References

[1] MOTZKIN, T.S.: 'Beiträge zur Theorie der Linearen Ungleichungen', *PhD Thesis Azriel, Jerusalem* (1936).

[2] PADBERG, M.: *Linear optimization and extensions*, Vol. 12 of *Algorithms and Combinatorics*, Springer, 1995.

[3] SCHRIJVER, A.: *Theory of linear and integer programming*, Wiley, 1986.

Arkadi Nemirovski
Fac. Industrial Engin. and Management
Technion: Israel Inst. Technol.
Technion-City, Haifa 32000, Israel
E-mail address: nemirovs@ie.technion.ac.il

Kees Roos
Dept. ITS/TWI/SSOR Delft Univ. Technol.
P.O. Box 356
2600 AJ Delft, The Netherlands
E-mail address: C.Roos@twi.tudelft.nl

MSC2000: 15A39, 90C05

Key words and phrases: inequality systems, duality, certificate, transposition theorem.

MULTI-INDEX TRANSPORTATION PROBLEMS, *MITP*

An ordinary *transportation problem* has variables with two indices, typically corresponding to sources (or origins, or supply points) and destinations (or demand points). A *multi-index transportation problem* (MITP) has variables with three or more indices, corresponding to as many different types of points or resources or other factors. Multi-index transportation problems were considered by T. Motzkin [22] in 1952; an application involving the distribution of different types of soap was presented by E. Schell [35] in 1955. MITPs are also known as *multidimensional transportation problems* [4]. There are several versions and special cases of MITPs:

- The number k of dimensions may be fixed to a small value; the resulting MITP is called a *k-index transportation problem*, kITP. Quite naturally, the best studied cases are the *three-index transportation problems* (3ITPs), also known as *three-dimensional*, or *3D transportation problems*.
- The type of constraints is determined by an integer m with $0 < m < k$, defining m-fold kITPs (called *symmetric MITPs* in [16]; see also [41, Chapt. 8]). The most common cases are *axial MITPs*, when $m = k-1$; and *planar MITPs*, when $m = 1$; see below for details.
- Integer solutions may or may not be required. Integrality requirements, which give rise to *integer MITPs*, may be necessary since MITPs lack the integrality property enjoyed by ordinary transportation problems (but see [22] for an exception).
- Unit right-hand sides, in conjunction with integrality requirements, give rise to *multi-index assignment problems* (MIAPs). (Some authors use this term for integer MITPs with integer right-hand sides; the present terminology, consistent with that for ordinary assignment and transportation problems, seems preferable.) MIAPs are hard to solve: the 3IAP is already NP-hard by reduction from the 3-dimensional matching problem [17]. Even worse [6]: no polynomial time algorithm for the 3IAP can achieve a constant performance ratio, unless $P = NP$.
- The objective function is usually a simple linear combination of the variables, normally a total cost to be minimized as in equation (1) below. Alternatives, not considered in this article, may include bottleneck objectives ([36], [11]), more general nonlinear objectives such as in [34], or multicriteria problems [38].
- There may be additional constraints, such as upper bounds on the variables, (capacitated MITPs), variables fixed to the value zero (MITPs with forbidden cells), or constraints on certain partial sums of variables (MITPs with generalized capacity constraints).

MITPs with linear objectives and without integrality restrictions are linear programming problems with a special structure. The most extensively studied *integer MITPs* are *three-index assignment problems* (3IAPs); see also **Three-index assignment problem**.

Formulations. The following compact notation ([34], [31]) avoids multiple summations and multiple layers of subindices. Let $k \geq 3$ denote the number of dimensions or indices, and $K = \{1, \ldots, k\}$. For $i \in K$ let A_i denote the set of values of the ith index. Let $A = \otimes_{i \in K} A_i = A_1 \times \cdots \times A_k$ denote the Cartesian product of these index sets, that is, the set of all joint indices (k-tuples) $a = (a(1), \dots, a(k))$ with $a(i) \in A_i$ for all $i \in K$. One variable x_a is associated with each joint index $a \in A$. Thus, for example in a 3ITP with index sets I, J and L, the variable x_a stands for $x_{ij\ell}$ when the joint index is $a = (i, j, \ell)$.

Given unit costs $c_a \in \mathbf{R}$ for all $a \in A$, a linear objective function is

$$\min \sum_{a \in A} c_a x_a \tag{1}$$

and the variables are usually restricted to be nonnegative:

$$x_a \geq 0 \quad \text{for all } a \in A. \tag{2}$$

Given the integer m with $0 < m < k$, the demand constraints of the m-fold kITP are defined as follows. Let $\binom{K}{k-m}$ denote the set of all $(k-m)$-element subsets of K; an $F \in \binom{K}{k-m}$ is interpreted as a set of $k-m$ 'fixed indices'. Given such an F and a $(k-m)$-tuple $g \in A_F = \otimes_{f \in F} A_f$ of 'fixed values', let

$$A(F, g) = \{a \in A \colon a(f) = g(f),\ \forall f \in F\}$$

be the set of k-tuples which coincide with g on the fixed indices. The m-fold demand constraints are

$$\sum_{a \in A(F,g)} x_a = d_{Fg} \tag{3}$$

$$\text{for all } F \in \binom{K}{k-m},\ g \in A_F,$$

where the right-hand sides d_{Fg} are given positive demands associated with the values g for fixed index subset F. These 'demands' may also denote supplies or capacities when the indices represent sources or some other resource type. When some of these resources are in excess, the equality in constraints (3) may be replaced with inequalities. Problem (1)–(3) is a kITP. Adding the integrality restrictions

$$x_a \in \mathbf{N} \quad \text{for all } a \in A, \tag{4}$$

yields an integer MITP.

As mentioned above, the most common cases are $m = k - 1$, defining axial MITPs; and $m = 1$, defining planar MITPs. For the axial problems, the notation may simplified by letting $d_{ig} = d_{Fg}$ when $F = \{i\}$. Note that each variable x_a appears in the same number k of axial and planar demand constraints; however there are only $\sum_{i \in K} |A_i|$ axial constraints, versus $\sum_{i \in K} \prod_{f \in K \setminus \{i\}} |A_f|$ planar constraints. Of course, it is possible to combine demand constraints with different values of m, so as to formulate different types of restrictions (e.g., see [5] and [16]).

Reductions between MITPs are presented in [16], where it is shown in particular that an m-fold kITP can be reduced to a 1-fold kITP for any m (with $0 < m < k$), thereby generalizing a result in [14]. Thus, an algorithm that solves planar kITPs is in principle capable of solving m-fold kITPs for any m (with $0 < m < k$).

Notice that any MITP with arbitrary right hand sides can be transformed to a MITP with right hand sides 1. This is a (pseudopolynomial) transformation and simply involves duplicating a resource with a supply of q units by q unit-supply resources. There seems to be little advantage in doing so, except perhaps in converting an integer MITP into one with 0–1 variables.

Another issue is the existence of feasible solutions. For an axial MITP the requirement of equal total demands $\sum_g d_{ig} = \sum_g d_{jg}$ for all $i, j \in K$ is a necessary and sufficient condition for the existence of feasible solutions. Feasibility conditions are more complicated for nonaxial problems; see [40] for a review of results for planar problems. See also [41, Chapt. 8] for properties of polytopes associated with (integer) MITPs, including issues of degeneracy.

Applications.

Transportation and Logistics. MITPs are used to model transportation problems that may involve different goods; such resources as vehicles, crews, specialized equipment; and other factors such as alternative routes or transshipment points. Thus index sets A_1 and A_2 may represent destinations and sources, respectively, and the other sets $A_3, A_4, \dots$ these additional factors. The type of 'demand' constraints used will reflect the availability of these factors and their interactions. Thus, for example, an axial demand constraint (3) with right-hand side d_{3i} will be used for a vehicle type $i \in A_3$ of which d_{3i} units are globally available (at identical cost) to all sources and destinations, while a constraint with $F = \{2,3\}$ will be used if there are d_{Fg} vehicles of type $g(3)$ available at the different sources $g(2)$.

Interesting cases arise when each resource or factor $\ell \in A_i$ corresponds to a point $P_{i,\ell}$ in a *metric space*, i.e., a set with a distance δ, and the unit costs c_a are 'decomposable' as defined below. Each joint index $a \in A$ may be interpreted as a *cluster* of points among which transportation and other activities are conducted. The unit cost c_a reflects the within-cluster transportation costs associated with these activities; it is *decomposable* if it can be expressed as a function of the distances between pairs of points in the cluster a. Examples include the *diameter* $\max_{i,j} \delta(P_{i,a(i)}, P_{j,a(j)})$, when all these activities are performed simultaneously; the sum costs $\sum_{i,j} \delta(P_{i,a(i)}, P_{j,a(j)})$ when all activities are performed sequentially; and the *Hamiltonian path* or *path* costs, when all points $P_{i\ell}$ in the cluster have to be visited in a shortest sequence.

Other interesting cases arise when one of the indices denotes time. A simple *dynamic location problem* [27] may be modeled as an axial kITP, where index set A_1 may denote the set of facilities (say, warehouses) to be located; A_2 that of candidate locations; and A_3 that of time periods. The costs c_{ijt} may include discounted construction and operating costs of these facilities. See [38] and [33] for other applications of this type.

Timetabling. Other problems involving time and which can be formulated as MITPs arise in timetabling or staffing applications. To illustrate, consider the following generic situation. Given are N employees (index i), each of which can be assigned to one of M tasks (index j) during each of T time periods (index k). Moreover, for each pair consisting of a task and a time period a number r_{jk} is given denoting the number of employees required for task j in period k. Also, a number r_{ij} is given denoting the number of periods that task j requires employee i. An employee can only be assigned to one task during each time period. Finally, there is a cost-coefficient c_{ijk} which gives the cost of employee i performing task j in period k. This problem is called the *multiperiod assignment problem* in [21] (see also the references contained therein). To model this as a planar 3ITP, let A_1 be the set of employees; A_2 the set of tasks; A_3 the set of time periods;

$$d_{Fg} = \begin{cases} r_{jk} & \text{for } F = \{2,3\}, \quad \forall g = (j,k); \\ 1 & \text{for } F = \{1,3\}, \quad \forall g = (i,k); \\ r_{ij} & \text{for } F = \{1,2\}, \quad \forall g = (i,j); \end{cases}$$

and require the decision variables to be in $\{0,1\}$. A special case arises when $r_{jk} = 1$ for all j,k and $N = M$. The polyhedral structure of the resulting planar 3ITP is investigated in [7]. Other references dealing with timetabling problems formulated as MITPs are [15], [10] and [12].

Multitarget Tracking. Consider the following (idealized) situation. N objects move along straight lines in the plane. At each of T time instants a *scan* has been made, and the approximate position of each object is observed and recorded. From such a scan it is not possible to deduce which object generated which observation. Also, a small error may be associated with each observation. A *track* is defined as a T-tuple of observations, one from each scan. For each possible track a cost is computed based on a least squares criterion associated with the observations in the track. The problem is now to identify N tracks while minimizing the sum of the costs of these tracks. This problem is called the *data-association problem* in [25]. It can be modeled as an axial integer TIAP as follows: let A_i be the set of observations in scan i, $i = 1,\dots,T$, and let $d_{ig} = 1$, $i = 1,\dots,T$, $g = 1,\dots,N$. Not surprisingly, this problem is NP-hard already for $T = 3$ (see [37]; notice however that this does not follow from the NP-hardness of 3IAP due to the structure present in the cost-coefficients in the objective function of multitarget tracking problems). Other references dealing with target tracking problems formulated as axial MIAPs are [23] and [24]; see also [20].

Tables with Given Marginals. Other statistical applications of MITPs require finding multidimensional tables with given sums across rows or higher-dimensional planes, as specified in constraints (3). The right-hand sides d_{Fg} of such constraints are often known as *marginals*. In a simple application [3] arising in the *integration of surveys* and *controlled selection*, each index set represents a population from which a sample is to be drawn. A (joint) sample is a k-tuple, one from each population. The marginals are specified marginal probability distributions over each population, giving rise to axial demand constraints. Given sample costs c_a, the problem is to find a joint probability distribution, defined by (x_a), of all the samples, consistent with these marginal distributions and of minimum expected cost (1).

In contrast, problems of *updating input-output matrices* (see [34] and references therein) typically have nonlinear objectives. In such problems, given are a k-dimensional array B of data (for example, past input-output coefficients) and arrays d of marginals (for example, forecast aggregate coefficients) with appropriate dimensions. The problem is to determine values x_a, the updated array entries, satisfying the demand constraints corresponding to the given marginals, and such that the resulting updated array $X = (x_a)$ differs as little as possible from the given array B, as specified by an appropriate (nonlinear) objective function. A (nonlinear) MITP arises when the values x_a are constrained to be nonnegative, a natural requirement in many contexts.

Other Applications. include an axial integer 3ITP model for planning the launching of weather satellites [27], and an axial integer 5IAP arising in routing meshes in circuit design [9].

Solution Methods. As noted above, MITPs are linear programming problems with a special structure. There are several proposals for extensions of LP (transportation) algorithms to MITPs (e.g., [13], [4] for 3ITPs and [1] for a 4ITP).

As also mentioned earlier, integer MITPS are hard to solve. Exact algorithms have been proposed for the axial integer 3IAP (see **Three-index assignment problem**) and for the planar integer 3IAP (see [39] and [19]). Other exact approaches for integer MITPs rely on structure that is present in the particular application considered (see, e.g., [12]).

Several methods have been proposed to obtain good approximate solutions to integer MITPs. In [21] results are reported for a *rounding heuristic* on some medium-sized planar integer 3ITPs. A *tabu search* algorithm for this problem is described in [18]. Heuristic solution approaches based on *Lagrangian relaxation* are proposed in [26], [28] and [29] for multitarget tracking problems.

One major difficulty with these exact or approximate solution methods may be the sheer size of MITP formulations; if, for example, all $|A_i| = n$ then an m-fold kITP has n^k variables and $\binom{k}{m} n^{k-m}$ constraints. In contrast, the two approaches sketched below yield feasible solutions to *axial* MITPs much more quickly than simply writing down all the cost coefficients. In particular, these algorithms only produce the nonzero variables x_a and their values; all other variables are zero in the solution. In addition, this solution is integral if all demands are integral. Of course, the effectiveness of these methods relies on some assumptions on the cost coefficients c_a, assumptions which are verified in several applications.

A Greedy Algorithm for Axial MITPs. The *greedy algorithm* below (a multi-index extension of the *North-West corner rule*) finds a feasible solution to axial MITPs in $O(k \sum_i |A_i|)$ time, which is (for fixed k) linear in the size of the demand data d_{ig}. This solution is in fact optimal if the cost coefficients are known to satisfy a 'Monge property' [3], [31], [32] defined below. (For $k = 3$, this greedy algorithm is already described in [4] to obtain a basic feasible solution).

Consider the axial kITP with equality constraints (3) and assume that each $A_i = \{1, \dots, |A_i|\}$. Recalling that the demands are denoted d_{ig}, assume that $\sum_{g \in A_i} d_{ig} = \sum_{g \in A_1} d_{1g}$ for all $i \in K$, a necessary and sufficient condition for the problem to be feasible.

```
PROCEDURE greedy MITP algorithm
  WHILE (∑_{g∈A_i} d_ig > 0 for all i ∈ K) DO
    let a(i) = min{g ∈ A_i: d_ig > 0};
    let Δ = min{d_{i,a(i)}: i ∈ K};
    let x_a = Δ;
    FOR i ∈ K DO let d_{i,a(i)} = d_{i,a(i)} − Δ;
RETURN x
END
```

A greedy algorithm for axial MITPs.

A Monge Property. The *join* $a \vee b$ and *meet* $a \wedge b$ of $a, b \in A$ are

$$(a \vee b)_i = \max\{a(i), b(i)\},$$
$$(a \wedge b)_i = \min\{a(i), b(i)\} \quad \text{for all } i \in K.$$

The cost coefficients (c_a) satisfy the *Monge property* if

$$c_{a\vee b} + c_{a\wedge b} \leq c_a + c_b \quad \text{for all } a, b \in A.$$

Note that this is just the *submodularity* of the function $c\colon A \to \mathbf{R}$ defined on the product lattice A, see [3], [31], [32]. These references show that the above greedy algorithm returns an optimal solution for *all* feasible demands if and only if the cost function satisfies the Monge property. The latter two references also extend the greedy algorithm

i) to the case of forbidden cells when the non-forbidden cells form a *sublattice* of A; and

ii) so that it returns an optimal dual solution.

They also show that optimizing a linear function over a *submodular polyhedron* is special case of the dual problem. It is shown in [32] that the primal problems are equivalent to the 'submodular linear programs on forests' of [8].

Cost functions c with the Monge property include typical decomposable costs (as defined above) when all the points are located on a same line or on parallel lines (one line for each factor type A_i). For these problems, the greedy algorithm above amounts to a 'left to right sweep' across the points.

Hub Heuristics for Axial MITPs. The basic idea ([30], extending earlier work on axial 3IAPS [6] and MIAPs [2] with decomposable costs) is to solve a small number of ordinary transportation problems and to expand their solutions into a feasible solution to the original MITP. For a large collection of decomposable costs arising from applications, the objective value of this feasible solution is provably within a constant factor of the optimum.

Given an index h, called the *hub*, determine, for each index $i \neq h$, a feasible solution to the ordinary transportation problem defined by supplies $(d_{ij})_{j\in A(i)}$ and $(d_{hg})_{g\in A(h)}$. The Expand procedure below then takes as inputs these solutions $y^{(h)} = (y^i)_{i\neq h}$ and expands them into a feasible solution $x^{(h)}$ to the axial MITP. Its running time is $O(|A_h| \sum_{i\neq h} |A_i|)$.

```
PROCEDURE Expand(h, y^(h))
FOR g := 1 TO n_h DO
  q := 0;
  a(i) := 1 for i ∈ K \ h;
  WHILE (q < d_{h,g}) DO
    let ℓ be such that
    y^ℓ_{a(ℓ),g} = min{y^r_{a(r),g} : r ≠ h};
    x_a^(h) := y^ℓ_{a(ℓ),g};
    y^r_{a(r),g} := y^r_{a(r),g} − x_a^(h) for all r ∈ K \ h;
    a(ℓ) := a(ℓ) + 1;
    q := q + x_a^(h);
RETURN x^(h)
END
```

The Expand procedure for axial MITPs.

In the hub heuristics for decomposable costs, the ordinary transportation problems use as cost coefficients the distances $\delta(P_{ij}, P_{hg})$ between the corresponding points P_{ij} and P_{hg} in the metric space. The expanded MITP solution x^h would be optimum if the cost function was that of the star with center h, namely if $c_a = \sum_{i\neq h} \delta(P_{i,a(i)}, P_{h,a(h)})$. The *triangle-inequality* property of the distance δ allows one to bound the cost penalty from using this h-star cost function instead of the actual decomposable cost function.

In the *single hub heuristic*, one chooses a hub $h \in K$; solves these $k-1$ transportation problems; inputs their solutions $y^{(h)}$ to Expand; and simply outputs the resulting MITP solution $x^{(h)}$. If the distance δ satisfies the triangle inequality, the cost of this solution $x^{(h)}$ is no more than $k-1$ times the optimal cost, in the worst case, for many common decomposable cost functions. The *multiple-hub heuristic* is an obvious extension whereby one performs the single-hub heuristic k times, once for each $h \in K$, and retains the best solution. This amounts to solving $\binom{K}{2}$ ordinary transportation problems. Under the same assumptions as above and for many common decomposable cost functions, the cost of the resulting solution is less than twice the optimum cost in the worst case.

See also: **Motzkin transposition theorem; Minimum concave transportation problems; Stochastic transportation and location problems.**

References

[1] BAMMI, D.: 'A generalized-indices transportation problem', *Naval Res. Logist. Quart.* **25** (1978), 697–710.

[2] BANDELT, H.-J., CRAMA, Y., AND SPIEKSMA, F.C.R.: 'Approximation algorithms for multidimensional assignment problems with decomposable costs', *Discrete Appl. Math.* **49** (1994), 25–50.

[3] BEIN, W.W., BRUCKER, P., PARK, J.K., AND PATHAK, P.K.: 'A Monge property for the d-dimensional transportation problem', *Discrete Appl. Math.* **58** (1995), 97–109.

[4] CORBAN, A.: 'A multidimensional transportation problem', *Rev. Roumaine Math. Pures et Appl.* **IX** (1964), 721–735.

[5] CORBAN, A.: 'On a three-dimensional transportation problem', *Rev. Roumaine Math. Pures et Appl.* **XI** (1966), 57–75.

[6] CRAMA, Y., AND SPIEKSMA, F.C.R.: 'Approximation algorithms for three-dimensional assignment problems with triangle inequalities', *Europ. J. Oper. Res.* **60** (1992), 273–279.

[7] EULER, R., AND LE VERGE, H.: 'Time-tables, polyhedra and the greedy algorithm', *Discrete Appl. Math.* **65** (1996), 207–222.

[8] FAIGLE, U., AND KERN, W.: 'Submodular linear programs on forests', *Math. Program.* **72** (1996), 195–206.

[9] FORTIN, D., AND TUSERA, A.: 'Routing in meshes using linear assignment', in A. BACHEM, U. DERIGS, M. JÜNGER, AND R. SCHRADER (eds.): *Oper. Res. '93*, 1994, pp. 169–171.

[10] FRIEZE, A.M., AND YADEGAR, J.: 'An algorithm for solving 3-dimensional assignment problems with application to scheduling a teaching practice', *J. Oper. Res. Soc.* **32** (1981), 989–995.

[11] GEETHA, S., AND VARTAK, M.N.: 'The three-dimensional bottleneck assignment problem with capacity constraints', *Europ. J. Oper. Res.* **73** (1994), 562–568.

[12] GILBERT, K.C., AND HOFSTRA, R.B.: 'An algorithm for a class of three-dimensional assignment problems arising in scheduling applications', *IIE Trans.* **8** (1987), 29–33.

[13] HALEY, K.B.: 'The solid transportation problem', *Oper. Res.* **10** (1962), 448–463.

[14] HALEY, K.B.: 'The multi-index problem', *Oper. Res.* **11** (1963), 368–379.

[15] JUNGINGER, W.: 'Zurückführung des Stundenplanproblems auf einen dreidimensionales Transportproblem', *Z. Oper. Res.* **16** (1972), 11–25.

[16] JUNGINGER, W.: 'On representatives of multi-index transportation problems', *Europ. J. Oper. Res.* **66** (1993), 353–371.

[17] KARP, R.M.: 'Reducibility among combinatorial problems', in R.E. MILLER AND J.W. THATCHER (eds.): *Complexity of Computer Computations*, Plenum, 1972, pp. 85–103.

[18] MAGOS, D.: 'Tabu search for the planar three-index assignment problem', *J. Global Optim.* **8** (1996), 35–48.

[19] MAGOS, D., AND MILIOTIS, P.: 'An algorithm for the planar three-index assignment problem', *Europ. J. Oper. Res.* **77** (1994), 141–153.

[20] MAVRIDOU, T., PARDALOS, P.M., PITSOULIS, L., AND RESENDE, M.G.C.: 'A GRASP for the biquadratic assignment problem', *Europ. J. Oper. Res.* **105/3** (March 1998), 613–621.

[21] MILLER, J.L., AND FRANK, L.S.: 'A binary-rounding heuristic for multi-period variable-task duration assignment problems', *Computers Oper. Res.* **23** (1996), 819–828.

[22] MOTZKIN, T.: 'The multi-index transportation problem', *Bull. Amer. Math. Soc.* **58** (1952), 494.

[23] MURPHEY, R., PARDALOS, P.M., AND PITSOULIS, L.: 'A GRASP for the multitarget multisensor tracking problem': *DIMACS*, Vol. 40, Amer. Math. Soc., 1998, pp. 277–302.

[24] MURPHEY, R., PARDALOS, P.M., AND PITSOULIS, L.: 'A parallel GRASP for the data association multidimensional assignment problem': *IMA Vol. Math. Appl.*, Vol. 106, Springer, 1998, pp. 159–180.

[25] PATTIPATTI, K.R., DEB, S., BAR-SHALOM, Y., AND WASHBURN JR., R.B.: 'Passive multisensor data association using a new relaxation algorithm', in Y. BAR-SHALOM (ed.): *Multitarget-multisensor tracking: Advances and applications*, 1990, p. 111.

[26] PATTIPATTI, K.R., DEB, S., BAR-SHALOM, Y., AND WASHBURN JR., R.B.: 'A new relaxation algorithm passive sensor data association', *IEEE Trans. Autom. Control* **37** (1992), 198–213.

[27] PIERSKALLA, W.P.: 'The multidimensional assignment problem', *Oper. Res.* **16** (1968), 422–431.

[28] POORE, A.B.: 'Multidimensional assignment formulation of data-association problems arising from multitarget and multisensor tracking', *Comput. Optim. Appl.* **3** (1994), 27–57.

[29] POORE, A.B., AND RIJAVEC, N.: 'A Lagrangian relaxation algorithm for multidimensional assignment problems arising from multitarget tracking', *SIAM J. Optim.* **3** (1993), 544–563.

[30] QUEYRANNE, M., AND SPIEKSMA, F.C.R.: 'Approximation algorithms for multi-index transportation problems with decomposable costs', *Discrete Appl. Math.* **76** (1997), 239–253.

[31] QUEYRANNE, M., SPIEKSMA, F.C.R., AND TARDELLA, F.: 'A general class of greedily solvable linear programs', in G. RINALDI AND L. WOLSEY (eds.): *Proc. Third IPCO Conf. (Integer Programming and Combinatorial Optimization)*, 1993, pp. 385–399.

[32] QUEYRANNE, M., SPIEKSMA, F.C.R., AND TARDELLA, F.: 'A general class of greedily solvable linear programs', *Math. Oper. Res.* (to appear).

[33] RAUTMAN, C.A., REID, R.A., AND RYDER, E.E.: 'Scheduling the disposal of nuclear waste material in a geologic repository using the transportation model', *Oper. Res.* **41** (1993), 459–469.

[34] ROMERO, D.: 'Easy transportation-like problems on K-

dimensional arrays', *J. Optim. Th. Appl.* **66** (1990), 137–147.

[35] SCHELL, E.: 'Distribution of a product by several properties', in DIRECTORATE OF MANAGEMENT ANALYSIS (ed.): *Second Symposium in Linear Programming 2*, DCS/Comptroller HQ, US Air Force, Washington DC, 1955, pp. 615–642.

[36] SHARMA, J.K., AND SHARUP, K.: 'Time-minimizing multidimensional transportation problem', *J. Engin. Production* **1** (1977), 121–129.

[37] SPIEKSMA, F.C.R., AND WOEGINGER, G.J.: 'Geometric three-dimensional assignment problems', *Europ. J. Oper. Res.* **91** (1996), 611–618.

[38] TZENG, G., TEODOROVIĆ, D., AND HWANG, M.: 'Fuzzy bicriteria multi-index transportation problems for coal allocation planning of Taipower', *Europ. J. Oper. Res.* **95** (1996), 62–72.

[39] VLACH, M.: 'Branch and bound method for the three-index assignment problem', *Ekonomicko-Matematicky Obzor* **3** (1967), 181–191.

[40] VLACH, M.: 'Conditions for the existence of solutions of the three-dimensional planar transportation problem', *Discrete Appl. Math.* **13** (1986), 61–78.

[41] YEMELICHEV, V.A., KOVALEV, M.M., AND KRATSOV, M.K.: *Polytopes, graphs and optimization*, Cambridge Univ. Press, 1984.

Maurice Queyranne
Univ. British Columbia
Vancouver, B.C., Canada
E-mail address: Maurice.Queyranne@commerce.ubc.ca
Frits Spieksma
Maastricht Univ.
Maastricht, The Netherlands
E-mail address: spieksma@math.unimaas.nl

MSC2000: 90C35
Key words and phrases: transportation problem, three-dimensional transportation problem, greedy algorithm, Monge property, approximation algorithms.

MULTI-OBJECTIVE COMBINATORIAL OPTIMIZATION, *MOCO*

It is well known that, on the one hand, *combinatorial optimization* (CO) provides a powerful tool to formulate and model many optimization problems, on the other hand, a multi-objective (MO) approach is often a realistic and efficient way to treat many real world applications. Nevertheless, until recently, Multi-objective combinatorial optimization (MOCO) did not receive much attention in spite of its potential applications. One of the reason is probably due to specific difficulties of MOCO models. We can distinguish three main difficulties. The first two are the same as those existing for multi-objective integer linear programming (MOILP) problem (cf. **Multi-objective integer linear programming**), i.e.

- the number of *efficient solutions* may be very large;
- the nonconvex character of the feasible set requires to device specific techniques to generate the so-called 'nonsupported' efficient solutions (cf. **Multi-objective integer linear programming**).

A particular single CO problem is characterized by some specificities of the problem, generally a special form of the constraints; the existing methods for such problem use these specificities to define efficient ways to obtain an optimal solution. For MOCO problem, it appears interesting to do the same to obtain the set of efficient solutions. Consequently, and contrary to what is often done in MOLP and MOILP methods, a third difficulty is to elaborate methods avoiding to introduce additional constraints so that we preserve during all the procedure the particular form of the constraints.

The general form of a MOCO problem is

$$(\mathrm{P})\quad \begin{cases} {}'\min_{X\in S}{}' & z_k(X) = c_k X, \\ & k = 1,\dots,K, \\ \text{where} & S = D \cap B^n \\ \text{with} & X(n\times 1), \\ & B = \{0,1\} \end{cases}$$

and D is a specific polytope characterizing the CO problem: assignment problem, knapsack problem, traveling salesman problem, etc.

There exists several surveys on MOCO; some are devoted to specific problems (i.e., the particular form of D): the shortest path problem [8], transportation networks [2], and the scheduling problem [6], [7]; the survey [9] is more general examining successively the literature on MO assignment problems, knapsack problems, network flow problems, traveling salesman problems, location problems, set covering problems.

In the present article we put our attention on the existing methodologies for MOCO. First we examine how to determine the set $E(\mathrm{P})$ of all the efficient solutions and we distinguish three ap-

ENCYCLOPEDIA OF OPTIMIZATION

Volume III

Interior – M

C.A. Floudas and P.M. Pardalos, Eds.

Kluwer Academic Publishers
DORDRECHT / LONDON / BOSTON
2001
ISBN 0-7923-6932-7 (set)
0-7923-7028-7 (volume 3)