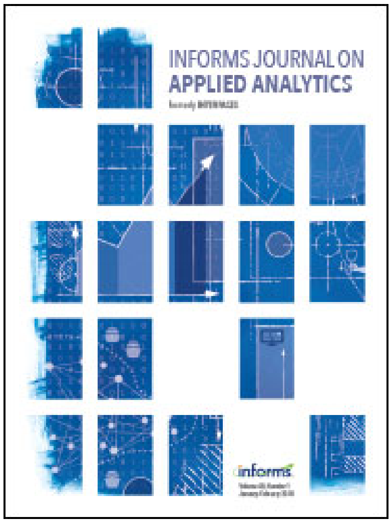

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Filling a Theater During the COVID-19 Pandemic

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Abstract. In the summer of 2020, Music Building Eindhoven (MBE) had to deal with the economic consequences of the COVID-19 pandemic for theater halls because governmental regulations were having a severe impact on the occupancy. In particular, MBE faced the challenge of determining how to maximize the number of guests in a theater hall while respecting social distancing rules. We have developed and implemented an optimization model based on trapezoid packings to address this challenge. The model showed that up to 40% of the normal capacity can be realized for a single show setting and up to 70% in cases where artists opt for two consecutive performances per evening without reusing seats. The solution was adopted by MBE with significant monetary and managerial benefits.

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Keywords: integer programming • COVID-19

Prologue

All around the world, the COVID-19 crisis has hit the cultural sector hard. Festivals are canceled, orchestras are at the brink of bankruptcy, choirs have stopped performing, and theaters are struggling to survive. Different countries or regions have imposed different rules in an attempt to stop the spread of the virus. We do not aim here to overview the precise (dynamic) contents of all these rules and their impact on the cultural sector; a number of descriptions of such rules and their impact can be found on governmental websites (e.g., see Bundesministerium für Gesundheit (2020) for Germany, Kaiser Family Foundation (2020) for the United States, Krisinformation.se (2020) for Sweden, NSW Government (2020) for Australia, and UK Department for Business, Energy & Industrial Strategy (2020) for the United Kingdom) and in other contributions, such as Jacobs (2020).

The situation in the Netherlands is not atypical from other countries or regions. Starting March 12, 2020, until June 1, 2020, all performances were canceled or suspended. From June 1 onward, a relaxation of the rules has allowed performances with at most 30 guests, as long as nonfamily members were seated at least 1.5 meters apart. The upper bound on the number of guests for indoor performances was eventually increased further to 100 on July 1, 2020; a description of the current rules can be found on the Government of the Netherlands (2020) website. Clearly, these rules have a dramatic impact on the operation of any theater, and despite governmental efforts, theaters are struggling to survive. As a consequence, many employees in this sector risk losing their jobs.

Indeed, for many theaters, the challenge is to find a way to welcome their guests while satisfying the distance rules and to still be commercially viable. Many creative efforts have resulted in a number of ideas that are being experimented with (e.g., the use of a so-called nebulizer device; see Greb and Wojcik (2020)). Here, we focus on the question to what extent large audiences can still be accommodated in a theater when distance rules must be satisfied. We describe and implement an optimization problem that, given the layout of the seats in a theater and the distribution of the demand, allows a theater to compute a safe seating arrangement that attains the maximum occupancy.

The Music Building Eindhoven (MBE), located in the city of Eindhoven in the Netherlands, features a "Grand Room" (1,250 seats) and a "Small Room" (400 seats). This theater has served as a motivation for this study, and all our computational efforts are based on its two theater rooms. Our findings have been implemented by MBE, allowing them to remain open (as long as governmental rules allow).

The rest of this paper is organized as follows. First, we give a precise description of the problem faced by MBE, and subsequently, we phrase the problem in terms of packing *trapezoids* and implement the associated mathematical models. We then discuss the outcomes of our models given input data provided by MBE and describe the benefits of the solutions of the model for MBE. Managerial suggestions conclude the paper.

Problem Description

Here, we describe crucial ingredients of the problem faced by MBE in a more general mathematical model. We first discuss seats, distances, and forbidden zones, and then we introduce the concept of *target profiles*; these allow us to arrive at our problem statement. For a more extensive description, we refer to Blom et al. (2020).

Seats, Distances, and Forbidden Zones

When a theater wants to offer a COVID-19–proof experience to its customers, a few constraints need to be taken into account. Obviously, safety is of utmost importance, and therefore, the subset of seats that can be used for reservations needs to be chosen according to the guidelines provided by the government. We realize that these guidelines vary for different countries. However, a common denominator between different countries is that members of distinct families (or *bubbles* or *households*) should keep a prespecified distance from each other to prevent the spread of the COVID-19 virus. In the Netherlands, this minimum distance is fixed to 1.5 meters, as established by the Dutch government (see Government of the Netherlands (2020)).

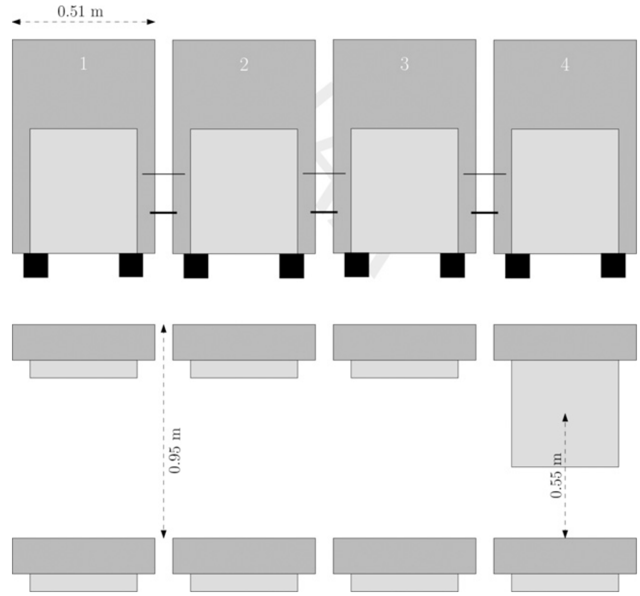
Figure 1 shows a sketch of four consecutive seats, viewed from the front and from above, and the corresponding interseat distances of seats in MBE. In particular, the *width* of a seat is 0.51 meters, the *row distance* is equal to 0.95 meters, and the distance between the midpoint of a seat and the back of the seat directly in front of it is 0.55 meters.

As distance rules do not apply to members of the same family, guests from the same family are allowed to sit next to each other, within the 1.5-meter bound. We assume from now on all members of the same family occupy consecutive seats on the same row. Furthermore, we associate with each seat a *row label* r, indicating the rth row (as seen from the stage) and its *position label* s in row r (as seen from the left side of the row). For reasons of convenience for our mathematical models, we assume that the seats in each row are numbered such that all seats with some position label s are positioned on a straight line. Figure 2 illustrates this convention for $s = 3$.

In many typical theaters (such as MBE), consecutive rows are staggered for reasons of visibility. This feature is also illustrated in Figure 2.

Based on the distances introduced in Figure 1 and using a separating distance of 1.5 meters (as stipulated by the Dutch government), a simple calculation reveals that when occupying a single seat, there is a "ring" of 12 other seats around it that are forbidden for use by a member of another family. In general, as

Figure 1. Front and Upper View of Four Seats in MBE, with Corresponding Measures



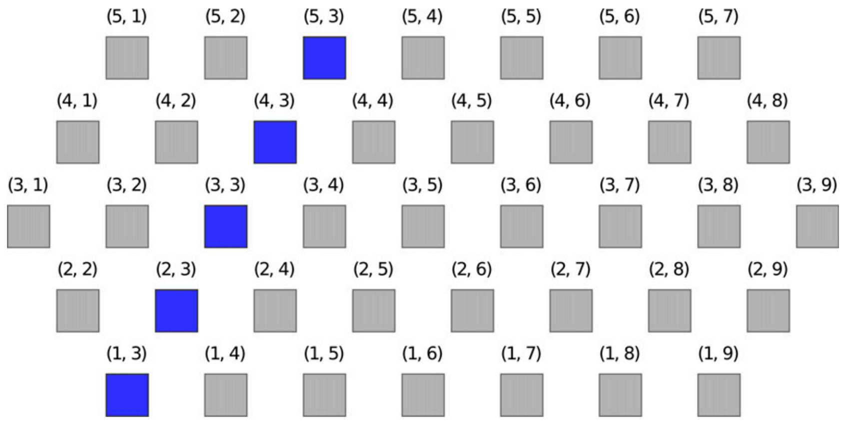
a family consisting of t members whose leftmost member is located at seat (r, s) will occupy the consecutive seats $(r,s),(r,s+1)$ up to and until $(r,s+t-1)$, the corresponding *forbidden zone* for a family of size t consists of $2t+10$ seats—namely, four seats in the same row (two on the left and two on the right and $t + 3$ in each of the adjacent rows). This calculation silently assumes that all these seats indeed exist—that the family of size t occupies seats in "the middle of" the theater. The forbidden zones corresponding to families of sizes 1, 2, and 3 are depicted by the crossed squares in Figure 3.

We view a *seating arrangement* as a set of labels (r, s, t), meaning a family of size t occupies consecutive seats starting from seat (r, s). It can be regarded as a plan to fill the theater within the social distancing rules. We call a seating arrangement *safe* if no two guests from different families are seated within 1.5 meters of each other—or, in other words, when no member of a family is in the forbidden zone of another family. A relevant property of a seating arrangement is the number of guests it contains, which we refer to as the *size* of the seating arrangement; thus, the size of a seating arrangement is nothing else but the number of guests present in the theater.

Seating Decisions

Apart from providing a safe environment for the audience while enjoying a performance, a theater needs to consider its booking strategy. In general, multiple factors play a role when deciding on such a strategy (see Baldin and Bille (2018) and the references contained therein). One option is to sell the individual seats (perhaps after segmentation into classes) chosen by

Figure 2. (Color online) Numbering of Seats in Which Seats with Position $s = 3$ in Each Row Are Situated on a Straight Line



customers in a first-come, first-serve manner. The risk of such a strategy is that customers choose seats that do not lead to a maximum *occupancy*. Another option is to simply sell tickets and only reveal very shortly before the start of the performance which particular seats are assigned to which individual customers. This allows the theater flexibility to find a maximum occupancy, yet customers might find it unattractive not to be able to choose their specific seats. Without going into the details of the various considerations, we have opted, in collaboration with MBE, for a policy that (i) allows customers to choose their seats and (ii) uses a target profile to take the size of families visiting the performance into account.

Let us elaborate on this notion of target profiles. There are different types of shows at MBE (i.e., jazz, popular, or classic) that have different types of audiences. Indeed, one can well imagine that the type of show determines the relative frequencies with which families of sizes 1, 2, 3, and 4 are present in the audience. As an example, a show suitable for families with small children will have a larger frequency of families of size 4 in the audience than a show featuring classical music. We will assume that the relative frequency of the family sizes is prespecified. Thus, as an example, we assume, for a particular show, that the audience consists of 10% families of size 1, 70% families of size 2, 5% families of size 3, and 15% families of size 4. When finding our seating arrangement, we are allowed to deviate from these frequencies by a given, small amount. Such a target profile can be determined through statistical analysis or machine learning models applied on historical data. We return to this issue when discussing the experimental outcomes.

In summary, we have arrived at the following optimization problem: given the theater characteristics, the separating distance, and a target profile,

maximize the number of customers present in the theater (size of a seating arrangement)

by selecting occupied/unoccupied seats

subject to COVID-19 distancing and family-group constraints.

Trapezoid Packings

In this section, we establish a nontrivial connection between finding a safe seating arrangement and the problem of packing a maximum number of *trapezoids* in a polygonal shape directly behind (r, s). As an example, consider the seat $(r,s) = (3,3)$. The trapezoid based at seat (3, 3) is the collection of seats formed by the seat (3, 3) itself, the seat (3, 2) directly on the left,

Figure 3. (Color online) Crossed Seats Form the Forbidden Zone Whenever a Family of Size 1, 2, and 3 Occupies the Middle Seat(s)

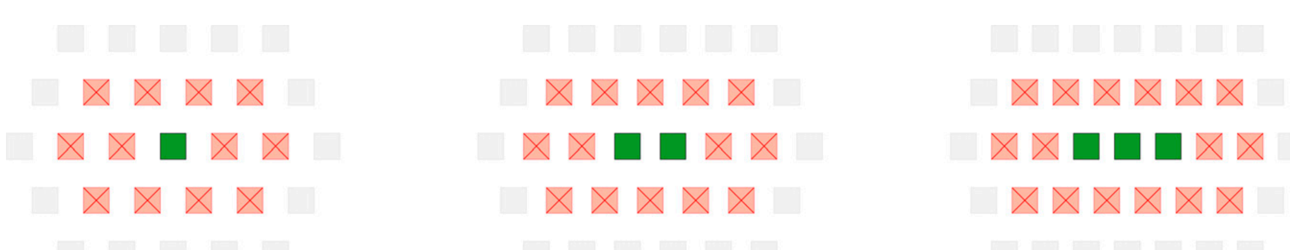
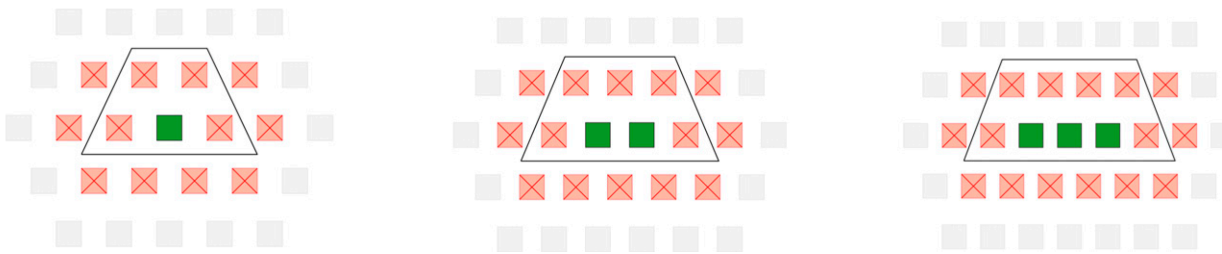


Figure 4. (Color online) Trapezoids Corresponding to Families of Sizes 1, 2, and 3



seat (3, 4) directly on the right, and seats (4, 2), (4, 3) directly behind it. In summary, the corresponding trapezoid consists of the collection of seats

$$\{(3,2),(3,3),(3,4),(4,2),(4,3)\}.$$

This notion directly translates to the trapezoid corresponding to a set of t consecutive seats. Observe that a trapezoid for a family of size t contains $2t+3$ seats (again assuming we consider a family in the middle of the theater). As an illustration, we depict in Figure 4 trapezoids that correspond to families of sizes 1, 2, and 3.

Observation 1. A collection of *pairwise disjoint* trapezoids corresponds to a seating arrangement that is safe, and vice versa. We refer to the appendix, where we phrase this observation as Theorem A.1.

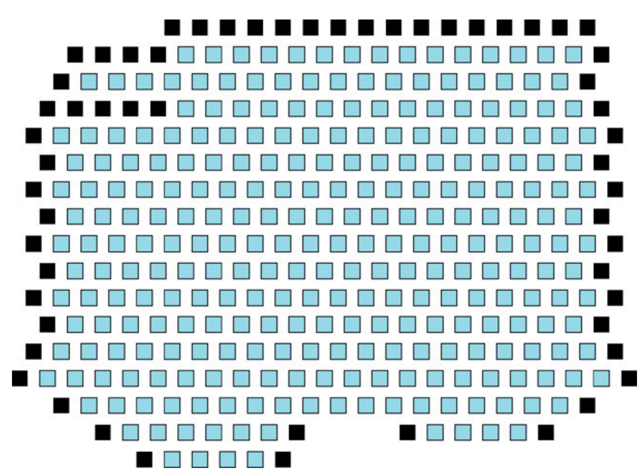
Thus, when (virtually) placing trapezoids around the seats in a theater such that there is no intersection between any pair, one has found a safe seating arrangement. This statement is illustrated in Figure 5, where the trapezoids are placed in a regular pattern—the green seats will be occupied by members of size 2 families (*pairs*) only. Because the corresponding trapezoids do not have an overlap, this constitutes a safe seating arrangement. This observation forms the basis of our *integer programming* models, which solve the main problem of MBE and which, in principle, apply to theaters with a set of seats positioned in any given shape or irregular form.

Figure 5. (Color online) Safe Seating Arrangement as a Collection of Pairwise Disjoint Trapezoids

On the Capacity Induced by a Separating Distance

Under normal circumstances, it is clear how guests of a theater use the available resources: each guest needs one seat. With social distancing rules, it is not immediately clear to what extent guests claim the resources. There will be many empty seats in any safe seating arrangement, and it is not obvious which guest to blame or charge. However, the equivalence between a safe seating arrangement and a disjoint packing of trapezoids reveals that each family with t members at (r, s) blocks the seats within its associated trapezoid and so is responsible at least for the emptiness of these seats. Therefore, a family of size t blocks at most $2t+3$ seats. That means that a family of size 1 blocks at most 4 seats (while paying for 1), whereas a family of size 4 blocks at most 11 seats (while paying for 4). This is

Figure 6. (Color online) Floor Plan of the Ground Floor of the Grand Room of MBE



Note. The set of actual seats in the Grand Room is light-colored, with the virtual rim of seats being dark-colored.

also an indication that it is more beneficial to welcome larger families because a large family blocks relatively fewer seats compared with a small family.

To precisely analyze the impact of the boundary seats on the capacity, we need to take the so-called *rim* into account. We define the rim as the set of "virtual seats" that do not actually exist but that do occur in the trapezoid associated with an actual seat. We refer to Figure 6 for an illustration of the rim of the Grand Room of MBE. Clearly, for large theaters with a relatively simple boundary, adding the rim makes the set of seats only marginally larger. This basically means that for large regular theaters, there is only a marginal effect of the boundary on the capacity. By intersecting an optimally dense regular arrangement such as in Figure 5 with the actual layout of such a large and regular theater, one obtains a seating arrangement whose occupancy—the percentage of seats that are occupied by a customer—tends to the theoretical maximum attained by the regular arrangement (see Blom et al. (2020)). This construction will yield satisfactory seating arrangements for, say, a stadium, but for an irregular theater of intermediate size, such as MBE, a more careful analysis is in order. Using integer programming, we will provide just that.

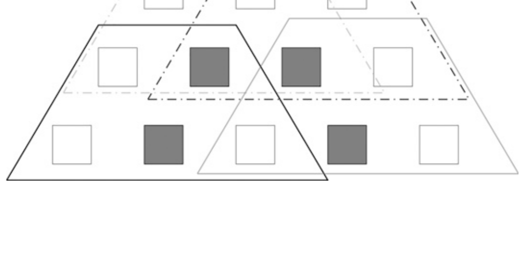
An Integer Programming Model to Find Maximum Size Seating Arrangements

We describe the basic ingredients of an *integer linear programming* (ILP) formulation for our problem. The complete formulation can be found in the appendix.

Building the Model

Placing a trapezoid that corresponds to a family of size t, based at seat (r, s), is a decision that can be modeled using a binary variable, $y_{r,s,t}$. Indeed, we set $y_{r,s,t}$ to be 1 if and only if (r, s, t) is part of the seating arrangement. Using these variables, it is straightforward to come up with constraints that ensure the disjointedness of the trapezoids, thereby guaranteeing a safe seating arrangement. In addition, proximity to a given target profile can be formulated; we refer to the appendix for the corresponding mathematical formulation.

Figure 7. Only One of the Four Trapezoids Corresponding to Size 1 Families on the Gray Seats Can Be Selected



Consecutive Shows

One of the ideas that MBE (along with many other theaters) has implemented to remain commercially viable is to perform the same show during the same evening twice, each time for a different audience. We refer to this phenomenon as *consecutive shows*. Clearly, this puts a burden on the performing artist(s), but in many cases, this is a realistic option. MBE, however, is not able to clean the seats in between the shows. There is simply not enough time and manpower between consecutive shows to perform this task. This creates an interdependence between the two seating arrangements for each individual show because, for obvious reasons, each seat can be used at most once in each of the two seating arrangements. It is, however, relatively straightforward to extend the ILP formulation to accommodate consecutive shows; we refer to the appendix for more details.

Speeding Up the Solution Process

For some instances of our integer linear programming formulation, the corresponding *linear programming* (LP) relaxation leads to long running times of the solver. We will now mention two methods that we implemented to improve solver performance.

Strengthening the Linear Relaxation. Our model contains a constraint stating that for each seat (r, s) in the theater, only one trapezoid including (r, s) can be picked in our packing. We add a class of valid inequalities to our formulation that exploits the structure of trapezoids even more: for any triple of seats, one can pick only one trapezoid that contains at least two seats of the triple. Figure 7 illustrates this new class of constraints.

Symmetry-Breaking Techniques. The presence of symmetry in a (mixed) ILP formulation often poses a computational challenge (e.g., Margot 2010 and Hojny and Pfetsch 2019). Indeed, naive implementations can be unsuccessful, as many equivalent problems need to be solved in the branch-and-bound procedure to ensure optimality. We have identified valid inequalities that, for the case of consecutive shows, remove the symmetry that arises from interchanging seating arrangements between consecutive shows.

Implementation Details

We now describe the details of the implementation of the model we developed. First, we provide a timeline of our contact with MBE. Next, we solve our models and report on the computational outcomes for the theater rooms of MBE. These results gave rise to further considerations because MBE wanted to investigate the impact of seating arrangements that leave every second row empty, as such solutions are more practical from a logistical point of view. We conclude by comparing both settings, thereby providing insights that help MBE to decide which policy to pursue.

Timeline

The first contact with the management of MBE was established in early May 2020. We agreed on the basic structure of our models and their required input—the target profiles. From then on, we provided regular updates on our work, and two months later, we presented our preliminary findings to an MBE team that included the booking director and the manager of operations. This resulted in a follow-up question regarding the impact on the occupancy of seating arrangements in which rows are used alternatingly. We adjusted our models accordingly and communicated the results to the management of MBE.

Preliminary Analysis

We implemented our integer linear programming models in Julia 1.3.0, using the modeling language JuMP to build the optimization model, with Gurobi as the lower-level LP and ILP solver. Experiments were run on a computer equipped with an Intel Core i7-7700HQ CPU @ 2.8 GHz with 32 GB of RAM.

In order to obtain an indication of the relative frequencies of family sizes to consider, the management of MBE selected 30 shows, mainly from the year 2019, that it considered representative. As MBE keeps track of the sizes of the families that visit the shows in MBE, it is straightforward to use these historical data to obtain average relative frequencies. More concretely, based on these data, we found that 18% of the families present in the audience are of size 1, 70% are of size 2, 6% are of size 3, and 6% are of size 4. This is reflected in target profile `mge1`. Furthermore, the management of MBE designed three other target profiles (`mge2`, `mge3`, and `mge4`) to study the effect of other particular compositions of family sizes. In summary, we considered four different target profiles that specify the relative frequencies of families of sizes 1, 2, 3, and 4:

- Historical data on reservations: `mge1` : (0.18, 0.7, 0.06, 0.06)
- Families of size 2 only: `mge2` : (0, 1, 0, 0)
- Families of sizes 1 and 2: `mge3` : (0.2, 0.8, 0, 0)
- Families of sizes 2 and 4: `mge4` : (0, 0.5, 0, 0.5)

Given such a target profile, we now can solve our integer programming models for both the Grand Room and the Small Room. To find out the impact of having consecutive shows, we consider two scenarios: a single show or consecutive shows. Both the basic versions of both models (`vanilla`) and the versions with the speedup techniques (`speedup`) are considered and compared.

Let us define the *occupancy* of a seating arrangement $\mathcal{A}$ as the percentage of all seats occupied by a customer and denote it by $occ(\mathcal{A})$. Tables 1 and 2 provide the occupancies $occ(\mathcal{A})$ for the Grand Room and the Small Room, respectively, where $\mathcal{A}$ is an optimal safe seating arrangement with respect to the corresponding target profile, for both the single show and the consecutive show cases.

Let us first comment on the occupancies found in Tables 1 and 2 (Grand Room and Small Room, respectively). For each of the four target profiles, the differences in occupancy between the Grand Room and the Small Room are small, for both the single show and for the consecutive show situations. This is to be expected, as the interseat distances from Figure 1 apply to both rooms. Also, in the case of a single show, the occupancies found are rather similar for the four different target profiles, with the exception of `mge4`. Target profile `mge4` has a relatively large fraction of families of size 4 (the largest family size considered), which is beneficial for finding seating arrangements with a large occupancy. However, in the case of a single show, all profiles allow an occupancy of about 33%—this corresponds to a setting with one-third of the seats being occupied in a single show.

When analyzing the outcomes for consecutive shows, we observe that the presence of large families (`mge4`) leads to better occupancies—this effect is more pronounced than it is for a single show. Another interesting observation is that the occupancies almost double when compared with a single show. Hence, the effect of the constraint that a seat can be used at most

Table 1. Occupancies $occ(\mathcal{A})$ (in Percent) of Maximum Safe Seating Arrangements $\mathcal{A}$ in the Grand Room, According to the Target Profiles

	Single show			Consecutive show		
Target profile	Occupancy (%)	`vanilla`	`speedup`	Occupancy (%)	`vanilla`	`speedup`
`mge1`	32	3.39	1.50	63	532.69	48.28
`mge2`	29	0.28	0.10	56	6.67	2.49
`mge3`	30	1.39	0.97	58	2,107.68	6.05
`mge4`	36	5.29	1.10	70	4,485.33	726.11

Note. The reported numbers in the columns "`vanilla`" and "`speedup`" represent time in seconds (rounded to two decimal places).

Table 2. Occupancies $occ(\mathcal{A})$ (in Percent) of Maximum Safe Seating Arrangements $\mathcal{A}$ in the Small Room, According to the Target Profiles

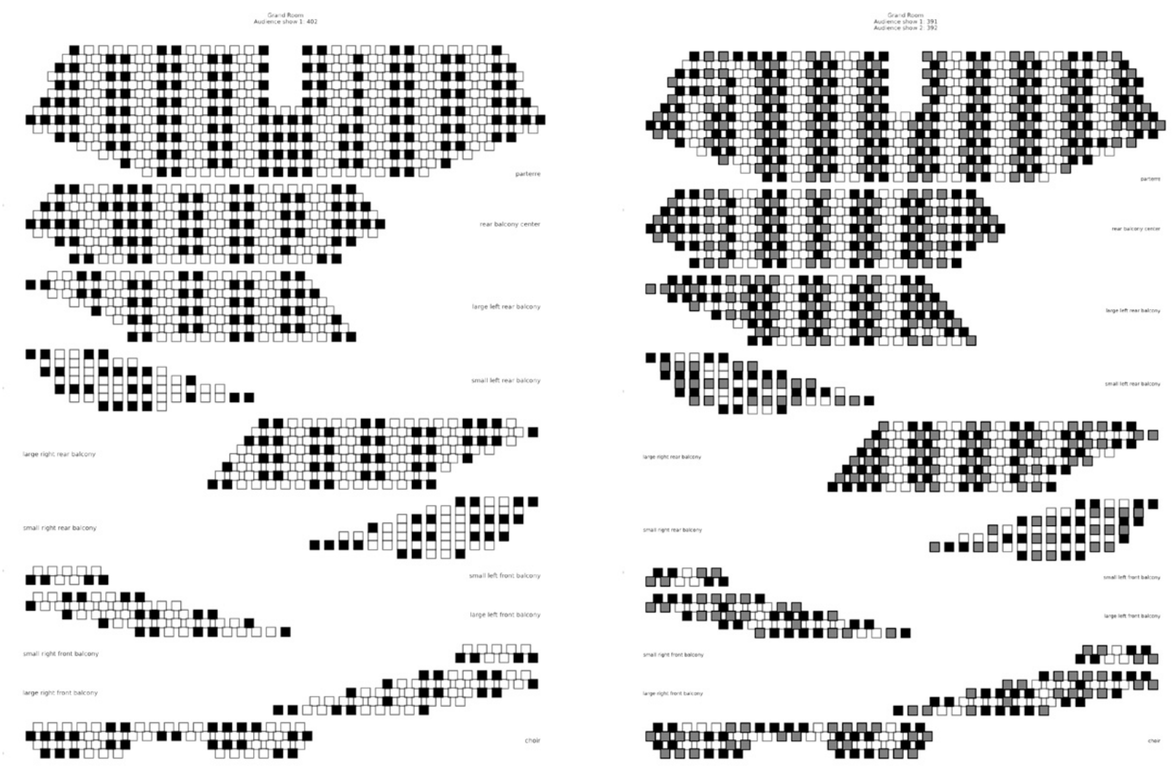
	Single show			Consecutive show		
Target profile	Occupancy (%)	`vanilla`	`speedup`	Occupancy (%)	`vanilla`	`speedup`
`mge1`	34	1.19	0.41	64	8.18	13.82
`mge2`	31	0.02	0.02	58	0.30	0.35
`mge3`	31	0.22	0.07	59	2.19	0.89
`mge4`	37	0.08	0.11	70	5.46	9.17

Note. The reported numbers in the columns "`vanilla`" and "`speedup`" represent time in seconds (rounded to two decimal places).

once in two shows is negligible; in other words, the model is able to find two single-show seating arrangements with no seats in common such that the numbers of seats occupied in both shows is (almost) balanced. For the target profile based on historical data, `mge1`, the model is able to find seating arrangements that use almost two-thirds of the available seats. This is an important finding, as it gives MBE an idea of the consequences of having consecutive shows. The optimal safe seating arrangements for the Grand Room, for both the single show setting and the setting with consecutive shows, are illustrated in Figure 8. Here, we use white squares to indicate seats that are forbidden for use by customers and black (black and gray for two consecutive shows) squares to indicate seats that can be occupied by customers.

Let us now comment on the computation times. In particular, we see that adding a second show to the model drastically increases the computation time of the solver, which can probably be explained by the fact that additional symmetries are introduced in the problem by adding a second show and that the number of variables and constraints both increase linearly in $|\mathcal{S}|$. Furthermore, the choice for the target profile also largely influences the running time of the algorithm. It is striking to see that the instances for which

Figure 8. Visualization of Optimal Safe Seating Arrangements for the Grand Room of MBE with Target Profile `mge1`



(a) (b)

Notes. (a) Maximum number of occupied seats (in black) of the Grand Room for a single show with target profile `mge1`. (b) Maximum number of occupied seats (in black and gray) of the Grand Room for two consecutive shows with target profile `mge1`.

the algorithm has the worst performance are also the ones for which the target profiles are further away from intuitively optimal—that is, relatively large proportions of small families and relatively small proportions of large families. For the instances of the Grand Room, we see that the impact of adding the speedup techniques is rather large. This can be explained by the fact that these instances have a rich variety of symmetries.

Further Considerations During Implementation

After having found solutions as depicted in Figure 8, we discussed extensively with the management of MBE about the practicality of these solutions. Indeed, in such a solution, there are occupied seats on each row of the theater, and as a consequence, a relevant question becomes how to steer the guests to their seats. An irregular solution—that is, a solution in which occupied seats are present in each row—is quite labor intensive to uphold. The staff needs to pay attention to ensure that guests will sit in the designated seats. The management of MBE posed the question to what extent a solution that consists of alternating empty rows would decrease the number of guests. Indeed, for such solutions featuring alternate empty rows, it will be much easier to guide guests to their seats, and in fact, it becomes possible to rearrange the order in which families take place in their row on the fly. This translates into a lesser need for personnel to host guests. Also, because the safety of an arrangement depends only on a condition within each row, finding safe arrangements becomes so easy that it almost can be done manually or with software applying straightforward strategies. Without the need to incorporate a "black box" advanced solver in the process of selling seats, the flexibility of this process may be greatly increased.

Tables 3 and 4 report for the Grand Room and the Small Room, respectively, the optimum occupancies of safe seating arrangements $\mathcal{A}'$ that have the property of using only one of two consecutive rows at a time per show. For the consecutive show case, this means in the first show, all odd-numbered rows could be used and all even-numbered rows in the second show. We only report computation times for the `vanilla` descriptions, as the speedup techniques now only yield redundant inequalities. The column called "Loss (%)" indicates the percentage loss of occupied seats, which can be seen as a proxy for the loss in revenue.

Clearly, as the results in Tables 3 and 4 correspond to a more restricted setting of our problem, the realized occupancies are always smaller than those achieved for the setting where all rows can be used for all shows. Indeed, we observe that for all instances, especially the ones based on the Small Room, the percentage loss of occupied seats is not insignificant. However, for the Grand Room, losses are always bounded by 10%.

Computation times for this setting are much smaller. This is caused by the much smaller size of the resulting instances and much fewer dependencies between the variables.

The solutions that correspond to the occupancies for the target profile `mge1` in which rows are used in an alternating fashion are given in Figure 9.

Managerial Benefits

The results of our models allowed MBE to draw a number of conclusions:

- On the basis of the outcomes of our models (see Tables 1–4), MBE concluded that consecutive shows almost double the occupancy.
- The impossibility of cleaning between consecutive shows poses no obstacle for organizing consecutive shows.
- Solutions with alternating empty rows greatly facilitate logistical efficiency, whereas the decrease in the number of guests remains bounded by 10%.

These conclusions have convinced the management of MBE to actively pursue a strategy of having consecutive shows. In an evaluation session in August 2020, it also became clear that MBE generally chose to organize consecutive shows with seating arrangements that use rows alternatingly.

Conclusion

The 1.5-meter constraint has a huge impact on the occupancy when filling a theater. In the case of a typical theater such as MBE, occupancy of its rooms will not exceed 40% when performing a single show on an

Table 3. Occupancies $occ(\mathcal{A}')$ (in Percent) of Maximum Safe Seating Arrangements $\mathcal{A}'$ in the Grand Room, According to the Target Profiles and Alternating Empty Rows

	Single show			Consecutive show		
Target profile	Occupancy (%)	Loss (%)	`vanilla` (s)	Occupancy (%)	Loss (%)	`vanilla` (s)
`mge1`	29	−8.5	0.15	57	−9.4	0.33
`mge2`	27	−7.8	0.01	52	−7.5	0.05
`mge3`	27	−9.1	0.05	52	−9.8	0.07
`mge4`	34	−5.8	0.04	65	−6.0	0.12

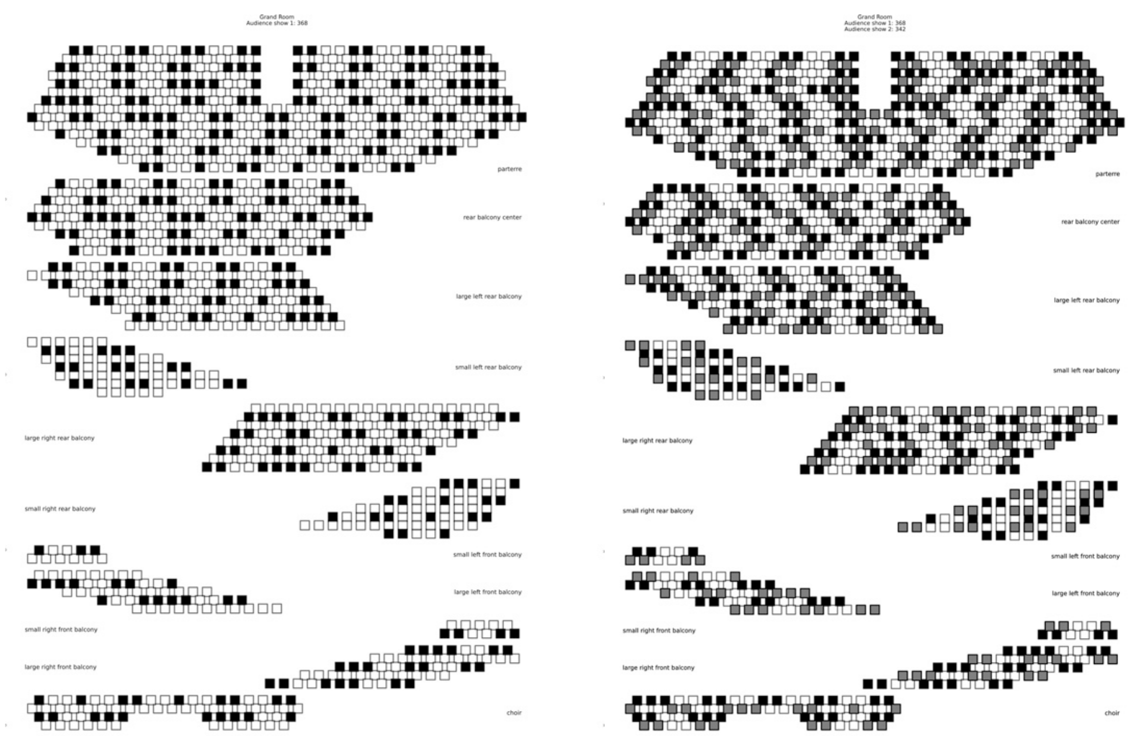
Table 4. Occupancies $occ(\mathcal{A}')$ (in Percent) of Maximum Safe Seating Arrangements $\mathcal{A}'$ in the Small Room, According to the Target Profiles and Alternating Empty Rows

	Single show			Consecutive show		
Target profile	Occupancy (%)	Loss (%)	`vanilla` (s)	Occupancy (%)	Loss (%)	`vanilla` (s)
`mge1`	30	−15.7	0.06	58	−9.8	0.11
`mge2`	26	−14.8	0.00	52	−9.6	0.01
`mge3`	26	−16.8	0.02	52	−11.8	0.08
`mge4`	34	−13.5	0.01	66	−5.7	0.03

evening. However, by allowing two consecutive shows per evening, outcomes of our models show that it is possible to reach an occupancy of 70% while satisfying the constraint that no seat is used twice during an evening. Even more, our models estimate that the impact of imposing solutions that allow the efficient handling of seating guests by using alternating empty rows comes at the cost of losing between 5% and 10% of the number of occupied seats. These insights and the models they are based on, together with other innovations, may offer some hope for theaters to remain competitive.

There are a number of possible avenues to further explore. One is to consider the existing segments in the theater and their corresponding prices. Indeed, different seat grades exist in MBE (balcony seats are priced differently from last-row seats), and from a revenue-maximizing point of view, it makes sense to incorporate these prices and develop a weighted version of our ILP models. Another issue is the usage of our models in a dynamic context. Here, the seat allocation can be dynamically updated during the booking process, thereby achieving a maximum flexibility. This allows MBE to learn the target profile during the booking process. One may even combine these two issues and use the prices to steer the target profile. Both of these options have been discussed with the management of MBE and may become more relevant once stability in the COVID-19 measures has been achieved.

Figure 9. Visualization of Optimal Safe Seating Arrangements for the Grand Room of MBE with Target Profile `mge1` and Alternating Empty Rows



(a) (b)

Notes. (a) Maximum number of occupied seats (in black) of the Grand Room using a single show with the target profile `mge1`. (b) Maximum number of occupied seats (in black and gray) of the Grand Room using two consecutive shows with the target profile `mge1`.

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Appendix
Let us denote by $\mathcal{F}_{r',s'}$ ($\mathcal{F}_{r',s',t'}$) the set of "forbidden" seats for other families whenever (r',s') is occupied by a family (of size t).

Theorem A.1. *Let $\mathcal{A} \subseteq \mathcal{S} \times T$ be a seating arrangement. Then, $\mathcal{A}$ is safe if and only if*

$$\{\mathcal{T}_{r,s,t} : (r,s,t) \in \mathcal{A}\}$$

is a collection of pairwise disjoint trapezoids.

Proof. The proof takes the form of two lemmas.

Lemma A.1. *Let $(r,s),(r',s') \in \mathcal{S}$. Then, $(r,s) \in \mathcal{F}_{r',s'} \iff \mathcal{T}_{r,s} \cap \mathcal{T}_{r',s'} \neq \emptyset$.*

Proof of Lemma A.1. *Necessity:* Suppose $(r,s) \in \mathcal{F}_{r',s'}$. Because $\mathcal{F}_{r',s'} = (r',s') + \mathcal{F} = (r',s') + \mathcal{T} + (-\mathcal{T})$, by Minkowski, it follows that there are $(u,v),(u',v') \in \mathcal{T}$ so that $(r,s) = (r',s') + (u',v') - (u,v)$. Then,

$$\mathcal{T}_{r,s} \ni (r,s) + (u,v) = (r',s') + (u',v') \in \mathcal{T}_{r',s'},$$

so that $\mathcal{T}_{r,s} \cap \mathcal{T}_{r',s'} \neq \emptyset$, as required.

Sufficiency: Suppose $\mathcal{T}_{r,s} \cap \mathcal{T}_{r',s'} \neq \emptyset$. Then, $(r,s) + (u,v) = (r',s') + (u',v')$ for some $(u,v),(u',v') \in \mathcal{T}$. Then, $(r,s) = (r',s') + (u',v') - (u,v) \in (r,s) + \mathcal{T} + (-\mathcal{T}) = \mathcal{F}_{r',s'}$, as required. □

Lemma A.2. *Let $(r,s,t),(r',s',t') \in \mathcal{S} \times T$. Then,*

$$\mathcal{S}_{r,s,t} \cap \mathcal{F}_{r',s',t'} \neq \emptyset \iff \mathcal{T}_{r,s,t} \cap \mathcal{T}_{r',s',t'} \neq \emptyset.$$

Proof of Lemma A.2. We have $\mathcal{S}_{r,s,t} \cap \mathcal{F}_{r',s',t'} \neq \emptyset$ if and only if there are $i \in \{0,\ldots,t-1\}, i' \in \{0,\ldots,t'-1\}$, so that $(r,s+i) \in \mathcal{F}_{r',s'+i'}$. By Lemma A.1, this is equivalent to

$$\mathcal{T}_{r,s+i} \cap \mathcal{T}_{r',s'+i'} \neq \emptyset \text{ for some } i \in \{0,\ldots,t-1\}, i' \in \{0,\ldots,t'-1\}.$$

In turn, this is equivalent to $\mathcal{T}_{r,s,t} \cap \mathcal{T}_{r',s',t'} \neq \emptyset$. □

By Lemma A.2, we may replace the asymmetrical condition $\mathcal{S}_{r,s,t} \cap \mathcal{F}_{r',s',t'} = \emptyset$ in the definition of a safe arrangement by the equivalent symmetrical condition $\mathcal{T}_{r,s,t} \cap \mathcal{T}_{r',s',t'} = \emptyset$. This proves Theorem A.1. □

The IP Formulation (Single Show)
With any collection $\mathcal{A} \subseteq \mathcal{S} \times T$, where $T \subseteq \mathbb{N}_+$ is a finite collection of allowed family sizes, we can associate a characteristic vector $\mathbf{y} \in \{0,1\}^{\mathcal{S}\times T}$ with $y_{r,s,t} = 1$ if and only if $(r,s,t) \in \mathcal{A}$. Then, $\mathcal{A}$ is a seating arrangement in $\mathcal{S}$ if and only if

$$y_{r,s,t} = 0 \qquad \text{whenever } \mathcal{S}_{r,s,t} \subseteq \mathcal{S}. \tag{A.1}$$

This seating arrangement $\mathcal{A}$ is safe if and only if

$$\sum_{(r',s',t'):\mathcal{T}_{r',s',t'} \ni (r,s)} y_{r',s',t'} \le 1, \qquad \text{for each } (r,s) \in \mathcal{S} + \mathcal{T}. \tag{A.2}$$

From a geometric point of view, Constraint (A.2) ensures that each seat $(r,s) \in \mathcal{S} + \mathcal{T}$ is covered by at most one of the trapezoids in which it is contained. Finally, $\mathcal{A}$ accommodates n_t families of each size $t \in T$ if

$$\sum_{(r,s):\mathcal{S}_{r,s,t} \subseteq \mathcal{S}} y_{r,s,t} = n_t, \qquad \text{for each } t \in T. \tag{A.3}$$

Thus, the feasibility of a safe seating arrangement that simultaneously accommodates n_t families of size t for $t \in T$ translates to an integer linear feasibility problem in variables $y_{r,s,t}$ and n_t. However, without a priori conditions on the number of families of each size t, the optimal solutions of this problem will tend toward including many large families and few small families. This is intuitively clear, because a family of t together "wastes" a trapezoid of $2t+3$ seats, so that $2 + 3/t (= 1/d_t)$ seats are taken per person in a family of size t. In the extreme case that T includes large enough sizes to fill entire rows of seats with a single family, then a solution in which the even rows are empty and each odd row is filled with a single family is feasible—similar for leaving the odd rows empty and filling the even rows. One of these solutions then is optimal and uses at least half of the seats in $\mathcal{S}$. Indeed, now that we are letting our imagination roam free, we can fill the entire theater with a single large enough family if we also let go of our restriction that families must be seated in the same row. To ensure that we find safe seating arrangements that approximately correspond to the typical sizes of families that book seats for a performance, we use the target profile. Recall from the problem statement that the target profile imposes the condition

$$(p_t - \epsilon)\sum_{t\in T} n_t \le n_t \le (p_t + \epsilon)\sum_{t\in T} n_t \qquad \text{for each } t \in T. \tag{A.4}$$

In this way, we obtain an integer linear program that maximizes the size of a seating arrangement over all safe seating arrangements in $\mathcal{S}$:

$$\max\left\{\sum_{t\in T} t n_t : \text{(A.1), (A.2), (A.3), (A.4)}, \quad \mathbf{y} \in \{0,1\}^{\mathcal{S}\times T}, \quad \mathbf{n} \in \mathbb{Z}^T\right\}. \tag{A.5}$$

Notice that the LP relaxation of Problem (A.5) gives an upper bound that, by the safety constraints (A.2), is informed that each family of size t occupies at least $2t+3$ seats from $\mathcal{S} + \mathcal{T}$.

The IP Formulation (Consecutive Shows)
However, it is relatively straightforward to extend our model to find k consecutive seating arrangements $\mathcal{A}_v$ for $v \in V = \{1,\ldots,k\}, k \in \mathbb{Z}_+$, so that no seat is used in two different arrangements—that is, if $v, v' \in V$ are distinct, then

$$\mathcal{S}_{r,s,t} \cap \mathcal{S}_{r',s',t'} = \emptyset$$

for all $(r,s,t) \in \mathcal{A}_v$ and $(r',s',t') \in \mathcal{A}_{v'}$.

To model the problem of finding such consecutive seating arrangements, we use binary variables $\mathbf{y} \in \{0,1\}^{\mathcal{S}\times T\times V}$ and integer variables $\mathbf{n} \in \mathbb{Z}^T$. The condition that each $\mathcal{A}_v$ is a seating arrangement of $\mathcal{S}$ becomes

$$y_{r,s,t,v} = 0, \qquad \text{whenever } \mathcal{S}_{r,s,t} \subseteq \mathcal{S}. \tag{A.6}$$

The safety of each $\mathcal{A}_v$ is modeled by

$$\sum_{(r',s',t'):\mathcal{T}_{r',s',t'} \ni (r,s)} y_{r',s',t',v} \leq 1, \qquad \text{for each } (r,s) \in \mathcal{S} + \mathcal{T}, v \in V. \tag{A.7}$$

We also need to ensure that no seat is used more than once:

$$\sum_{v \in V} \sum_{(r',s',t'):\mathcal{S}_{r',s',t'} \ni (r,s)} y_{r',s',t',v} \leq 1, \qquad \text{for each } (r,s) \in \mathcal{S}. \tag{A.8}$$

Letting the n_t count the overall number of families of size t is accomplished by writing

$$\sum_{v \in V} \sum_{(r,s) \in \mathcal{S}} y_{r,s,t,v} = n_t, \qquad \text{for each } t \in T. \tag{A.9}$$

The profiling condition (A.4) need not change at all.

Maximizing the number of guests in consecutive arrangements in $\mathcal{S}$ while respecting a profile $\boldsymbol{p} \in \mathbb{R}^T$ up to a fixed $\epsilon > 0$ is then modeled as the following ILP:

$$\max\left\{\sum_{t \in T} t n_t : \text{(A.4), (A.6), (A.7), (A.8), (A.9)}, \; \mathbf{y} \in \{0,1\}^{\mathcal{S}\times T \times V}, \;\; \mathbf{n} \in \mathbb{Z}^T\right\}. \tag{A.10}$$

This model is rather flexible: many additional wishes can be formulated. For instance, upper bounds on n_t for some $t \in T$, or a balance between the distribution in different shows, or specific (monetary) weights to maximize the revenue that could be gained, seats can all be arranged through standard modifications of the integer linear program.

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Verification Letter

Wim Vringer, Managing Director, Muziekgebouw Frits Philips Eindhoven, Heuvel 140, 5611 DK Eindhoven, Netherlands, writes:

"With this letter I wish to confirm that there has been an intense collaboration between our organization, concert hall Muziekgebouw Frits Philips Eindhoven, and Frits Spieksma from Eindhoven University of Technology and his team from the Mathematics and Computer Science Department. The goal of this collaboration was to calculate the maximum seat occupancy for our two concert halls and the general public areas, in the new circumstances that the Covid-19 virus caused. The process and outcome of the analyses are well explained in the manuscript 'Packing Theatres.'

"— The mathematic analyses helped us create business models which make it possible to sell the maximum amount of tickets, with variable composed audiences. Organizing two shows per evening, it is now possible to reach an occupancy of 70% while no seat is sold twice. We incorporated the models in our ticket system.

"— In addition, the calculations helped us to reorganize the public areas and create a safe and spacious environment for our guests.

"We thank Prof. Spieksma for the pleasant cooperation and wish him and his team all the best for the future."

Danny Blom is a PhD candidate in the Combinatorial Optimization group at Eindhoven University of Technology. His research focuses on mathematical models for various challenges in the domain of kidney exchange. In particular, he is interested in mechanisms for transnational collaborations in kidney exchange and applying techniques from robust optimization to this problem field.

Rudi Pendavingh is affiliated with the Department of Mathematics and Computer Science at Eindhoven University of Technology. He obtained a PhD on the topic of

topological graph theory and has worked on very diverse topics in combinatorics and geometry since. In recent years, his research has focused on matroid theory.

Frits Spieksma is a full professor of combinatorial optimization in the Department of Mathematics and Computer Science at Eindhoven University of Technology. His key field of expertise is operations research, especially combinatorial optimization problems and applications thereof. He combines an interest in theoretical results with the desire to affect practice. Within his field, Frits is particularly interested in various scheduling, routing, and clustering problems.