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A mathematical analysis of fairness in shootouts

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A shootout is a popular mechanism to identify a winner of a match between two teams. It consists of rounds in which each team gets, sequentially, an opportunity to score a point. It has been shown empirically that shooting first or shooting second in a round has an impact on the scoring probability. This raises a fairness question: is it possible to specify a sequence such that identical teams have equal chance of winning? We show that, for a sudden death, no repetitive sequence can be fair. In addition, we show that the so-called Prohuet–Thue–Morse sequence is not fair. There is, however, an algorithm that outputs a fair sequence whenever one exists. We also analyze the popular best-of-k shootouts and show that no fair sequence exists in this situation. In addition, we find explicit expressions for the degree of unfairness in a best-of-k shootout; this allows sports administrators to asses the effect of the length of the shootout on the degree of unfairness.

Keywords: sports; fairness; shootout; prouhet-thue-morse sequence.

1. Introduction

We consider the following situation. Two teams, called A and B, play a match. To decide upon a winner of an otherwise tied game, a so-called *shootout* takes place. The shootout has two phases. Phase 1 consists of k rounds, and in every round each of the teams shoots once (what it means to shoot, depends on the particular sport). Shooting leads either to success (scoring) or to failure (missing). The team that has the most successes after Phase 1 (which we call a best-of-k) wins the match. If both teams have the same number of successes, the shootout continues with Phase 2 (which we call sudden death). This phase consists of individual rounds and ends only when in a particular round one team scores and the other does not.

Many popular sports use shootouts to identify a winner, though the setup can differ per sport. For example, in football, FIFA rules FIFARules prescribe that in each round the same team shoots first both in Phase 1 as well as in Phase 2. The resulting sequence is denoted as AB/AB/AB/AB/AB‖$(\mathrm{AB})^\infty$, where the symbol '‖' is used to separate Phase 1 from Phase 2 and '/' is used to separate the rounds. This sequence is called the penalty sequence. Another example of a shootout is the tiebreaker in tennis - the serving player can be seen as the team shooting a penalty in football. The rules of the tiebreak (see ITFRules) stipulate that Phase 1 is a best-of-6, with the first serving player alternating in each round, while in Phase 2 the first serving player also alternates in each round: AB/BA/AB/BA/AB/BA‖$(\mathrm{AB/BA})^\infty$; this sequence is called the ABBA sequence. Other sports using shootouts are field hockey, ice hockey, rugby and water polo.

In a seminal paper by Apesteguia & Palacios-Huerta (2010), the penalty sequence used in football is shown to give an advantage to the team that starts, Team A. This is generally explained by the pressure of lagging behind exercised on the second shooting team B, resulting in a so-called First Mover Advantage (FMA). Indeed, the consecutive nature of the two penalties in a round gives an asymmetry between the first shooting team and the second shooting team. Thus, even when the two teams are equally strong, i.e. even when their probabilities of scoring are the same in all situations, their chances of winning the match may differ. This difference in win probabilities depends on the particular sequence of the shootout. In this note we investigate the existence of so-called fair sequences, both for the best-of-k and for the sudden death.

1.1 *Related literature*

Wright (2014) gives an overview of rules in various sports that affect fairness, see also Kendall & Lenten (2017) and Haigh (2009). The subject of shootouts, and the possible presence of FMA in football shootouts, is heavily debated in the literature. Since the work of Apesteguia & Palacios-Huerta (2010), the presence of a FMA has been confirmed in Palacios-Huerta (2014), Anbarci *et al.* (2015), Vandebroek *et al.* (2018) and Rudi *et al.* (2019), while it has not been found in Kocher *et al.* (2012) and Arrondel *et al.* (2019). We quote Csató (2020): 'To summarize, while the empirical evidence remains somewhat controversial, it seems probable that the team kicking the first penalty enjoys an advantage'.

Even for those that doubt the existence of an FMA in football, it is well established that psychological factors have an impact on the probability of scoring. This is shown by Jordet *et al.* (2007) and Arrondel *et al.* (2019). In the latter study, they identify three different situations (called 'survival', 'catch-up' and 'break point'), which are shown to have different impacts on the scoring probability.

Different models have been proposed to capture the FMA. A frequently used model explains the existence of an FMA by using a probability to score when trailing and a probability to score when not trailing (see the Appendix of Apesteguia & Palacios-Huerta, 2010). They derive corresponding win probabilities, and Vandebroek *et al.* (2018) show that within this model, when using the penalty sequence, the FMA is irrespective of the length of the shootout. Csató (2020) derives win probabilities for various (dynamic) sequences within this model (see also Csató & Petróczy, 2020 for a more comprehensive analysis).

Brams & Ismail (2018) specifically focus on fairness in shootouts. While accepting the existence of an FMA, they propose and analyze rules where the first shooting team in a round is determined by the outcomes of previous rounds. Among such dynamic sequences are the Catch Up Rule and the Behind First Alternating Order Rule (see also Anbarci *et al.*, 2015; Csató, 2020; Csató & Petróczy, 2020, for analyses of dynamic sequences).

It is interesting to note that other sports than football, have other scoring probabilities. For instance, the probability of scoring a penalty in ice hockey equals around 33% (see Kolev *et al.*, 2015). Here, as missing a penalty is the expected outcome, the pressure moves to the goalie; one can view the goalie in ice hockey as the one taking the penalty; the goalie 'scores' when the goalie stops the penalty. In line with the existence of an FMA, Kolev *et al.* (2015) claim there is advantage in shooting second.

A popular sequence in the scientific literature is the so-called Prohuet–Thue–Morse (PTM) sequence. This sequence has many applications in various branches of science, see Allouche & Shallit (1999) for a rigorous analysis. In shootouts, it has been studied in Brams & Taylor (1999), Palacios-Huerta (2012), Rudi *et al.* (2019) and Cohen-Zada *et al.* (2018).

1.2 *Our contribution*

We analyze a shootout with a prescribed format, i.e. we do not allow the sequence to depend on outcomes during the shootout. Teams A and B have the same scoring probabilities in the same situation, i.e. teams

A and B are *equally strong*; these scoring probabilities remain constant during the shootout. Following literature (see, e.g. Brams & Ismail, 2018), we define the concept of a fair sequence in Section 2. We present the following results in this note:

- for sudden death, no sequence that is a repetition of any finite sequence is fair (Section 3.2.1);
- the PTM sequence is unfair (Section 3.2.2);
- we give an algorithm that outputs a fair sequence for sudden death (Section 3.3);
- for best-of-k, we show how to find least unfair sequences (Section 4.2); and
- we empirically compute these least unfair sequences for relevant values for p and q when $k = 5$, and for different values of k when $p = \frac{3}{4}$ and $q = \frac{2}{3}$ (Sections 4.3 and 4.4).

We conclude in Section 5.

2. Preliminaries

As sketched in Section 1, a shootout between teams A and B consists of a best-of-k, followed by a sudden death. Each of these two phases consists of rounds, and every round has one of the two teams shooting first. The problem is to specify, prior to the start of the shootout, for each of the upcoming rounds both in the best-of-k, as well as in the sudden death, which team will shoot first; a specification for the sudden death will be referred to as a *sequence*, and as a *finite sequence* if it is only a specification for a finite number of rounds. For instance, Phase 1 of the penalty sequence can be represented by the finite sequence AAAAA, while Phase 2 of the ABBA sequence can be represented by the sequence ABAB $\cdots$.

Throughout this note we assume Team A and Team B are equally strong. We say a team wins a round, if it scores in that round while the other team does not. If neither of the teams wins the round, the round is tied. The phenomenon of the first shooting team having higher probability of winning the round than the second, is called the First Mover Advantage (FMA).

To model the sudden death, we introduce the probabilities $\mathcal{P} = \{P_+, P_-, P_\pm\}$, where P_+ (P_-) equals the probability that the first (second) shooting team wins the round, and where $P_\pm$ equals the probability of a tie. This set satisfies $P_+ + P_- + P_\pm = 1$. The FMA is then defined as $\lambda = P_+ - P_-$.

The *length* of a finite sequence is the number of rounds for which *the sequence* specifies the first shooting team. We denote the set of finite sequences of length n by $\mathcal{S}_n$. We say that a sequence S is *repetitive* if it consists of the concatenation of a finite sequence $S_n \in \mathcal{S}_n$, i.e. $S = S_n S_n S_n \cdots$. A sequence S may or may not be a repetitive, in any case, it should specify for all rounds $r \in \mathbb{N}$ which team shoots first. For a given sequence, we define the concept of being *fair* in the following way.

DEFINITION 2.1 Let teams A and B be equally strong. Given $\mathcal{P}$, a sequence S is called *fair*, if:

$$\mathbb{P}(\text{Team A wins}) = \mathbb{P}(\text{Team B wins}). \tag{1}$$

If S is not fair, it is *unfair*.

We will now investigate the existence of fair sequences for sudden death in Section 3 and for best-of-k in Section 4.

3. Sudden death

In this section, we consider sequences that specify the first shooting team in the sudden death phase of the shootout. We introduce the characteristic polynomial of a sequence in Section 3.1. Next, we show in Section 3.2 that, when $P_\pm \in \mathbb{Q}$, no repetitive sequence is fair. We also show that the so-called Prohuet-Thue-Morse sequence is unfair for all $P_+ > P_-$. In Section 3.3, we introduce an algorithm that for a given $\mathcal{P}$, returns a fair sudden death sequence if it exists.

3.1 *The characteristic polynomial of a sequence*

In a sudden death shootout, starting each round, the score is a draw—if one team gains an advantage by winning a round, that team wins the sudden death and the game is over. Thus, for a team to win Round r, the first $r-1$ rounds resulted in a draw. This leads to the following observation:

$$\mathbb{P}(\text{Team A wins}) = \sum_{r=1}^{\infty} \mathbb{P}(r-1 \text{ rounds drawn})\mathbb{P}(\text{Team A wins round } r),$$

$$\mathbb{P}(\text{Team B wins}) = \sum_{r=1}^{\infty} \mathbb{P}(r-1 \text{ rounds drawn})\mathbb{P}(\text{Team B wins round } r).$$

The probability for team A to win in a certain Round r, depends on who shoots first in that round. We take I to be the index set of all rounds in which A is allowed to shoot first. In the popular $AAAA$-series, this means $I = \mathbb{N}$, while in the $ABBA$-series, this would mean $I = \{1, 3, 5, \dots\}$. Of course, in the remaining rounds $\bar{I} = \mathbb{N} \setminus I$, team B shoots first.

Recall that P_+, P_- are the probabilities that the first respectively second shooting team wins a round, with $P_\pm$ as the probability the round will be tied, and $\lambda = P_+ - P_-$ as FMA. We have the following lemma.

LEMMA 3.1 Let $\mathcal{P}$ be given. Teams A and B, shooting first in rounds $I, \bar{I}$ respectively, win the sudden death with equal probability if and only if

$$\sum_{r\in I} P_\pm^{r-1} = \sum_{r\in \bar{I}} P_\pm^{r-1}. \tag{2}$$

Proof. We prove this by a straightforward calculation.

$$\begin{aligned}
\mathbb{P}(\text{Team A wins}) &= \sum_{r=1}^{\infty} \mathbb{P}(r-1 \text{ rounds drawn})\mathbb{P}(\text{Team A wins Round } r) \\
&= \sum_{r\in I} P_\pm^{r-1} P_+ + \sum_{r\in \bar{I}} P_\pm^{r-1} P_- \\
&= \sum_{r\in I} P_\pm^{r-1} (P_- + \lambda) + \sum_{r\in \bar{I}} P_\pm^{r-1} P_- \\
&= \sum_{r=1}^{\infty} P_\pm^{r-1} P_- + \lambda \sum_{r\in I} P_\pm^{r-1} \\
&= \frac{P_-}{1-P_\pm} + \lambda \sum_{r\in I} P_\pm^{r-1}.
\end{aligned}$$

Similarly, we find for Team B:

$$\mathbb{P}(\text{Team B wins}) = \frac{P_-}{1-P_\pm} + \lambda \sum_{r\in\bar{I}} P_\pm^{r-1}.$$

Since the first term $\frac{P_-}{1-P_\pm}$ is equal for both teams, the lemma follows. □

DEFINITION 3.1 Given an index set I, we define $h_I(x) = \sum_{i\in I} x^{i-1}$. For a (finite) sequence S, with team A shooting first in rounds I, we define the characteristic polynomial $f_S(x) = h_I(x) - h_{\bar{I}}(x)$.

Using this definition, we state the following corollary to Lemma 3.1.

COROLLARY 3.1 Let $\mathcal{P}$ be given. A sequence S is fair if and only if $f_S(P_\pm) = 0$, i.e. if $P_\pm$ is a zero of $f_S(x)$.

3.2 *Unfair sequences*

3.2.1 *Repetitive sequences are unfair.* Although Lemma 3.1 and Corollary 3.1 provide a necessary and sufficient condition for a sequence to be fair, it is not clear when (2) is satisfied. The following theorem clarifies the status for repetitive sequences.

THEOREM 3.1 Let $P_\pm \in \mathcal{P}$ be rational, i.e. $P_\pm \in \mathbb{Q}_{(0,1)}$. Each repetitive sequence S is unfair.

Proof. Consider a repetitive sequence S that is a concatenation of some finite sequence S_n. Let I_n be the index set indicating when Team A shoots first in this sequence S_n. The characteristic polynomial of S_n is then given by $f_{S_n}(x) = \sum_{i\in I_n} x^{i-1} - \sum_{i\in\bar{I}_n} x^{i-1}$, a polynomial of degree $n-1$. Since S is repetitive, it follows that

$$f_S(x) = f_{S_n}(x)\cdot(1 + x^n + x^{2n} + \cdots) = \frac{f_{S_n}(x)}{1-x^n}.$$

Clearly,

$$f_S(P_\pm) = 0 \iff f_{S_n}(P_\pm) = 0.$$

The following claim states that, for $x \in (0,1)$, there is no rational solution to $f_{S_n}(x) = 0$.

CLAIM 1. Any finite degree polynomial $u(x)$ with coefficients in $\{-1, 1\}$ has no rational zero's in $(0, 1)$.

Proof. Consider $u(x) = \sum_{i=0}^{n} c_i x^i$ with $c_i \in \{-1, 1\}$, and let $\frac{w}{v} \in \mathbb{Q}_{(0,1)}$ with $\gcd(w, v) = 1$, be such that $u(\frac{w}{v}) = 0$. Then, it must hold that $v^n u(\frac{w}{v}) = 0 \mod v$ as well. However, $v^n u(\frac{w}{v}) = c_n w^n \mod v$. As $\gcd(w, v) = 1$, this leads to $c_n w^n = 0 \mod v$ and this is not possible unless $w = v = 1$ or $w = 0$. □

As $P_\pm \in \mathbb{Q}$, there are no $P_\pm$ that satisfy $f_{S_n}(P_\pm) = 0$ and we conclude that S is unfair. □

Remark. It follows easily from Theorem 3.1 that, given rational $\mathcal{P}$, no finite sequence will be fair. In addition, observe that the popular *AAAA*- and *ABAB*-sequences are unfair for all real values of $\mathcal{P}$ when $\lambda = P_= - P_- > 0$. Indeed, this follows as $f_{AAAA}(x) = \frac{1}{1-x} > 0$ and $f_{ABAB}(x) = \frac{1}{1+x} > 0$ for $x \in (0, 1)$.

The relevance of Theorem 3.1 lies in the guaranteed presence of unfairness; when picking/determining $\mathcal{P}$ from empirical data and deciding upon a finite sequence to be repeated, any resulting sequence is unfair.

3.2.2 *The PTM sequence is unfair.* In order to mitigate the First Mover Advantage, the PTM sequence is suggested in Brams & Taylor (1999), *The Win-Win solution*. It was also proposed by Palacios-Huerta (2012) and, recently, further analyzed in Rudi *et al.* (2019). Also, for the tennis tiebreak, it was discussed by Cohen-Zada *et al.* (2018). In contrast to their observation, we show that in our model, for any $P_{\pm} \in \mathcal{P}$, the PTM-sequence is an unfair sequence.

Definition 3.2 The PTM sequence is a sequence containing 0 and 1, obtained in the following way. Start with a 1. Invert the sequence, meaning that 1 becomes 0 and vice versa. Add the newly obtained sequence at the end of the old one. Repeat this process.

- Step 0: 1
- Step 1: 1 0
- Step 2: 1 0 0 1
- Step 3: 1 0 0 1 0 1 1 0

When looking at this as a penalty sequence, one can let team A start the Round n when $PTM(n) = 1$ and team B when $PTM(n) = 0$.

Theorem 3.2 The PTM sequence is unfair for all $\mathcal{P}$.

Proof. We will construct the characteristic polynomial of the PTM-sequence and show that is bounded away from 0 in the interval $(0, 1)$.

Let S_n be the first n rounds of the PTM-sequence, and for notational purposes, let $f_n(x)$ be its characteristic polynomial. Clearly, for $n = 2$, $f_2(x) = 1 - x$. By construction, the first 2^k terms of the characteristic polynomial of the PTM-sequence are inverted to obtain the terms $2^k + 1$ up to 2^{k+1}. For the characteristic polynomial, inversion of a coefficient is just multiplication by -1, which leads to the following expression:

$$f_{2^{k+1}}(x) = f_{2^k}(x)(1 - x^{2^k}).$$

To explain this equality, notice that the first 2^k terms of $f_{2^{k+1}}$ terms are the same as $f_{2^k}(x)$. The next 2^k terms all have degree x^{2^k} or higher, and are the inverse of $f_{2^k}(x)$—hence the multiplication of $f_{2^k}(x)$ with the term $-x^{2^k}$.

Applying this expression iteratively for $2^1, 2^2, 2^4 \ldots$, we get the following (Allouche & Shallit, 1999):

$$f_{PTM}(x) = (1 - x)(1 - x^2)(1 - x^4) \cdots = \prod_{k=0}^{\infty} \left(1 - x^{2^k}\right). \tag{3}$$

We now prove that there is no $x \in (0, 1)$ for which $f_{PTM}(x) = 0$, implying there is no $P_{\pm}$ for which the PTM-sequence is fair.

Claim 2. For $x \in (0, 1), f_{PTM}(x) > 0$.

Proof. Clearly, $f_{PTM}(x)$ is decreasing on $(0, 1)$. Notice that $f_{PTM}(x) = (1 - x)f_{PTM}(x^2)$, for all x. Suppose there is an $x \in (0, 1)$ for which $f_{PTM}(x) = 0$. Consequently, as $1 - x > 0$, this implies that $f_{PTM}(x^2) = 0$. As $x^2 < x$, this implies that if there is an $x \in (0, 1)$ for which $f_{PTM}(x) = 0$, we can pick an arbitrarily small x' for which $f_{PTM}(x') = 0$. As the function f_{PTM} is strictly decreasing on $(0, 1)$, and attaining $f(1) = 0$, either $f(x) = 0$ for all $x \in (0, 1)$, or $f(x) > 0\ \forall x \in (0, 1)$.

Take $x = \frac{1}{2}$. Then

$$f_{PTM}\left(\frac{1}{2}\right) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2^2}\right)\cdots \qquad > 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{16}\cdots$$
$$> 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} \qquad = \frac{1}{8} > 0.$$

□

Using Claim 2, we conclude that there are no values $x \in (0, 1)$ for which the PTM-sequence is fair, and hence the proof is complete. □

3.3 *An algorithm generating a fair sequence*

We established in Section 3.2.1 that, for given values $\mathcal{P} \subset \mathbb{Q}$, there are no finite fair sequences. However, it is possible to construct infinite sequences that are fair, for any value of $P_\pm \geq 0.5$ (indeed, notice that if $P_\pm < 0.5$, the team shooting first will have an insurmountable FMA and no fair sequence exists). Consider now the following algorithm which takes $P_\pm \geq 0.5$ as input.

Let Team A shoot first in Round 1 and set $I_1 = \{1\}$. Starting from $n = 1$, construct I_{n+1} from I_n in the following way:

Step 1: If $f_{I_n}(P_\pm) < f_{\bar{I}_n}(P_\pm)$ then $I_{n+1} := I_n \cup \{n + 1\}$, else $I_{n+1} := I_n$.

Step 2: n:=n+1. Go to Step 1.

We prove the following theorem.

THEOREM 3.3 Let $\mathcal{P}$ be such that $P_\pm \geq 0.5$. Then, Algorithm 3.3 returns a fair sequence.

Proof. Define $d_n \equiv |f_{I_n}(P_\pm) - f_{\bar{I}_n}(P_\pm)|$ and write $d_n = |f_n - \bar{f}_n|$ By induction on n, we will prove that $d_n \leq P_\pm^{n-1}\ \forall n$ if $P_\pm \geq 0.5$. When $n = 1$, $d_n \leq P_\pm^{n-1} = 1$. Suppose that $d_k \leq P_\pm^{k-1}$ holds for $k = 1, \ldots, n - 1$ and suppose that $f_{n-1} > \bar{f}_{n-1}$. Then $\bar{f}_n = \bar{f}_{n-1} + P_\pm^{n-1}$ and $f_n = f_{n-1}$. Either $\bar{f}_n \geq f_n$, or $f_n > \bar{f}_n$.

In the first case, clearly $d_n \leq P_\pm^{n-1}$.

In the second case, $d_n = d_{n-1} - P_\pm^{n-1} \leq P_\pm^{n-2} - P_\pm^{n-1} \leq P_\pm^{n-1}$ as $P_\pm \geq 0.5$.

In both cases, $d_n \leq P_\pm^{n-1}$.

Thus, $\lim_{n\to\infty} f_{I_n}(P_\pm) - f_{\bar{I}_n}(P_\pm) = 0$, and as a consequence Algorithm 3.3 constructs a fair sequence. □

We point out that Algorithm 3.3 can be criticized from the point of view that it does not end. That property however, seems necessarily linked to its goal, namely finding a fair sequence, which by Theorem 3.1 cannot be achieved by a sequence of finite length. In any case, when one is prepared to specify $P_\pm$, it is certainly possible to precompute a very large number of entries, and use that as proxy for a fair sequence.

Of course, the resulting sequence depends on the chosen values for $P_+, P_-, P_\pm$. Thus, to decide which particular sequence to use in practice, one would have to decide on the values for these probabilities. When choosing $P_+ = \frac{1}{4}, P_- = \frac{3}{16}$ (see Apesteguia & Palacios-Huerta, 2010; Vandebroek *et al.*, 2018), this results in a sequence that starts with: *ABBBABABBAA* .... Finally, we mention that, even though the algorithm identifies a single fair sequence, the index set I is not necessarily a unique index set, i.e. multiple distinct fair sequences may exist.

4. The best-of-k

As described in Section 1, the first phase of a shootout consists of a best-of-k, where the winner is the team that scored the most penalties after Round k. In case of a tie, there is a Phase 2 which consists of a sudden death. Apesteguia & Palacios-Huerta (2010) model the existence of an FMA in a best-of-k by assuming that the pressure of being behind affects the scoring probability negatively, see Section 4.1. In Section 4.2 we show how to compute the degree of unfairness in this model and we apply this to the best-of-5 in Section 4.3 and to the best-of-k when $p = \frac{3}{4}, q = \frac{2}{3}$ in Section 4.4.

4.1 *Modeling psychological pressure*

The main feature of the model is to encompass the added pressure of shooting while lagging behind in a fitting and realistic way. Recall that we assume Team A and B are equally strong. Next, we introduce parameters $p, q \in (0, 1)$ as the probability of a team scoring. A team has probability p of scoring, if it is equal or ahead, and probability q if the team is trailing. We assume $p > q$.

If the score, at the beginning of the round, is equal, and Team A shoots first, Team B second, this results in the following possible outcomes with probabilities:

$$\begin{aligned}
P_+ &= \mathbb{P}(\{\text{A scores, B does not}\}) && = p(1-q), \\
P_- &= \mathbb{P}(\{\text{B scores, A does not}\}) && = (1-p)p, \\
P_\pm &= \mathbb{P}(\{\text{A and B score equally often}\}) && = pq + (1-p)^2.
\end{aligned}$$

Notice that the probability that Team B scores after Team A scored, is q. If Team A missed, however, the probability that Team B would score is equal to p. This makes sense as in the first case, the score would have been in favor of Team A, adding pressure to Team B to catch up. As we assume that $p > q$, we immediately see that $P_+ > P_-$, indicating the existence of (FMA) λ, where $\lambda := P_+ - P_-$.

If, at the start of a round, one team (say A) leads, the possible outcomes with probabilities are slightly different:

$$\begin{aligned}
Q_+ &= \mathbb{P}(\{\text{Team A extends lead with 1}\}) && = p(1-q), \\
Q_- &= \mathbb{P}(\text{Team B decreases lead with 1}) && = (1-p)q, \\
Q_\pm &= \mathbb{P}(\{\text{A and B score equally often}\}) && = pq + (1-p)(1-q).
\end{aligned}$$

The values of p, q are, of course, not known, but they can be estimated from real world results. In football, $p = \frac{3}{4}, q = \frac{2}{3}$ is a common pick in the literature (see Brams & Ismail, 2018; Csató, 2020).

4.2 *Degree of unfairness*

Theorem 3.3 tells us that it is possible to create a sudden death sequence that is fair. Hence, when the shootout is tied after k rounds, we can use Theorem 3.3 to construct a sequence such that both teams have an equal probability of winning. That leaves us with only k choices to be made, indicating who shoots first in the k rounds of the best-of-k. To model our choice of who shoots first, we introduce $\sigma_i \in \{-1, 1\}, i = 1, \ldots, k$, where $\sigma_i = 1$ indicates Team A shooting first in round i, $\sigma_i = -1$ indicates Team B shooting first. We will derive formulas for W_i ($\overline{W}_i$), indicating the probability of team A (B) winning the shootout given that there is a tie at the start of round i ($1 \leq i \leq k$). We also define and calculate $\Delta_i = W_i - \overline{W}_i$, the difference between the probability of winning for team A and B, $i = 1, \ldots, k$. The objective is to find values for the σ_i that minimize Δ_1, as the difference in winning probabilities between Team A and Team B at the start of the shoot out is then minimized. Lastly, we define the auxiliary term K_j as follows: given that one team is ahead by 1 at the start of the round, K_j is the probability that the score will be leveled for the first time after j rounds.

Suppose the score is tied at the beginning of Round i ($1 \leq i \leq k$). One of the following three events occurs:

Event 1: The teams draw this round, and Round $i+1$ will start with a tied score.

Event 2: One of the teams wins the Round, and somewhere between the beginning of Round $i+2$ and Round $k+1$, the score is leveled again.

Event 3: One of the teams wins the round and stays ahead for the rest of the shootout.

Only in the case of Event 3, the model gives the team starting the round an advantage over the other team, as that team is more likely to take the lead because of FMA, and by staying ahead profiting from that lead.

We can see Team A's winning probability W_i as the sum of the probability of winning in each of these events. For each of these events, we list the probability of occurring and the winning probability for Team A.

1. Event 1 occurs with probability $P_\pm$. The probability of winning after drawing Round i, is given by W_{i+1}. Hence the probability of winning in this fashion equals $P_\pm W_{i+1}$.
2. The probability that some team wins Round i equals $1 - P_\pm$. The probability that after the following j rounds, $j = 1, \ldots, k-i$, the score is leveled for the first time, is K_j. The total probability of Team A winning in this fashion, is thus given by $(1 - P_\pm) \sum_{j=1}^{k-i} K_j W_{i+j+1}$.
3. The probability that Team A takes the lead in Round i, is given by $\frac{1}{2}(1 + \sigma_i)P_+ + \frac{1}{2}(1 - \sigma_i)P_-$. The probability that the leading team stays ahead after this round, is given by $1 - \sum_{j=1}^{k-i} K_j$, so the total probability of Team A winning in this fashion is the product of both terms.

This leads to the following winning probability for Team A:

$$
\begin{aligned}
W_i = {} & P_\pm W_{i+1} + (1 - P_\pm) \sum_{j=1}^{k-i} K_j W_{i+j+1} \\
& + \left(1 - \sum_{j=1}^{k-i} K_j\right) \frac{1}{2} \big((1 + \sigma_i)P_+ + (1 - \sigma_i)P_-\big) .
\end{aligned}
$$

From this, we derive, as $\varDelta_i = W_i - \overline{W}_i$:

$$\begin{aligned}\varDelta_i &= P_\pm \varDelta_{i+1} + (1-P_\pm)\sum_{j=1}^{k-i} K_j \varDelta_{i+j+1} + \left(1 - \sum_{j=1}^{k-i} K_j\right)\sigma_i(P_+ - P_-)\\ &= P_\pm \varDelta_{i+1} + (1-P_\pm)\sum_{j=1}^{k-i} K_j \varDelta_{i+j+1} + \left(1 - \sum_{j=1}^{k-i} K_j\right)\sigma_i\lambda.\end{aligned}$$

Notice that we can write $\varDelta_i$ as a linear combination of terms $\lambda\sigma_j$, with $j \geq i$. Thus $\varDelta_i = \sum_{j=i}^{k} D_{i,j}\lambda\sigma_j$. Applying this iteratively, we find the following expression for $\varDelta_i$:

$$\varDelta_i = \sum_{j=i+1}^{k}\left(P_\pm D_{i+1,j} + (1-P_\pm)\sum_{\ell=1}^{j-i-1} K_\ell D_{i+1+\ell,j}\right)\lambda\sigma_j + \left(1 - \sum_{j=1}^{k-i} K_j\right)\lambda\sigma_i.$$

For $\varDelta_1$, which represents the unfairness that we want to minimize, we arrive at the following:

$$\varDelta_1 = \sum_{j=2}^{k}\left(P_\pm D_{2,j} + (1-P_\pm)\sum_{\ell=1}^{j-2} K_\ell D_{2+\ell,j}\right)\lambda\sigma_j + \left(1 - \sum_{j=1}^{k-1} K_j\right)\lambda\sigma_1.$$

4.3 *Results for the best-of-5*

We can apply these general setting to the popular case of $k = 5$. This results in

$$\begin{aligned}
i = 6:&\quad \varDelta_6 = 0.\\
i = 5:&\quad \varDelta_5 = \lambda\sigma_5.\\
i = 4:&\quad \varDelta_4 = P_\pm\lambda\sigma_5 + \big(1 - K_1\big)\lambda\sigma_4.\\
i = 3:&\quad \varDelta_3 = (P_\pm^2 + (1-P_\pm)K_1)\lambda\sigma_5 + P_\pm(1-K_1)\lambda\sigma_4 + (1 - K_1 - K_2)\lambda\sigma_3.\\
i = 2:&\quad \varDelta_2 = (P_\pm D_{3,5} + (1-P_\pm)K_1 D_{4,5} + (1-P_\pm)K_2)\lambda\sigma_5\\
&\qquad + (P_\pm D_{3,4} + (1-P_\pm)K_1 D_{4,4})\lambda\sigma_4 + P_\pm D_{3,3}\lambda\sigma_3 + (1 - \sum_{j=1}^{3} K_j)\lambda\sigma_2.\\
i = 1:&\quad \varDelta_1 = \big(P_\pm D_{2,5} + (1-P_\pm)K_1 D_{3,5} + (1-P_\pm)K_2 D_{4,5} + (1-P_\pm)K_3\big)\lambda\sigma_5\\
&\qquad + (P_\pm D_{2,4} + (1-P_\pm)K_1 D_{3,4} + (1-P_\pm)K_2 D_{4,4})\lambda\sigma_4\\
&\qquad + (P_\pm D_{2,3} + (1-P_\pm)K_1 D_{3,3})\lambda\sigma_3 + P_\pm(1 - \sum_{j=1}^{3} K_j)\lambda\sigma_2 + (1 - \sum_{j=1}^{4} K_j)\lambda\sigma_1.
\end{aligned}$$

All the unknowns that remain, are the terms K_j. These terms represent the probability that the leading team gives away the lead after exactly j rounds. In our model, the probabilities for these events are given

TABLE 1 *Least unfair sequences for given p and q in best-of-5*

p/q	0,6	0,61	0,62	0,63	0,64	0,65	0,66	0,67	0,68	0,69	0,7	0,71	0,72	0,73	0,74	0,75	0,76	0,77	0,78	0,79	0,8
0,6	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,61	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,62	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,63	ABBBA	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,64	ABABB	ABBBA	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,65	ABABB	ABABB	ABBBA	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,66	ABBAB	ABABB	ABBBA	ABBBA	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,67	ABBAB	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x	x
0,68	ABBAB	ABBAB	ABBBA	ABABB	ABBBA	ABBBA	ABBBA	ABBBA	x	x	x	x	x	x	x	x	x	x	x	x	x
0,69	ABBAB	ABBAB	ABBBA	ABBBA	ABABB	ABBBA	ABBBA	ABBBA	AABBB	x	x	x	x	x	x	x	x	x	x	x	x
0,7	ABBAB	ABBAB	ABBAB	ABBBA	ABBBA	ABABB	ABABB	ABBBA	ABBBA	AABBB	x	x	x	x	x	x	x	x	x	x	x
0,71	ABBBA	ABBAB	ABBAB	ABBBA	ABBBA	ABABB	ABABB	ABABB	ABBBA	ABBBA	AABBB	x	x	x	x	x	x	x	x	x	x
0,72	ABBBA	ABBBA	ABBBA	ABBAB	ABBBA	ABBBA	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	x	x	x	x	x	x	x	x	x
0,73	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBBA	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	x	x	x	x	x	x	x	x
0,74	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBBA	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	x	x	x	x	x	x	x
0,75	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	x	x	x	x	x	x
0,76	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	x	x	x	x	x
0,77	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	x	x	x	x
0,78	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	x	x	x
0,79	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	AABBB	x	x
0,8	ABBBA	ABBBA	ABBBA	ABBBA	ABBBA	ABBAB	ABBAB	ABBAB	ABBAB	ABBAB	ABABB	ABABB	ABABB	ABABB	ABABB	ABABB	AABBB	AABBB	AABBB	AABBB	x

by

$$K_1 = Q_-.$$
$$K_2 = Q_\pm Q_-.$$
$$K_3 = (Q_\pm^2 + Q_+Q_-)Q_-.$$
$$K_4 = (Q_\pm^2 + 2Q_+Q_-)Q_\pm Q_-.$$

The eventual aim is to have an assignment that minimizes $|\Delta_1|$, the unfairness at the start of Round 1. We assume that Team A shoots first in Round 1, which gives $\sigma_1 = 1$. Given values for p, q, we compute which of 16 possible assignments of $\sigma_2, \sigma_3, \sigma_4, \sigma_5$, minimizes $|\Delta_1|$. The corresponding sequences are presented in the following Table 1, with equal colors for equal sequences. We choose a grid with $0.6 \leq q < p \leq 0.8$ to reflect realistic probabilities in football.

It is interesting to see the variety of preferred sequences, even when fixing the value of p or q, or their difference. This can be explained as the underlying calculation of the unfairness in a best of k shootout, is a polynomial of degree $2k$. The code to calculate which sequence is fairest, is given in Appendix A.

4.4 *Results for best-of-k with fixed p, q*

The expressions obtained in Section 4.3 to calculate the unfairness of a sequence of any length, given p, q, is used to calculate the fairness of shootouts of another length than Best-of-5. The fairest sequences for $p = \frac{3}{4}, q = \frac{2}{3}$ for shootouts of length $k = 2, 3, \ldots, 10$ are shown in Table 2. It is important to stress that the values reported in the column 'measure of unfairness' are the difference in winning chances between both teams, assuming that in the event of a tie after k rounds, both teams have an equal chance of winning the subsequent sudden death. It is remarkable that comparing two fairest sequences of different length, the longer penalty sequence is not automatically less unfair, even if the p, q is constant for both sequences. The code that was used to calculate is given in Appendix A.

TABLE 2 *Least unfair sequence and their offset for various shootout lengths with* $p = \frac{3}{4}, q = \frac{2}{3}$

k	Sequence	Measure of unfairness (10^{-3})
2	AB	16.93
3	ABB	7.62
4	ABBA	1.47
5	ABABB	0.21
6	AABBBB	0.24
7	ABBABAB	0.01
8	ABAABBBB	0.04
9	AABBBBBBA	0.03
10	ABABABBABB	0.002

5. Conclusion

Many sports use shootouts to identify a winner in an otherwise tied game. Popular examples include football, field hockey, ice hockey, tennis, rugby and water polo. A shootout has rounds, where in each round two teams, in an alternating fashion, have the possibility to score a point. Such a shootout has two phases. Phase 1 is a best-of-k (in soccer $k = 5$, the tie-break in tennis has $k = 6$), and if the shootout is tied after Phase 1, it continues with Phase 2, which is a sudden death. It is widely accepted that shooting when behind impacts the chances of scoring a point compared to shooting when not behind. We consider the problem to specify a sequence that determines which team shoots first in each round of the shootout such that identical teams have equal chance of winning the shootout; such sequences are called fair.

Using a common way to model the discrepancy between scoring chances for both teams, we show that in a sudden death, repetitive sequences are not fair, for any choice of the parameters; we also show that the PTM-sequence is not fair for any choice of parameters. There is, however, an algorithm that outputs a fair sequence.

For a shootout decided over best-of-k, we show that no fair sequence exists. Using the popular choice $p = \frac{3}{4}, q = \frac{2}{3}$ (reflecting the probabilities of scoring when not behind, and when behind, respectively), the least unfair sequence in a best-of-5 is AB/BA/AB/BA/BA. More generally, we show that the degree of unfairness depends on the length of the shootout: longer shootouts can be significantly fairer than shorter ones.

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A.

```
totalRounds = N
Sequences = set(list(combinations([0,1]*N,N)))

  for _p in range(60,81):
    for _q in range(60,_p):
      p = 0.01*_p
      q = 0.01*_q

      for sequence in Sequences:
        sigma = sequence
        best = 1
        bestseq = None
        value = f(0,0)

        if abs(0.5-value) < abs(0.5 - best):
          best = value
          bestseq = sigma

      print("For " p,q, "the best sequences is ", bestseq, " with
          unfairness best)


  def f(round,score):
    if round < totalRounds:
      if score > 0:
        result = p*(1-q)*f(round+1,score+1) +
              (1-p)*q*f(round+1,score-1) +
              (p*q + (1-p)*(1-q))*f(round+1,score)
      elif score < 0:
        result = p*(1-q)*f(round+1,score-1) +
        (1-p)*q*f(round+1,score+1) +
        (p*q + (1-p)*(1-q))*f(round+1,score)
      elif score==0:
        if sigma[round]==1:
          result = p*(1-q)*f(round+1,1)+p*(1-p)*f(round+1,-1)+
          (p*q-(1-p)*(1-p))*f(round+1,0)
        elif sigma[round]==0:
          result = p*(1-q)*f(round+1,-1)+p*(1-p)*f(round+1,1)+
          (p*q-(1-p)*(1-p))*f(round+1,0)

    elif round == totalRounds:
      if score > 0:
        result = 1
      elif score < 0:
        result = 0
      elif score == 0:
        result = 0.5

    return result
```