given an alphabet Act of actions, a branching probabilistic domain $\mathbb{P}_b$ can be specified by the domain equation*

$$\mathbb{P}_b \quad \simeq \quad Meas(Act \times id_{\frac{1}{2}}(\mathbb{P}_b) + Act).$$

We have that $Act \times id_{\frac{1}{2}}(\bullet)$ is locally contractive and so is $Act \times id_{\frac{1}{2}}(\bullet) + Act$ and thus $Meas(Act \times id_{\frac{1}{2}}(\bullet) + Act)$ is also locally contractive by local nonexpansiveness of Meas. This means that the domain equation above must have a unique solution.

The domain $\mathbb{P}_b$ captures branching behavior of probabilistic processes. For readability the examples below use the notation $\rho \cdot x$ for $\rho \cdot \Delta_x$. The process $1 \cdot a$ produces action a with probability 1. The process p_1 given by $p_1 = \frac{1}{2} \cdot \langle a, 1 \cdot a\rangle + \frac{1}{2} \cdot \langle a, 1 \cdot b\rangle$ describes a fair choice between executing a followed by a, and a followed by b. In the process p_2 given by $p_2 = 1 \cdot \langle a, \frac{1}{2} \cdot a + \frac{1}{2} \cdot b\rangle$ the action a is executed before choosing between a and b for the next action.





The domain $\mathbb{P}_b$ contains a process p_3 which satisfies $p_3 = \frac{1}{2} \cdot a + \frac{1}{2} \cdot \langle a, p_3 \rangle$ in other words $p = \frac{1}{2} \cdot a + \frac{1}{2} \cdot \langle a, \frac{1}{2} \cdot a + \frac{1}{2} \cdot \langle a, \ldots \rangle \rangle$. A process p_4, $p_4 = \frac{1}{2} \cdot a + \frac{1}{4} \cdot \langle a, 1 \cdot a \rangle + \frac{1}{8} \cdot \langle a, 1 \cdot \langle a, 1 \cdot a \rangle \rangle + \ldots$, which is similar to process p_2 of example 4.1 is also an element of $\mathbb{P}_b$.

The example p_4 illustrates that structures can be built that are similar to elements of $\mathbb{P}_l$. However, p_1 and p_2 show that distinctions can be made that are not possible in the linear domain. For these examples it is clear which linear process corresponds to the branching process. In general additional structure is required on the domain be able to obtain a linear probabilistic process from a branching probabilistic process [15, chapter 7].

In the metric setting bisimilarity usually corresponds with equality on a metric branching domain [9]. Similarly, in a coalgebraic setting bisimilarity corresponds to equality in the final coalgebra [28]. For the basic probabilistic metric branching domain $\mathbb{P}_b$ in the example above we have that equality on this domain corresponds to the commonly used notion of Larsen-Skou bisimulation [23]. (See [33].) To be more precise: Giving a semantics for a basic probabilistic process language in a standard way using this domain yields an equivalence which is the same as Larsen-Skou bisimulation.

In this paper we have defined the *Meas* functor yielding Borel probability measures with compact support. As can be seen from the examples above (see [15] for a few more examples) the functor *Meas* can be used to build metric structures for modeling probabilistic languages.

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