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Orthogonal schedules in single round robin tournaments

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We dedicate this work to the memory of Gerhard Woeginger

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ABSTRACT

A measure for the flexibility of a Home-Away Pattern set (HAP-set) is the width. The width of a HAP-set equals the size of the largest set of schedules compatible with the HAP-set, for which no match is scheduled in the same round in any two schedules. We prove lower and upper bounds on the width, and identify HAP-sets with largest possible width when the number of teams is a power of 2.

1. Introduction

A basic and popular format for a competition is the well-known Single Round Robin (SRR) format. Given a set of teams $\mathcal{T}$, with $|\mathcal{T}|$ even, an SRR format prescribes that each team plays against each other team once in a set of $|\mathcal{T}|-1$ rounds, such that each team plays once in each round. Moreover, our setting prescribes that in each match there is one team that plays home, and one team that plays away - sometimes abbreviated with H and A respectively.

When faced with the task of deciding upon the fixtures, i.e., to come up with a *schedule* that specifies which match is played in which round, and which team plays home and away in each match, various strategies have been described in literature that come up with a schedule; here, we simply refer to the surveys [6] and [1]. We also mention [2] as an important source of references.

In this note we focus on a question that is relevant for a set of strategies that are known as first-break-then-schedule. These are strategies that follow a 2-step approach: first decide upon the home/away designation of each team in each round (thereby specifying a home-away pattern (HAP) for each team), then allocate all the matches in a way that is compatible with these HAPs, see [8] and [7] for early references. A key question is to what extent specifying the HAP-set in Step 1 impacts the set of possible schedules in Step 2. Or in other words, what is the diversity of schedules compatible with a given HAP-set?

This issue has been investigated in [3] where various measures for the flexibility of a HAP-set are proposed and analyzed. One such measure is called the *width*. Informally, the width of a HAP-set equals the size of the largest set of compatible schedules, such that no match is scheduled in the same round in any two schedules of the set (see Section 2 for precise definitions). Clearly, the larger the width, the more flexible the HAP-set. In [3], it is established that the width of a popular HAP-set known as the canonical HAP-set equals 1. Moreover there exists a match that, in every schedule compatible with the canonical HAP-set, is always scheduled in the same round.

Here we focus on the following questions: do there exist HAP-sets that have large width? And how large can the width actually be?

Section 2 gives the preliminaries, and in Section 3 we give upper and lower bounds for the width. In Section 4, we prove that the upper bound on the width from Section 3 can be achieved for a particular HAP-set when the number of teams is a power of 2. Section 5 details a construction that allows one to combine HAP-sets, and their corresponding schedules, in a way that preserves the width. We close with describing an extension in Section 6.

2. Preliminaries and notation

We consider a set of teams $\mathcal{T}$, with $2n := |\mathcal{T}|$. To avoid trivialities, we assume $n \geq 2$. A single round robin format uses $2n-1$ rounds that we denote by the set R. Throughout the paper, we take $R = \{1, 2, \ldots, 2n-1\}$. For each team $t \in \mathcal{T}$, its *Home-Away Pattern* (HAP) is given by $H(t) = (H_r(t))_{r \in R}$, with $H_r(t) \in \{0, 1\}$, where $H_r(t) = 0$ indicates team t playing Home and $H_r(t) = 1$ in-

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dicates team t playing Away in round $r \in R$. We define a *HAP-set* $\mathcal{H} = \{(H_r(t))_{r\in R} : t \in \mathcal{T}\}$ to consist of a HAP for every team $t \in \mathcal{T}$. Given two teams t, t', we define $\Delta(t, t') = \#\{r : H_r(t) \neq H_r(t')\}$ to be the number of rounds where t, t' differ in Home/Away-allocation.

A partition M of the set of all matches into $2n - 1$ rounds is given by $M = \cup_{r\in R} S_r$ where each S_r is a matching of the $2n$ teams, in such a way that for every pair of distinct teams $t, t' \in \mathcal{T}$, there exists a round $r \in R$ such that $(t, t') \in S_r$, ensuring that every team meets every other team once. A *schedule* S can be generated from such a partition by assigning a Home-team and Away-team in every match in the partition. As notation, we write $S(t, t') = r$ to indicate that in schedule S, the match between teams t, t' is scheduled in round $r \in R$.

Thus, to any schedule S we can associate a corresponding HAP-set $\mathcal{H}(S)$. Two distinct schedules can have equal HAP-sets. We say a HAP-set $\mathcal{H}$ is *feasible* if there exists a schedule S such that $\mathcal{H} = \mathcal{H}(S)$.

We define $\mathbb{H}_n$ to be the set of all feasible HAP-sets on $2n$ teams. As stated earlier, we define $\mathcal{H}(S)$ to be the HAP-set corresponding to schedule S. We also define $\mathcal{S}(\mathcal{H}) = \{S : \mathcal{H}(S) = \mathcal{H}\}$ to be the set of all schedules S that have HAP-set $\mathcal{H}$. We say that schedule S is *compatible* with HAP-set $\mathcal{H}$ if $S \in \mathcal{S}(\mathcal{H})$.

Definition 2.1. Two schedules S, S' are **orthogonal**- $S\perp S'$ - if for every pair of distinct teams $t, t' \in \mathcal{T}$, the round $S(t, t')$ is different from $S'(t, t')$.

In words, schedules S, S' being orthogonal means that no match is scheduled in the same round for S, S'.

Definition 2.2. Two schedules S, S' are **rotational orthogonal** - $S\perp_{\mathrm{rot}} S'$ - if there is a permutation of the rounds $\sigma : R \to R$ without fixed elements, that is, σ is such that there is no round $r \in R$ with $\sigma(r) = r$, and $S_r = S'_{\sigma(r)}$ $\forall r \in R$.

Clearly, any two schedules S, S' that are rotational orthogonal are also orthogonal. Using these definitions of orthogonality, we define the following measures of a given HAP-set $\mathcal{H}$.

Definition 2.3. [3] Given a HAP-set $\mathcal{H}$ for a set $\mathcal{T}$ of $2n$ teams, we define:

- $\mathrm{opp}(\mathcal{H}) = \min_{t,t'\in\mathcal{T}} |\{r : H_r(t) \neq H_r(t')\}|$,
- $\mathrm{width}(\mathcal{H}) = \max_{\mathcal{S}\subset\mathcal{S}(\mathcal{H})} |\{\mathcal{S} : S\perp S' \ \forall S, S' \in \mathcal{S}\}|$,
- $\mathrm{rotw}(\mathcal{H}) = \max_{\mathcal{S}\subset\mathcal{S}(\mathcal{H})} |\{\mathcal{S} : S\perp_{\mathrm{rot}} S' \ \forall S, S' \in \mathcal{S}\}|$.

In words, for a given $\mathcal{H} \in \mathbb{H}$, the measures $\mathrm{opp}(\mathcal{H}), \mathrm{width}(\mathcal{H})$, $\mathrm{rotw}(\mathcal{H})$ are defined as:

- $\mathrm{opp}(\mathcal{H})$: The minimum number of rounds over any pairs of teams t, t' such that they have a different (opposite) Home/Away-assignment in $\mathcal{H}$ - i.e., $\min_{t,t'} \Delta(t, t')$.
- $\mathrm{width}(\mathcal{H})$: The maximum number of schedules compatible with HAP-set $\mathcal{H}$ that are pairwise orthogonal, i.e., where no match (t, t') is played in the same round in two different schedules.
- $\mathrm{rotw}(\mathcal{H})$: The maximum number of schedules with HAP-set $\mathcal{H}$ that are pairwise rotational orthogonal.

It is not difficult to see that $\mathrm{opp}(\mathcal{H}) \geq \mathrm{width}(\mathcal{H}) \geq \mathrm{rotw}(\mathcal{H})$ for each feasible $\mathcal{H} \in \mathbb{H}$.

Remark 2.1. For notational purposes, we define the rotational width of a schedule $S = (S_r)_r$ compatible with $\mathcal{H}$, denoted $\mathrm{width}_{rot}(S)$ instead of $\mathrm{width}_{rot}(\mathcal{H})$ as follows. $\mathrm{width}_{rot}(S)$ equals the cardinality of the largest set of pairwise orthogonal schedules that can be obtained by permuting the rounds S_r, with all schedules compatible to $\mathcal{H}$. Obviously $\mathrm{width}_{rot}(S) \leq \mathrm{width}_{rot}(\mathcal{H})$.

In Definition 2.3, we have defined three measures of a given HAP-set $\mathcal{H}$. Our main goal here is to find HAP-sets with extremal values for these measures. Therefore, for each of these measures, we define o_n, w_n, x_n as follows:

Definition 2.4. For each $2n \geq 4$, we define:

- $o_n = \max_{\mathcal{H}\in\mathbb{H}_n} \mathrm{opp}(\mathcal{H})$,
- $w_n = \max_{\mathcal{H}\in\mathbb{H}_n} \mathrm{width}(\mathcal{H})$,
- $x_n = \max_{\mathcal{H}\in\mathbb{H}_n} \mathrm{rotw}(\mathcal{H})$.

Simply put, o_n, w_n, x_n equals the value of a HAP-set that scores best on the respective measure for a given n.

With these definitions, we get to the following fundamental question:

Question 2.1. *For a given value of n, what is w_n? And what are o_n, x_n?*

3. Upper and lower bounds for the width

We establish the following lower and upper bounds on the width:

Theorem 3.1. *For each $n \geq 2$:*

$$2 \leq x_n \leq w_n \leq o_n \leq n.$$

Proof. The inequalities $x_n \leq w_n \leq o_n$ are immediate, as for each $\mathcal{H} \in \mathbb{H}_n$, we have $\mathrm{opp}(\mathcal{H}) \geq \mathrm{width}(\mathcal{H}) \geq \mathrm{rotw}(\mathcal{H})$.

We now argue that $o_n \leq n$. Consider any HAP-set $\mathcal{H} \in \mathbb{H}_n$. As $\mathcal{H}$ is feasible, it follows that in every round, there are n teams that play at home and n teams that play away, so there are n^2 pairs of teams with a different Home/Away-allocation, leading to a total sum of different Home/Away-allocations equal to $\sum_{(t,t')\in\binom{\mathcal{T}}{2}} \Delta(t, t') = (2n - 1)n^2$. As there are $\binom{2n}{2} = n(2n - 1)$ pairs of teams, there must be a pair of teams $t, t' \in \mathcal{T}$ with $\Delta(t, t') \leq n$. Thus, for any HAP-set $\mathcal{H} \in \mathbb{H}_n$, we have $\mathrm{opp}(\mathcal{H}) \leq n$, which implies $o_n \leq n$.

We now show that $2 \leq x_n$. We prove this inequality by first considering a partition of the set of all matches into $2n - 1$ rounds; next, we construct a HAP-set such that there exist two schedules compatible with it, such that all matches in round r in one schedule, are scheduled in round $r + 1$ in the other schedule, where we work modulo $2n - 1$.

There are many ways to partition the set of $\binom{2n}{2}$ matches into $2n - 1$ rounds $S_1, \ldots, S_{2n-1}$ such that each round consists of n matches featuring each team exactly once; let's call such a partition a *feasible partition* of the matches. One possibility to find a feasible partition is the well-known Circle Method, see e.g. [5] and [9]; it is given by any proper edge coloring of K_{2n} with $2n - 1$ colors. Thus, we can assume we are given a set of rounds $S_1, \ldots, S_{2n-1}$; notice that the Home/Away assignment for the matches in these rounds has not been specified. We will give a procedure that constructs a HAP-set $\mathcal{H}$ in a round-by-round fashion, in such a way that there exist two schedules compatible with $\mathcal{H}$ that are rotational orthogonal.

Fix a round $r \in R$, and construct a simple undirected graph $G_{r,r+1} = (V = \mathcal{T}, E_{r,r+1})$, where $(t, t') \in E_{r,r+1}$ iff match (t, t') is played in round r or $r + 1$ (indices are read modulo $2n - 1$, thus $2n = 1$). Clearly, $G_{r,r+1}$ is a regular graph of degree 2, where every

Table 1
A HAP-set with rotational width at least 2.

	Rnd 1	Rnd 2	Rnd 3	Rnd 4	...	Rnd $2n-1$
Team 0	A	A	A	A	...	A
Team 1	**H**	**H**	A	H	...	A
Team 2	A	**H**	**H**	A	...	H
Team 3	H	A	**H**	**H**	...	A
⋮	⋮	⋮	⋮	⋱	⋱	⋮
Team $2n-1$	**H**	A	H	A	...	**H**

node (team) is incident to an edge corresponding to its match-up in round r and to an edge corresponding to its match-up in round $r+1$, $r \in R$. As $G_{r,r+1}$ is the edge-union of two perfect matchings, it can be seen as a collection of even cycles. For every cycle, define an orientation (clockwise or counter-clockwise), traverse every cycle, and for every edge (match) that is scheduled in round $r+1$, assign the first node (with respect to the orientation) to be the Home playing team and the second node the team that plays Away in round $r+1$.

Notice that this Home/Away assignment accommodates the matches in S_{r+1} as the nodes corresponding to each pair of teams that are matched in S_{r+1} are connected in $G_{r,r+1}$, and the construction ensures that these nodes receive a different Home/Away assignment. It is also true that this Home/Away assignment simultaneously accommodates the matches in S_r. Indeed, again the nodes corresponding to each pair of teams that are matched in S_r are connected in $G_{r,r+1}$, and hence receive a different Home/Away assignment.

When we perform this procedure sequentially for $r = 1, 2, \dots, 2n-1$, we thereby specify a HAP-set $\mathcal{H}$. By construction, this HAP-set is compatible with the schedule S, and it is also compatible with the schedule S', where S' is obtained from S by moving every match one round earlier (modulo $2n-1$).

Concluding, we have constructed a HAP-set $\mathcal{H}$, and we have shown that there exist two schedules S and S' that are compatible with it while $S \perp_{\text{rot}} S'$. This concludes the proof. □

Remark 3.1. Notice that, if n is odd, the upper bound from Theorem 3.1 can be improved. Indeed, any feasible HAP-set $\mathcal{H}$ on $2n$ teams has exactly $n(2n-1)$ Home assignments, which is an odd number when n is odd. When two teams t, t' both have an even number of Home's assigned, they have a different Home/Away-allocation in an even number of rounds, so $\Delta(t, t')$ is even. Also, when two teams t, t' both have an odd number of Home's assigned, $\Delta(t, t')$ is even. When there are more than 2 teams, there must be a pair of teams t, t' that have the same parity number of Home games, so they will have $\Delta(t, t')$ even. As n is presumed odd, and the only way to make a HAP-set with $\text{opp}(\mathcal{H}) = n$ is if every pair of teams t, t' has $\Delta(t, t') = n$, we see that $\text{opp}(\mathcal{H}) < n$.

Notice also that the procedure sketched in the second part of the proof of Theorem 3.1 starts from any feasible partition of the matches into rounds. In case one would start from the partition that is generated by the well-known circle method, we can, using this procedure identify an explicit HAP-set. In fact, we claim that the HAP-set from Table 1 arises, and it follows that this HAP-set has rotational width at least 2. One might comment that this HAP-set is unbalanced in the sense that Team 0 only has Away matches, whereas each other team plays only $n-1$ away matches. However, as the operation of inverting all Home/Away assignments in a single round does not impact the (rotational) width, one can improve this balance by inverting the Home/Away assignments in $\frac{n}{2}$ rounds, leading to a HAP-set such that no team plays more than $\frac{3}{2}n - 1$ matches away.

4. HAP-sets with maximum width

In this section, we identify a family of HAP-sets whose width equals the upper bound established in Section 3 in case the number of teams is a power of 2.

Theorem 4.1. *When $n = 2^\ell$ ($\ell \in \mathbb{N}$), there is a HAP-set $\mathcal{H}^* \in \mathbb{H}_n$ with width$(\mathcal{H}^*) = n$.*

Proof. We prove this by constructing HAP-set $\mathcal{H}^*$ and providing n orthogonal schedules that are compatible with $\mathcal{H}^*$.

Constructing $\mathcal{H}^*$ Let $\mathcal{P}$ be the set of all $2^{\ell+1}$ subsets of $\{1, 2, \dots, \ell+1\}$, and let $\mathcal{P}_0 = \mathcal{P} \setminus \{\emptyset\}$. The $2n$ teams are represented by $\mathcal{T} := \{T_A : A \in \mathcal{P}\}$, and the rounds by $\mathcal{R} := \{R_B : B \in \mathcal{P}_0\}$. For each $A \in \mathcal{P}$ and for each $B \in \mathcal{P}_0$ we choose:

$$H_{R_B}(T_A) := |A \cap B| \pmod 2.$$

When $H_{R_B}(T_A) = 0$, this implies a Home match for team $T_A \in \mathcal{T}$ in round $R_B \in \mathcal{R}$, otherwise when $H_{R_B}(T_A) = 1$ this implies an away match. We claim that the HAP-set $\mathcal{H}^* = \{(H_{R_B}(T_A))_{B \in \mathcal{P}_0} : A \in \mathcal{P}\}$ is a HAP-set with (rotational) width n.

Generating the schedules Note that two teams T_{A_1} and T_{A_2} can play each other at round R_B if and only if $|A_1 \cap B|$ and $|A_2 \cap B|$ have different parities. This is equivalent to the condition that $|A_1 \cap B| + |A_2 \cap B|$ is odd, which is also the same as $|(A_1 \triangle A_2) \cap B|$ being odd.

Therefore, the following is true:

$$T_{A_1} \text{ and } T_{A_2} \text{ can be scheduled at round } R_B \text{ iff } |(A_1 \triangle A_2) \cap B| \text{ is odd.} \tag{1}$$

Next, for each $K \in \mathcal{P}_0$, we define a set of (unordered) pairs of teams, each being a perfect matching on the teams:

$$M_K := \{\{X, X \triangle K\} : X \in \mathcal{P}\}.$$

If $X = Y \triangle K$, then $Y = X \triangle K$. Moreover, for any $X, Y \in \mathcal{P}$, there is a unique K such that $\{X, Y\} \in M_K$, namely $K = X \triangle Y$. Thus, $\mathcal{M} = \{M_K : K \in \mathcal{P}_0\}$ is a feasible partition of matches into rounds.

Crucially, all the matches in M_K can be scheduled at round R_B if and only if $|K \cap B|$ is odd by (1) - and there are exactly n rounds R_B for which this holds.

To find the n rotational orthogonal schedules, construct the bipartite graph $G = (\mathcal{M} \cup \mathcal{R}, E)$ where $(M_K, R_B) \in E$ if and only if $|K \cap B|$ is odd. Since each nonempty set has equal number of odd sized and even sized subsets, G is n-regular. Hence, there exists a 1-factorization of G - each 1-factor is a perfect matching giving one of the n rotational orthogonal schedules. □

Example 4.1. *Table 2 contains the HAP-set that follows from the construction in Theorem 4.1 when $2n = 2 \cdot 2^2 = 8$.*

We find the four pairwise orthogonal schedules for the HAP-set in Table 3.

Proof of Theorem 4.1 gives a HAP-set on $2n$ teams with complete rotational width in the case $n = 2^\ell$. Next, we will show that this complete rotational width is reserved only for powers of two.

Theorem 4.2. *If $x_n = n$ for some $n \geq 2$, then $n = 2^\ell$ ($\ell \in \mathbb{N}$).*

Proof. Let $\mathcal{H}$ be a HAP-set on $\mathcal{T} = \{1, 2, \dots, 2n\}$ such that $\text{rotw}(\mathcal{H}) = n$. Denote the rounds as $R = \{1, 2, \dots, 2n-1\}$, and let

Table 2
HAP-set on $2n = 8$ teams.

Teams \ Rounds	1	2	12	3	13	23	123
∅	0	0	0	0	0	0	0
1	1	0	1	0	1	0	1
2	0	1	1	0	0	1	1
12	1	1	0	0	1	1	0
3	0	0	0	1	1	1	1
13	1	0	1	1	0	1	0
23	0	1	1	1	1	0	0
123	1	1	0	1	0	0	1

Table 3
Four pairwise orthogonal schedules for 8 teams.

Schedule	Round 1	2	3	4	5	6	7
S_1	0-1	0-2	0-5	0-6	0-3	0-4	0-7
	2-3	1-3	3-6	1-7	2-1	1-5	3-4
	4-5	4-6	4-1	2-4	5-6	6-2	5-2
	6-7	5-7	7-2	3-5	7-4	7-3	6-1
S_2	0-3	0-6	0-2	0-7	0-1	0-5	0-4
	2-1	1-7	3-1	1-6	2-3	1-4	3-7
	4-7	4-2	4-6	2-5	5-4	6-3	5-1
	6-5	5-3	7-5	3-4	7-6	7-2	6-2
S_3	0-5	0-7	0-1	0-4	0-6	0-3	0-2
	2-7	1-6	3-2	1-5	2-4	1-2	3-1
	4-1	4-3	4-5	2-6	5-3	6-5	5-7
	6-3	5-2	7-6	3-7	7-1	7-4	6-4
S_4	0-7	0-3	0-6	0-5	0-4	0-2	0-1
	2-5	1-2	3-5	1-4	2-6	1-3	3-2
	4-3	4-7	4-2	2-7	5-1	6-4	5-4
	6-1	5-6	7-1	3-6	7-3	7-5	6-7

A_i be the set of rounds in which team i plays its game at Home. By Theorem 3.1, we have $n = \text{rotw}(\mathcal{H}) \le \text{opp}(\mathcal{H}) \le o_n \le n$, which implies $\text{opp}(\mathcal{H}) = n$. As previously explained in Remark 3.1, this shows

$$|A_x \triangle A_y| = \Delta(x, y) = n \text{ for any teams } x \neq y \tag{2}$$

On the other hand, as $rotw(\mathcal{H}) = n$, the set of all pairs of teams can be partitioned into $2n-1$ perfect matchings $M_1, M_2, \ldots, M_{2n-1}$ such that each M_i can be played in at least n rounds. Then, by (2), we can conclude that each M_i can be played at exactly n rounds. On the other hand, if $(x, y), (z, w) \in M_i$ for some i, then $A_x \triangle A_y$ and $A_z \triangle A_w$ contain the rounds in which all the matches M_i can be played. In other words, we obtain $|(A_x \triangle A_y) \cap (A_z \triangle A_w)| \ge n$, which gives $A_x \triangle A_y = A_z \triangle A_w$ by (2). Hence, for each perfect matching M_i, there is a subset $S_i \subseteq R$ of size n such that $A_x \triangle A_y = S_i$ whenever $(x, y) \in M_i$. For any $i, j \in R$ with $i \neq j$, pick $(x, y) \in M_i$ and $(x, z) \in M_j$. Since $A_x \triangle A_y = S_i$, $A_x \triangle A_z = S_j$, we have $A_y \triangle A_z = S_i \triangle S_j$. On the other hand, we know $(y, z) \in M_k$ for some $k \notin \{i, j\}$, then we obtain $S_i \triangle S_j = S_k$ for some $k \in R$. As a result, if $x_n = n$, we can find $S_1, S_2, \ldots, S_{2n-1} \subseteq R$ such that

$$\text{For any } i, j \in R \text{ with } i \neq j, \text{ there exists } k \in R \text{ with } S_i \triangle S_j = S_k. \tag{3}$$

We will show that such sets can be found only if n is a power of two. For each $i \in R$, consider the indicator vector v_i of S_i that is defined as

$$(v_i)_j := \begin{cases} 1, & \text{if } j \in S_i \\ 0, & \text{otherwise.} \end{cases}$$

Let v_0 be the zero vector of dimension $2n-1$, and write $\mathcal{C} = \{v_i : 0 \le i \le 2n-1\}$. We can think of $\mathcal{C}$ as a subset of the vector space $\mathbb{F}_2^{2n-1}$, under usual addition and scalar multiplication. On the other hand, it can be easily seen that (3) is equivalent to the following:

$$\text{For any } i, j \in R, \text{ we have } v_i + v_j \in \mathcal{C}.$$

This implies that $\mathcal{C}$ is closed under addition. Moreover, it is closed under scalar multiplication, as this is trivial for $\mathbb{F}_2^{2n-1}$. So we can conclude that $\mathcal{C}$ is a subspace of $\mathbb{F}_2^{2n-1}$. Therefore, its order $2n$ should be a power of 2, which implies that n is a power of 2, which completes the proof. □

5. Maximum opposing HAP-sets

We call a HAP-set $\mathcal{H}$ on $2n$ teams a *maximum opposing* HAP-set if $\text{opp}(\mathcal{H}) = n$. In this section we introduce a procedure to create a HAP-set $\mathcal{H}$ that is maximum opposing, provided that we have two other HAP-sets, $\mathcal{H}_1, \mathcal{H}_2$ on $2n_1, 2n_2$ teams with $n = 2n_1 n_2$ and $\mathcal{H}_1, \mathcal{H}_2$ both maximum opposing.

For the matrices $A_{m\times n}$ and $B_{p\times q}$, the Kronecker product of A and B, denoted by $A \otimes B$, is defined as the $pm \times qn$ block matrix:

$$\begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}$$

Lemma 5.1. *Suppose $A_{m\times m}$ and $B_{n\times n}$ are orthogonal matrices. Then, $A \otimes B$ is also orthogonal.*

Proof. It follows from the well-known properties of the Kronecker product:

$$\begin{aligned}(A \otimes B) \cdot (A \otimes B)^T &= (A \otimes B)\left(A^T \otimes B^T\right) \\ &= \left(AA^T\right) \otimes \left(BB^T\right) = I_m \otimes I_n = I_{mn}. \quad \square\end{aligned}$$

For a given HAP-set $\mathcal{H} = \left\{(H_r(t))_{1\le r\le 2n-1} : t \in \{1, \ldots, 2n\}\right\}$, define the matrix $M^{\mathcal{H}}$ as follows:

$$M_{ij}^{\mathcal{H}} = \begin{cases} -1/\sqrt{2n}, & \text{if } j \ge 2 \text{ and } H_{j-1}(i) = 0, \\ 1/\sqrt{2n}, & \text{otherwise,} \end{cases}$$

for $1 \le i, j \le 2n$. For a given matrix $M_{2n\times 2n}$ with entries $\{1, -1\}$, define the HAP-set $\mathcal{H}(M) = \left\{(H_r(t))_{1\le r\le 2n-1} : t \in \{1, \ldots, 2n\}\right\}$ such that $H_r(t) = M_{r+1,t}$ holds for all $1 \le r \le 2n-1$ and $1 \le t \le 2n$.

Theorem 5.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ be HAP-sets on $2n_1, 2n_2$ teams and $\text{opp}(\mathcal{H}_i) = n_i$ for $i = 1, 2$. Then, there is a HAP-set on $4n_1n_2$ teams that is maximum opposing.*

Proof. It is important to recall that if $\mathcal{H}$ is a HAP-set with $\text{opp}(\mathcal{H}) = n$, then $\Delta(t, t') = n$ holds for all teams $t \neq t'$, which shows $M^{\mathcal{H}}$ is orthogonal. Conversely, if $M_{2n\times 2n}$ is a matrix with entries $\{1, -1\}$ such that its rows are orthogonal to each other and all the entries in its first column are the same, then $\text{opp}(\mathcal{H}(M)) = n$ holds. Suppose HAP-sets $\mathcal{H}_1, \mathcal{H}_2$ are given by:

$$\mathcal{H}_1 = \left\{(H_r^1(t))_{1\le r\le 2n_1-1} : t \in \{1, \ldots, 2n_1\}\right\}$$

$$\mathcal{H}_2 = \left\{(H_r^2(t))_{1\le r\le 2n_2-1} : t \in \{1, \ldots, 2n_2\}\right\}$$

By Lemma 5.1, $M^{\mathcal{H}_1} \otimes M^{\mathcal{H}_2}$ is orthogonal. Consider the matrix

$$M = 2\sqrt{n_1 n_2} \cdot \left(M^{\mathcal{H}_1} \otimes M^{\mathcal{H}_2}\right).$$

Table 4
A HAP-set $\mathcal{H}$ on 12 teams with $\text{opp}(\mathcal{H}) = 6$.

0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	0	0	0	0	0
1	1	1	0	0	0	1	1	1	0	0
1	1	0	1	0	0	1	0	0	1	1
1	0	1	0	1	0	0	1	0	1	1
0	1	1	0	0	1	0	0	1	1	1
0	0	1	1	1	0	1	0	1	0	1
0	0	1	1	0	1	1	1	0	1	0
0	1	0	1	1	0	0	1	1	1	0
0	1	0	0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	1	1	0	1
1	0	0	0	1	1	1	0	1	1	0

Observe that all the entries of M are ± 1 and any two rows of M are orthogonal to each other. Moreover, each entry in the first column of M is 1. Then, the HAP-set $\mathcal{H}(M)$ on $4n_1n_2$ teams satisfies $\text{opp}(\mathcal{H}(M)) = 2n_1n_2$. $\square$

Corollary 5.1. *When $o_n = n$, $o_{2n} = 2n$.*

Proof. If $o_n = n$, there must exist a HAP-set $\mathcal{H}$ on $2n$ teams, such that $\text{opp}(\mathcal{H}) = n$. Notice that the trivial HAP-set on $2n = 2$ teams, $\mathcal{H}^*$, has $\text{opp}(\mathcal{H}^*) = 1 = n_1$. By Theorem 5.1, the result follows. $\square$

The procedure in the proof of Theorem 5.1 preserves the property of having maximum opposing rounds of two HAP-sets. So far we have only shown for HAP-sets where the number of teams equals a power of two that they can have maximum opposing rounds. And in those cases, we can even explicitly construct the HAP-set, without using smaller sized HAP-sets, as shown in Section 4.

However, the property of having maximum opposing rounds, is not reserved for HAP-sets on 2^ℓ teams, as can be seen by the HAP-set $\mathcal{H}$ on 12 teams in Table 4. It has $\text{opp}(\mathcal{H}) = 6$, which is maximum. It is also the smallest possible number of teams n for which $n \neq 2^\ell$ and the HAP-set is maximum opposing.

The largest known set of orthogonal schedules compatible with the HAP-set in Table 4, obtained using a mixed integer program, has order 4.

6. An extension

It is clear that the width is a measure indicating to what extent a particular HAP-set can accommodate distinct schedules where an individual match is played in different rounds in distinct schedules. This is clearly relevant in a first-break-then-schedule approach, see Section 1. From a scheduling point of view, not only the round in which a match is scheduled matters, but also the other matches in that round. It could very well be that there is a preference to have pairs of matches in separate rounds. To capture this idea, we introduce *match-pair disjointness*. Given a HAP-set, two schedules are match-pair disjoint if each pair of matches that are in the same round in one schedule, are not in the same round in the other schedule.

Table 5
A HAP-set that allows two feasible match-pair disjoint schedules.

Teams/Rounds	1	2	3	4	5	6	7
0	H	H	H	H	H	H	H
1	A	A	A	H	A	H	A
2	H	A	H	A	H	H	H
3	A	H	A	H	H	A	H
4	A	H	H	A	H	A	A
5	H	H	H	A	A	H	A
6	H	A	A	H	A	A	H
7	A	A	A	A	A	A	A

Table 6
Two match-pair disjoint orthogonal schedules compatible with the HAP-set from Table 6.

Rounds	1	2	3	4	5	6	7
Schedule 1	0-1	0-2	0-2	0-4	0-5	0-6	0-7
	2-3	3-1	2-1	1-5	2-7	1-7	2-5
	5-4	4-6	4-7	3-7	6-2	3-6	3-4
	6-7	5-7	5-6	6-2	4-1	5-3	6-1
Schedule 2	0-4	0-1	0-7	0-2	0-6	0-3	0-5
	2-7	3-6	2-3	1-7	2-1	1-4	2-4
	5-3	4-7	4-6	3-4	3-7	2-6	3-1
	6-1	5-2	5-1	6-5	4-5	5-7	6-7

Notice that this property is different from orthogonality, i.e., a pair of schedules may be match-pair disjoint or not, and they may be orthogonal or not. Although we have no theoretical results for this property of match-pair disjointness, we provide, for $2n = 8$ teams, two schedules that are both match-pair disjoint, as well as orthogonal. (See Table 5.)

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