1 Normed spaces, Banach spaces

Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $E$ be a vector space over $\mathbb{K}$.

**Definitions:** A mapping $\| \cdot \|$ from $E$ to $\mathbb{R}$ is called a norm iff:

- $\|x\| \geq 0$ for all $x \in E$,
- $\|x\| = 0 \iff x = 0$,
- $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in E$, $\lambda \in \mathbb{K}$,
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$. (triangle inequality)

Then $E$ together with $\| \cdot \|$ is called a normed space.

A simple consequence of the triangle inequality is the so-called inverse triangle inequality: For all $x, y \in E$,

$$|\|x\| - \|y\|| \leq \|x - y\|.$$  \hfill (1.1)

(Check!)

The distance of two points (elements) in $E$ is

$$d(x, y) := \|x - y\|.$$  

The open ball around $x_0$ in $E$ with radius $r > 0$ is

$$B_{E}(x_0, r) := \{ x \in E \mid d(x, x_0) < r \}.$$  

The distance of a point $x \in E$ from a (nonempty) subset $A \subset E$ is

$$d(x, A) := \inf_{y \in A} d(x, y).$$  

The closure of a subset $A \subset E$ is

$$\overline{A} := \{ x \in E \mid d(x, A) = 0 \}.$$  

A subset $A \subset E$ is called closed iff $A = \overline{A}$. A sequence $(x_n)$ in $E$ is called convergent to $x^* \in E$ iff $d(x_n, x^*) \to 0$ as $n \to \infty$. $x^*$ is called the limit of the sequence $(x_n)$. We write $x_n \overset{E}{\to} x^*$ or $\lim_{n \to \infty} x_n = x^*$.

**Lemma 1.1** *(Elementary limit properties)*

Assume $x_n \overset{E}{\to} x^*$, $y_n \overset{E}{\to} y^*$, $\lambda_n \overset{E}{\to} \lambda^*$. Then

(i) $x_n + y_n \overset{E}{\to} x^* + y^*$,

(ii) $\lambda_n x_n \overset{E}{\to} \lambda^* x^*$,

(iii) $\|x_n\| \to \|x^*\|$.

**Proof:** (i) and (ii): Straightforward (Check!)

(iii) By (1.1),

$$|\|x_n\| - \|x^*\|| \leq \|x_n - x^*\| \to 0,$$

hence $\|x_n\| \to \|x^*\|$.

\[\square\]
Lemma 1.2 (Closure and limit points) Assume $A \subset E$. A point $y \in E$ belongs to $\overline{A}$ if and only if there is a sequence $(x_n)$ in $A$ with $x_n \xrightarrow{E} y$. In particular, $A$ is closed if and only if all convergent sequences in $A$ have their limit in $A$.

Definition: A subset $A \subset E$ is called dense in $E$ iff $\overline{A} = E$.

Lemma 1.3 (Dense sets) A subset $A \subset E$ is dense in $E$ if and only if for any $\varepsilon > 0$ and any $x \in E$ there is a $y \in A$ such that $d(x,y) < \varepsilon$.

Definition: A sequence $(x_n)$ in $E$ is called a Cauchy sequence iff $d(x_n,x_m) \rightarrow 0$ as $n,m \rightarrow \infty$. More precisely,

$$\forall \varepsilon > 0 \exists N = N(\varepsilon) \forall m,n > N : d(x_m,x_n) < \varepsilon.$$  

Every convergent sequence is a Cauchy sequence. (Check!) Every Cauchy sequence that has a convergent subsequence is convergent. (Check!)

Definition: A normed space $(E,\|\cdot\|)$ is called a Banach space if it is complete, i.e., if every Cauchy sequence in $E$ converges.

Definition: Let $(E,\|\cdot\|)$ be a normed space and $f_1,f_2,\ldots \in E$. The series $\sum_{n=1}^\infty f_n$ is called convergent if the sequence $\{s_N\}$ of its partial sums $s_N := \sum_{n=1}^N f_n$ is convergent. The series $\sum_{n=1}^\infty f_n$ is called absolutely convergent if $\sum_{n=1}^\infty \|f_n\| < \infty$.

Lemma 1.4 (Completeness criterion for normed spaces) Let $(E,\|\cdot\|)$ be a normed space in which every absolutely convergent series is convergent.

Then $(E,\|\cdot\|)$ is complete, i.e. a Banach space.

Proof: Let $\{g_n\}$ be a Cauchy sequence in $E$. This sequence has a subsequence $\{\tilde{g}_n\}$ such that

$$\|\tilde{g}_{n+1} - \tilde{g}_n\| \leq 2^{-n}.$$  

(Check!) Now set $f_n := \tilde{g}_{n+1} - \tilde{g}_n$. Then $\sum_{n=1}^\infty \|f_n\| < \infty \xrightarrow{\text{Assumption}} \{s_N\} = \{\sum_{n=1}^N f_n\} = \{\tilde{g}_{N+1} - \tilde{g}_1\}$ convergent $\Rightarrow \{\tilde{g}_n\}$ convergent $\Rightarrow \{g_n\}$ convergent. $\blacksquare$

2 Inner Product spaces, Hilbert spaces

Let $E$ be a vector space over $\mathbb{K}$. A mapping $(\cdot,\cdot)$ from $E \times E$ to $\mathbb{K}$ is called an inner product iff

- $(x,y) = (y,x)$ for all $x,y \in E$,
- $(\lambda x,y) = \lambda (x,y)$ for all $x,y \in E$, $\lambda \in \mathbb{K}$
- $(x+y,z) = (x,z) + (y,z)$ for all $x,y,z \in E$,
- $(x,x) \geq 0$, and $(x,x) = 0 \iff x = 0$.

Then $E$ together with $(\cdot,\cdot)$ is called an Inner Product space. Additionally we define

$$\|x\| := (x,x)^{\frac{1}{2}}.$$  

(2.1)
Lemma 2.1 (Elementary properties of Inner Product spaces)

(i) Cauchy-Schwarz inequality: For all \( x, y \in E \):
\[
|\langle x, y \rangle| \leq \|x\| \|y\|.
\]

(ii) \( \| \cdot \| \) given by (2.1) is a norm on \( E \), hence \( (E, \| \cdot \|) \) is a normed space.

(iii) Parallelogram identity: For all \( x, y \in E \):
\[
\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).
\]

Proof: (i): The result is obviously true for \( \langle x, y \rangle = 0 \). Assume \( \langle x, y \rangle \neq 0 \). Then \( \|x\|, \|y\| \neq 0 \) and
\[
0 \leq \left| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right|^2 = 2 - 2\Re(\langle x, y \rangle),
\]
hence
\[
\Re(\langle x, y \rangle) \leq \|x\| \|y\|.
\]
Replacing \( x \) by \( \bar{x} := \langle x, y \rangle x \) in the last inequality yields
\[
|\langle x, y \rangle|^2 \leq |\langle x, y \rangle| \|x\| \|y\|,
\]
and the result follows by dividing by \( |\langle x, y \rangle| \).

(ii): We have by (i)
\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\Re(\langle x, y \rangle) \leq \|x\|^2 + \|y\|^2 + 2\|x\| \|y\| = (\|x\| + \|y\|)^2,
\]
and this implies the triangle inequality. The other properties are easy to check.

(iii) Straightforward.

By (ii), all concepts defined for normed spaces (convergence, distance, closure etc.) are also defined for inner product spaces. In particular:

Definition: A complete inner product space is called Hilbert space.

Definition: Two elements \( x, y \in E \) are called orthogonal iff \( \langle x, y \rangle = 0 \). We write \( x \perp y \).
An element \( x \) is called orthogonal to a subset \( A \subset E \) iff \( x \perp y \) for all \( y \in A \). We write \( x \perp A \).

Lemma 2.2 (Pythagoras’ theorem)
Let \( x, y \in E \). If \( x \perp y \) then \( \|x + y\|^2 = \|x\|^2 + \|y\|^2 \).

Proof: Straightforward.

Lemma 2.3 (Orthogonality and closure)
Assume \( x \in E, A \subset E \). If \( x \perp A \) then \( x \perp \overline{A} \).

Proof: Straightforward.
3 The Hilbert space $L^2(\Omega)$

Let $\Omega \subset \mathbb{R}^m$ be a domain, i.e. an open and connected subset of $\mathbb{R}^m$. Let $\mu$ denote the Lebesgue measure (see Appendix). For brevity we write

$$\int_\Omega f := \int_\Omega f(x) \, d\mu(x).$$

**Definition:** $L^2(\Omega)$ is the set of all measurable functions $f : \Omega \rightarrow \mathbb{K}$ such that $|f|^2$ is integrable. Due to

$$\int_\Omega |f + g|^2 \leq \int_\Omega (|f| + |g|)^2 \leq 2 \left( \int_\Omega |f|^2 + \int_\Omega |g|^2 \right)$$

it is easy to see that $L^2(\Omega)$ is a vector space with respect to pointwise addition and scalar multiplication. Moreover,

$$\mathcal{N} := \{ f \in L^2(\Omega) \mid f(x) = 0 \text{ a.e.} \}.$$

$\mathcal{N}$ is a linear subspace of $L^2(\Omega)$ (Check!).

On $L^2(\Omega)$ we define the relation $\sim$ by

$$f \sim g \iff f - g \in \mathcal{N}.$$

This is an equivalence relation (Check!). Hence it defines equivalence classes

$$[f] := \{ g \in L^2(\Omega) \mid f \sim g \}, \quad f \in L^2(\Omega).$$

On these equivalence classes, we define an addition and a scalar multiplication by

$$[f] + [g] := [f + g], \quad \lambda[f] := [\lambda f],$$

$f, g \in L^2(\Omega), \lambda \in \mathbb{K}$. These operations are well-defined, i.e. independent of the representants of the equivalence classes (Check!).

**Theorem and Definition:** The set of these equivalence classes,

$$L^2(\Omega) := L^2(\Omega)/\mathcal{N} := \{ [f] \mid f \in L^2(\Omega) \},$$

is a linear space with respect to these operations (Check!).

**Remark:** In the sequel, we will write $f$ instead of $[f]$ for elements of $L^2(\Omega)$. Note, however, that then $f = g$ has to be interpreted as $f(x) = g(x)$ for almost all $x \in \Omega$.

**Lemma 3.1 (Inner product on $L^2(\Omega)$)**

The mapping $(\cdot, \cdot) : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{K}$ given by

$$(f, g) := \int_\Omega f \overline{g}$$

is an inner product on $L^2(\Omega)$.
Proof: At first we note that $(\cdot, \cdot)$ is independent of the chosen representants, i.e. if $f_1 \sim f_2$ and $g_1 \sim g_2$ then $(f_1, g_1) = (f_2, g_2)$ (Check!). To show that $fg$ is integrable we note that

$$|fg| = |f||g| \leq \frac{1}{2}(|f|^2 + |g|^2),$$

hence $fg$ is integrable because of $f, g \in L^2(\Omega)$.

Assume now $(f, f) = \int \Omega |f|^2 = 0$. Set

$$N := \{x \in \Omega \mid f(x) \neq 0\}, \quad A_n := \{x \in \Omega \mid |f(x)|^2 > \frac{1}{n}\}.$$

Then

$$0 \leq \frac{1}{n} \mu(A_n) \leq \int_{A_n} |f|^2 \leq \int \Omega |f|^2 = 0,$$

hence $\mu(A_n) = 0$ for all $n \in \mathbb{N}$. Furthermore, $N = \bigcup_{n=1}^{\infty} A_n$ and therefore

$$\mu(N) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0,$$

e.i. $f(x) = 0$ for almost all $x \in \Omega$, i.e. $f = 0$ in $L^2(\Omega)$.

The other properties are easy to check. \[\blacksquare\]

Corollary: Any function $f \in L^2(\Omega)$ is integrable on any subset $A \subset \Omega$ having finite measure. In particular, $f(x) < \infty$ for almost all $x \in \Omega$.

Proof: Define

$$g(x) := \begin{cases} \frac{f(x)}{|f(x)|} & x \in A \text{ and } f(x) \neq 0, \\ 0 & x \notin A \text{ or } f(x) = 0. \end{cases}$$

Then $\int \Omega |g|^2 \leq \mu(A) < \infty$, hence $g \in L^2(\Omega)$, and

$$\int_A |f| = (f, g) < \infty.$$

The norm generated by $(\cdot, \cdot)$ will be denoted by $\| \cdot \|_2$. Thus

$$\|f\|_2^2 := \int \Omega |f|^2.$$

Theorem 3.2 (Completeness of $L^2(\Omega)$)

The inner product space $(L^2(\Omega), (\cdot, \cdot))$ is a Hilbert space.

Proof: Consider a sequence $\{f_k\}$ in $L^2(\Omega)$ such that

$$\sum_{k=0}^{\infty} \|f_k\|_2 = M < \infty.$$

We will show that $\sum_{k=0}^{\infty} f_k$, i.e. $\lim_{n \to \infty} \sum_{k=0}^{n} f_k$ exists in $L^2(\Omega)$, this implies the theorem by Lemma 1.4.
For \( x \in \Omega, n \in \mathbb{N} \) define
\[
h_n(x) := \sum_{k=1}^{n} |f_k(x)|, \quad h(x) := \sum_{k=1}^{\infty} |f_k(x)|.
\]
(Note that we do not assume that \( h(x) \) is finite almost everywhere.) Then
\[
\|h_n\|_2 \leq \sum_{k=1}^{n} \|f_k\|_2 = \sum_{k=1}^{n} f_k \leq M.
\]
Moreover, \( h_n(x)^2 \uparrow h(x)^2 \) for all \( x \in \Omega \). By the monotone convergence theorem,
\[
\int_{\Omega} h^2 = \lim_{n \to \infty} \int_{\Omega} h_n^2 \leq M^2,
\]
hence \( h \in L^2(\Omega) \), and in particular \( h(x) < \infty \) a.e.. Set \( N = \{ x \in \Omega \mid h(x) = \infty \} \) and
\[
f(x) := \begin{cases} \sum_{k=0}^{\infty} f_k(x) \quad x \in \Omega \setminus N, \\ 0 \quad x \in N. \end{cases}
\]
Then \( |f(x)| \leq h(x) \) for all \( x \in \Omega \), hence \( f \in L^2(\Omega) \). Moreover, by the definition of \( f \),
\[
\left| f(x) - \sum_{k=1}^{n} f_k(x) \right|^2 \xrightarrow{n \to \infty} 0 \quad \text{a.e. in } \Omega
\]
and
\[
\left| f(x) - \sum_{k=1}^{n} f_k(x) \right|^2 = \left| \sum_{k=n+1}^{\infty} f_k(x) \right|^2 \leq \left( \sum_{k=1}^{\infty} |f_k(x)| \right) \leq h^2(x) \quad \text{a.e. in } \Omega.
\]
By the dominated convergence theorem,
\[
\int_{\Omega} \left| f(x) - \sum_{k=1}^{n} f_k(x) \right|^2 \xrightarrow{n \to \infty} 0,
\]
i.e. \( \|f - \sum_{k=1}^{n} f_k\|_2 \to 0 \) or \( f = \sum_{k=1}^{\infty} f_k \) in \( L^2(\Omega) \).

4 Test functions

**Definition:** Any vector \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \) is called an \( (m\text{-dimensional}) \) multiindex. We use multiindex notation for the description of partial derivatives and write
\[
\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_m} x_m}, \quad |\alpha| := \alpha_1 + \ldots + \alpha_m.
\]
For \( A, B \subseteq \mathbb{R}^m \) we define
\[
A + B := \{ x + y \mid x \in A, y \in B \}, \quad \text{dist}(A, B) := \inf\{d(x, y) \mid x \in A, y \in B \}.
\]
Let $\Omega \subset \mathbb{R}^m$ be a domain, $f : \Omega \rightarrow \mathbb{K}$. The \textbf{support} of $f$ is defined as

$$\text{supp } f := \{x \in \Omega \mid f(x) \neq 0\}.$$ 

$f$ is called \textbf{finite} (in $\Omega$) iff $\text{supp } f$ is bounded and $\text{supp } f \subset \Omega$. Note that then $\text{supp } f$ is compact and has a positive distance to the boundary of $\Omega$.

A function $\phi : \Omega \rightarrow \mathbb{K}$ is called \textbf{smooth} iff all its partial derivatives (of all orders) exist and are continuous on $\Omega$. The set of all smooth functions on $\Omega$ is denoted by $C^\infty(\Omega)$. It is a linear space.

A function $\phi : \Omega \rightarrow \mathbb{K}$ is called a \textbf{test function} (on $\Omega$) if $\phi$ is smooth and finite (in $\Omega$). The set of all test functions on $\Omega$ is denoted by $C^\infty_0(\Omega)$. It is a linear subspace of $L^2(\Omega)$. (Check!)

\textbf{Example:} Set

$$\psi(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

Then $\psi$ is a test function on all domains $\Omega$ that contain the closed unit ball $\overline{B(0,1)}$. (Check!)

For $h > 0$ and finite and continuous $f$ on $\mathbb{R}^m$ we define

$$\psi_h(x) := \frac{\psi \left( \frac{x}{h} \right)}{\int_{\mathbb{R}^m} \psi \left( \frac{2}{h} \right) \, dx},$$

$$f_h(x) := (f * \psi_h)(x) := \int_{\mathbb{R}^m} f(x - y) \psi_h(y) \, dy.$$ (4.1)

\textbf{Remark:} $f * \psi_h$ is called the \textbf{convolution product} of $f$ and $\psi_h$. Note that in general the convolution product of two finite continuous functions on $\mathbb{R}^m$ is well defined, finite and continuous. (Why?)

**Lemma 4.1** (Regularization by Convolution)

(i) $\partial^\alpha f_h = f * \partial^\alpha \psi_h$, in particular, $f_h \in C^\infty(\mathbb{R}^m)$,

(ii) $\text{supp } f_h \subset \text{supp } f + B(0, h)$,

(iii) $f_h \in L^2(\mathbb{R}^m)$ and $\|f - f_h\|_2 \xrightarrow{h \downarrow 0} 0$.

**Proof:** (i) Substituting $z := x - y$ we get

$$f_h(x) = \int_{\mathbb{R}^m} f(z) \psi_h(x - z) \, dz.$$ 

For $j = 1, \ldots, m$, the functions $(x, z) \mapsto f(z)\psi(x - z)$ and $(x, z) \mapsto f(z)\partial_j \psi(x - z)$ are continuous and have bounded support. Hence integration and differentiation with respect to the parameter $x$ may be interchanged, i.e.

$$\partial_j f_h(x) = \int_{\mathbb{R}^m} f(z) \partial_j \psi_h(x - z) \, dz = (f * \partial_j \psi_h)(x).$$

Now (i) follows by repeated application of the same arguments.
(ii) Exercise!

(iii) It follows from (i) and (ii) that \( f_h \in L^2(\mathbb{R}^m) \). Let \( \varepsilon > 0 \) be given. As \( f \) is uniformly continuous, there is a \( h_0 = h_0(\varepsilon) \in (0, 1) \) such that

\[
|f(x) - f(x - y)| < \varepsilon \quad x, y \in \mathbb{R}^m, |y| \leq h_0.
\] (4.2)

Because of \( \int_{\mathbb{R}^m} \psi_h(y) \, dy = 1 \) we have, for any \( x \in \mathbb{R}^m \) and any \( h \in (0, h_0) \)

\[
f(x) = \int_{\mathbb{R}^m} f(x) \psi_h(y) \, dy,
\]

\[
f(x) - f_h(x) = \int_{\mathbb{R}^m} (f(x) - f(x - y)) \psi_h(y) \, dy = \int_{|y| \leq h} (f(x) - f(x - y)) \psi_h(y) \, dy,
\]

\[
|f(x) - f_h(x)| \leq \int_{|y| \leq h} |f(x) - f(x - y)| \psi_h(y) \, dy \leq \varepsilon \int_{|y| \leq h} \psi_h(y) \, dy = \varepsilon.
\]

As \( f \) and \( f_h \) both vanish outside the bounded set \( \text{supp} f_1 \), we get

\[
\|f - f_h\|^2_2 = \int_{\mathbb{R}^m} |f(x) - f_h(x)|^2 \, dy = \int_{\text{supp} f_1} \varepsilon^2 \, dy = \varepsilon^2 \mu(\text{supp} f_1).
\]

Hence \( \|f - f_h\|_2 \to 0 \) as \( h \to 0 \).

We will use the previous lemma to prove the following fundamental result on the space \( L^2(\Omega) \):

**Lemma 4.2 (Approximation by test functions)**

The subspace \( C_0^\infty(\Omega) \) is dense in \( L^2(\Omega) \).

**Proof:** For any function \( f \in L^2(\Omega) \) and any \( \varepsilon > 0 \) we have to find a function \( g \in C_0^\infty(\Omega) \) such that \( \|f - g\|_2 < \varepsilon \) (cf. Lemma 1.3). Without loss of generality, we can assume that \( f \) is real and \( f \geq 0 \) (Why?). The function \( g \) will be constructed in four steps.

1. Approximate \( f \) by a bounded finite function \( g_1 \):

   For \( n \in \mathbb{N} \) set

   \[
   \Omega_n := \{ x \in \Omega \mid |x| < n, \text{dist}(x, \mathbb{R}^m \setminus \Omega) > \frac{1}{n}, f(x) \leq n \}; \quad f_n(x) := \begin{cases} f(x) & x \in \Omega_n, \\ 0 & x \in \Omega \setminus \Omega_n. \end{cases}
   \]

   Then all \( f_n \) are bounded, nonnegative, and finite. For all \( x \in \Omega \) we have \( f_n(x)^2 \uparrow f(x)^2 \) and, by the monotone convergence theorem,

   \[
   \|f - f_n\|^2_2 = \int_{\Omega} (f(x) - f_n(x))^2 \, dx = \int_{\Omega \setminus \Omega_n} f(x)^2 \, dx = \int_{\Omega} f(x)^2 \, dx - \int_{\Omega} f_n(x)^2 \, dx \to 0
   \]

   as \( n \to \infty \). Thus \( f_n \to f \) in \( L^2(\Omega) \). By choosing \( g_1 = f_n \) with \( n \) sufficiently large we get \( \|f - g_1\|_2 < \varepsilon/4 \).

2. Approximate \( g_1 \) by a step function \( g_2 \):

   For \( n \in \mathbb{N} \) define

   \[
   s_n(x) := \max \left\{ \frac{k}{n} \mid k \in \mathbb{N}, \frac{k}{n} \leq g_1(x) \right\}.
   \]
Then all $s_n$ are finite, nonnegative step functions (taking only a finite number of values). Furthermore,

$$\|g_1 - s_n\|_2^2 = \int_{\text{supp } g_1} |g_1 - s_n|^2 \leq \mu(\text{supp } g_1) \frac{1}{n^2} \to 0.$$ 

Thus $s_n \to g_1$ in $L^2(\Omega)$. By choosing $g_2 = s_n$ with $n$ sufficiently large we get $\|g_1 - g_2\|_2 < \varepsilon/4$.

3. Approximate $g_2$ by a finite continuous function $g_3$:

The step function $g_2$ can be written as

$$g_2 = \sum_{k=1}^{N} \alpha_k \chi_{A_k}, \quad 0 < \alpha_k \leq M, \quad \chi_{A_k}(x) := \begin{cases} 1 & x \in A_k, \\ 0 & x \notin A_k, \end{cases}$$

where all $A_k$ are measurable and bounded and $\overline{A}_k \subset \Omega$. We start by approximating a single step function $\chi_{A}$, $A = A_k$ for some $k$. From measure theory it is known that there is a compact set $K$ and an open set $U$ such that $K \subset A \subset U$, $\overline{U} \subset \Omega$ bounded, and $\mu(U \setminus K) < \varepsilon/(4MN)$. Define

$$\tilde{\chi}_A(x) := \begin{cases} 1 & x \in K \\ \max \left\{ 0, 1 - \frac{2 \text{dist}(x, K)}{\text{dist}(K, \Omega) - \text{dist}(\Omega \setminus U)} \right\} & x \in U \setminus K \\ 0 & x \in \Omega \setminus U \end{cases}$$

Then $\tilde{\chi}_A$ is continuous, has support in $\Omega$ and

$$\|\tilde{\chi}_A - \chi_{A}\|_2 \leq \mu(U \setminus K) < \frac{\varepsilon}{4MN}.$$ (Check!) Define

$$g_3 := \sum_{k=1}^{N} \alpha_k \tilde{\chi}_{A_k}.$$ 

Then $\|g_3 - g_2\|_2 < \varepsilon/4$, $g_3$ is finite and continuous.

4. Approximate $g_3$ by a test function $g$:

Define $g := (g_3)_h := g_3 * \psi_h$ with $h > 0$. Lemma 4.1 ensures that $g$ is a test function in $\Omega$ and $\|g - g_3\|_2 < \varepsilon/4$ if $h$ is chosen sufficiently small.

Summarizing, we get

$$\|f - g\|_2 \leq \|f - g_1\|_2 + \|g_1 - g_2\|_2 + \|g_2 - g_3\|_2 + \|g_3 - g\|_2 \leq \varepsilon.$$ 

We will need the following generalization of Lemma 4.1:

**Lemma 4.3** (Regularization of $L^2$-functions)

Assume $f \in L^2(\Omega)$. Then all statements of Lemma 4.1 remain true.

The proof uses the dominated convergence theorem to justify the interchanging of differentiation and convolution. For the details we refer to [Ev98], Appendix C, Theorem 6.
5 Weak derivatives

**Definition:** Assume \( f \in L^2(\Omega), \ i = 1, \ldots, m \). A function \( g \in L^2(\Omega) \) is called \( i \)-th weak partial derivative iff

\[
\int_\Omega f \partial_i v \, dx = - \int_\Omega gv \, dx \quad \forall v \in C_0^\infty(\Omega).
\]

Note that not all \( f \in L^2(\Omega) \) have weak derivatives.

**Lemma 5.1** *(Elementary properties of weak derivatives)*

(i) The weak partial derivatives are unique.

(ii) Assume \( f \in C^1(\Omega) \cap L^2(\Omega) \) and \( \partial_i f \in L^2(\Omega) \). Then \( \partial_i f \) is the weak derivative of \( f \).

**Proof:** (i) Assume \( f \) has two partial weak derivatives \( g_1, g_2 \in L^2(\Omega) \). Then

\[
\int_\Omega (g_1 - g_2)v \, dx = 0 \quad \forall v \in C_0^\infty(\Omega),
\]

i.e. \( g_1 - g_2 \perp C_0^\infty(\Omega) \). By Lemmas 2.3 and 4.2, this implies \( g_1 - g_2 \perp L^2(\Omega) \), therefore \( g_1 = g_2 \).

(ii) Assume without loss of generality \( i = 1 \). Fix \( v \in C_0^\infty(\Omega) \). Extend \( v \) outside \( \Omega \) by 0. Fix a bounded open set \( U \) such that \( \text{supp} \, v \subset U \) and \( \overline{U} \subset \Omega \). Set

\[
\chi_U(x) := \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}
\]

and \( \eta := \chi_U * \psi_h \) with \( 0 < h < \min(\text{dist}(\text{supp} \, v, \mathbb{R}^m \setminus U), \text{dist}(\overline{U}, \mathbb{R}^m \setminus \Omega)) \). Then \( \eta \in C_0^\infty(\Omega) \) and \( \eta \equiv 1 \) on \( \text{supp} \, v \) (Check!). Now define \( \tilde{f} \) by

\[
\tilde{f}(x) = \begin{cases} f(x) \eta(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}
\]

Then \( f = \tilde{f} \) and \( \partial_1 f = \partial_1 \tilde{f} \) on \( \text{supp} \, v \). Now, by Fubini's theorem,

\[
\int_\Omega f \partial_1 v \, dx = \int_{\text{supp} \, v} \tilde{f} \partial_1 v \, dx = \int_{\mathbb{R}^m} \tilde{f} \partial_1 v \, dx = \int_{\mathbb{R}^{m-1}} \left( \int_{\mathbb{R}} \tilde{f} \partial_1 v \, dx \right) dx_2 \ldots dx_m
\]

\[
= - \int_{\mathbb{R}^{m-1}} \left( \int_{\mathbb{R}} \partial_1 \tilde{f} v \, dx \right) dx_2 \ldots dx_m = - \int_{\mathbb{R}^m} \partial_1 \tilde{f} v \, dx = - \int_{\text{supp} \, v} \partial_1 \tilde{f} v \, dx
\]

\[
= - \int_{\text{supp} \, v} \partial_1 f v \, dx = - \int_\Omega \partial_1 f v \, dx.
\]

Lemma 5.1 allows us to speak about the weak partial derivative of a function \( f \in L^2(\Omega) \) and to write \( \partial_i f \) for it. Analogously, we call \( g \in L^2(\Omega) \) the weak partial derivative of order \( \alpha, \alpha \in \mathbb{N}^m, \) iff

\[
\int_\Omega f \partial^\alpha v \, dx = (-1)^{|\alpha|} \int_\Omega gv \, dx.
\]

Again, this weak partial derivative is unique and coincides with the strong one if the latter exists.
Lemma 5.2 (Vanishing weak gradient)

Assume $f \in L^2(\Omega)$, $\partial_i f = 0$ for $i = 1, \ldots, m$. Then there is a constant $c$ such that $f = c$ almost everywhere.

Proof: For (sufficiently small) $\delta > 0$ define

$$\Omega_\delta := \{ x \in \Omega | \text{dist}(x, \mathbb{R}^m \setminus \Omega) > \delta \}.$$

It is sufficient to show that $f = c$ a.e. on $\Omega_\delta$ for any $\delta > 0$.

Choose $v \in C_0^\infty(\Omega_\delta)$ and $h \in (0, \delta)$. Then by Lemma 4.1, $v * \psi_h \in C_0^\infty(\Omega)$ and $\partial_i(v * \psi_h) = \partial_i v * \psi_h$, $i = 1, \ldots, m$. Therefore

$$0 = \int_{\Omega} \partial_i (v * \psi_h) \, dx = - \int_{\Omega} f \partial_i(v * \psi_h) \, dx = - \int_{\Omega} f (\partial_i v * \psi_h) \, dx$$

$$= \int_{\Omega} f(x) \int_{\Omega_\delta} \partial_i v(y) \psi_h(x-y) \, dy \, dx \quad \text{Fubini} = - \int_{\Omega_\delta} \left( \int_{\Omega} f(x) \psi_h(y-x) \, dx \right) \partial_i v(y) \, dy$$

$$= - \int_{\Omega_\delta} (f * \psi_h)(y) \partial_i v(y) \, dy.$$

By Lemma 4.3, $f * \psi_h$ is smooth and in $L^2(\Omega)$, and $\partial_i (f * \psi_h) = f * \partial_i \psi_h$ is in $L^2(\Omega)$ as well. The above calculation shows that the weak derivatives of $f * \psi_h$ vanish on $\Omega_\delta$, and by Lemma 5.1 (ii) this is the usual derivative. Hence there is a constant $c_h$ such that $f * \psi_h = c_h$ a.e.. Letting $h \downarrow 0$ and using Lemma 4.3 again gives

$$f = \lim_{h \downarrow 0} f * \psi_h = \lim_{h \downarrow 0} c_h = c \quad \text{in } L^2(\Omega_\delta),$$

because the $L^2$-limit of a sequence of constant functions has to be (a.e.) constant. (Why?)

For $m = 1$, we can generalize the Fundamental Theorem of Calculus in the following way:

Lemma 5.3 ("Weak" fundamental theorem of calculus)

Let $I \subset \mathbb{R}$ be an interval, $g \in L^2(I)$, $x_0 \in \overline{T}$ and

$$f(x) := \int_{x_0}^x g(t) \, dt, \quad x \in \overline{T}.$$

Then $f$ is continuous and has weak derivative $g$.

Proof: Continuity follows from the dominated convergence theorem (Check!). For $v \in C_0^\infty(I)$, fix $a, b \in \overline{T}$ such that $\text{supp } v \subset [a, b]$ and $x_0 \in [a, b]$. Then we have

$$\int f' \, dx = - \int_a^{x_0} \left( \int_x^{x_0} g(t)v'(x) \, dt \right) \, dx + \int_{x_0}^b \left( \int_{x_0}^b g(t)v'(x) \, dx \right) \, dt \quad \text{Fubini}$$

$$= - \int_a^{x_0} \left( \int_a^t g(t)v'(x) \, dx \right) \, dt + \int_{x_0}^b \left( \int_{x_0}^b g(t)v'(x) \, dx \right) \, dt$$

$$= - \int_a^b g(t) \left( \int_a^t v'(x) \, dx \right) \, dt + \int_a^b g(t) \left( \int_t^b v'(x) \, dx \right) \, dt$$

$$= - \int_a^b g(t)v(t) \, dt.$$

This proves $f' = g$ as a weak derivative.
The Sobolev space $W^{1,2}(\Omega)$

**Definition:**

$W^{1,2}(\Omega) := \{ u \in L^2(\Omega) \mid \text{the weak derivatives } \partial_i u \text{ exist in } L^2(\Omega), \ i = 1, \ldots, m \}$

This subspace of $L^2(\Omega)$ is called a **Sobolev space**. (Check that $W^{1,2}(\Omega)$ is indeed a linear space!)

On $W^{1,2}(\Omega)$ we define the inner product $(\cdot, \cdot)_{W^{1,2}}$ by

$$(u, v)_{W^{1,2}} := \int_{\Omega} \left( uv + \sum_{i=1}^{m} \partial_i u \partial_i v \right) dx = (u, v) + \sum_{i=1}^{m} (\partial_i u, \partial_i v)$$

and the corresponding norm $\| \cdot \|_{W^{1,2}}$ by

$$\| u \|_{W^{1,2}} := \| u \|_{L^2(\Omega)} + \sum_{i=1}^{m} \| \partial_i u \|_{L^2(\Omega)}.$$

**Examples:**

1. If $\Omega$ has finite measure then any function in $C^1(\overline{\Omega})$ is in $W^{1,2}(\Omega)$. (Why?)

2. Assume $m = 2$,

   $$\Omega := \{ x \in \mathbb{R}^2 \mid |x| < \frac{1}{2} \},$$

and let $u : \Omega \to \mathbb{R}$ be given by $u(x) := \log |\log |x||$. We will show that $u \in W^{1,2}(\Omega)$:

   Clearly $u \in L^2(\Omega)$ (Check!). Assume $v \in C_0^\infty(\Omega)$ and set

   $$\Omega_\varepsilon := \{ x \in \mathbb{R}^2 \mid \varepsilon < |x| < \frac{1}{2} \}.$$

   Note that $u$ is smooth on $\Omega_\varepsilon$. By dominated convergence and integration by parts,

   $$\int_{\Omega} u \partial_i v dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} u \partial_i v dx = \lim_{\varepsilon \to 0} \left( \int_{\Omega_\varepsilon} \partial_i uv dx - \int_{\{ |x| = \varepsilon \}} \frac{x_i}{|x|} uv dS_{\varepsilon} \right),$$

   $i = 1, 2$. On $\Omega_\varepsilon$ we have $\partial_i u = g_i$ with

   $$g_i(x) := \frac{x_i}{|x|^2 \log |x|}, \quad x \in \Omega$$

   where $g_i \in L^2(\Omega)$ (Check!). Hence $g_i v$ is integrable on $\Omega$ and therefore

   $$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \partial_i uv dx = \int_{\Omega} g_i v dx.$$

   Furthermore,

   $$\left| \int_{\{ |x| = \varepsilon \}} \frac{x_i}{|x|} uv dS_{\varepsilon} \right| \leq C \varepsilon \log |\log \varepsilon| \xrightarrow{\varepsilon \to 0} 0.$$

   (Check!)

   Thus

   $$\int_{\Omega} u \partial_i v dx = \int_{\Omega} g_i v dx, \quad v \in C_0^\infty(\Omega),$$
i.e. \( \partial_t u = g_i \) in the sense of weak derivatives. This shows \( u \in W^{1,2}(\Omega) \).

**Danger:** This example shows that for \( m \geq 2 \) functions in \( W^{1,2}(\Omega) \) are *not necessarily continuous*!

For \( m = 1 \), however, we have a simpler situation:

**Theorem 6.1 (\( W^{1,2} \)-functions in 1D)**

Let \( -\infty \leq a < b \leq \infty \), \( I = (a, b) \).

(i) Any equivalence class of functions in \( W^{1,2}(I) \) contains a representant which is continuous on \( T \).

(ii) For this representant, we have

\[
\max_{x \in T} |u(x)| \leq C\|u\|_{W^{1,2}}
\]

with \( C \) depending only on the length of \( I \).

(iii) If \( a = -\infty \) then \( \lim_{x \to -\infty} u(x) = 0 \); if \( b = \infty \) then \( \lim_{x \to \infty} u(x) = 0 \).

**Proof:** (i) Fix \( u \in W^{1,2}(I) \), \( x_0 \in T \) and define \( \tilde{u} : I \to K \) by

\[
\tilde{u}(x) := \int_{x_0}^{x} u'(t) \, dt, \quad x \in T.
\]

By Lemma 5.3, \( \tilde{u} \) is continuous, \( \tilde{u}' = u' \), and therefore by Lemma 5.2 there is a \( c \in K \) such that \( u = \tilde{u} + c \) a.e. in \( T \). In particular, this implies that \( u \) is equivalent to a continuous function. We will keep the notation \( u \) for this continuous representant.

(ii) Note that

\[
|u(x) - u(y)| = \int_{y}^{x} u'(t) \, dt, \quad x, y \in T
\]

with \( u \) from the proof of (i). (Check!) Assume first \(-\infty < a < b < \infty \) and define

\[
\overline{u} := \frac{1}{b-a} \int_{a}^{b} u \, dx, \quad u_1 := u - \overline{u}
\]

Then \( \int u_1 \, dx = 0 \), \( u_1' = u' \), and as \( u_1 \) is continuous, there is an \( y_0 \in T \) such that \( u_1(y_0) = 0 \) (Check!). Now, for \( x \in T \),

\[
|u(x)| \leq |u_1(x)| + |\overline{u}| \overset{(6.1)}{=} \left| \int_{y_0}^{x} u_1'(t) \, dt \right| + \frac{1}{b-a} \left| \int_{a}^{b} u(t) \, dt \right|
\]

Cauchy-Schwarz

\[
\leq C(\|u_1\|_2 + \|u\|_2) \leq C\|u\|_{W^{1,2}}.
\]

This proves (ii) for bounded intervals.

If \( b = \infty \) then, for any \( x \in T \), we restrict \( u \) to the subinterval \((x, x+1) \subset I \) and use the result for bounded intervals which has just been proved. Thus,

\[
|u(x)| \leq C\|u\|_{W^{1,2}((x,x+1))} \leq C\|u\|_{W^{1,2}((a,\infty))}.
\]

(6.2)
If \(a = -\infty\) we proceed analogously.

(iii) Assume \(b = \infty\). By the dominated convergence theorem,

\[
\|u\|_{W^{1,2}((x,x+1))}^2 = \int_x^{x+1} |u|^2 \, dt + \int_x^{x+1} |u'|^2 \, dt \xrightarrow{x \to \infty} 0,
\]

and from the first inequality in (6.2) we get \(\lim_{x \to \infty} u(x) = 0\). If \(a = -\infty\) we proceed analogously.

To show completeness of \(W^{1,2}(\Omega)\) with respect to this norm we will use the following lemma:

**Lemma 6.2 (Convergence of weak derivatives)**

Let \((u_n)\) be a sequence in \(W^{1,2}(\Omega)\) such that \(u_n \to u\) in \(L^2(\Omega)\) and \(\partial_i u_n \to w_i\) in \(L^2(\Omega)\) for some \(u, w_i \in L^2(\Omega)\), \(i = 1, \ldots, m\). Then \(\partial_i u = w_i\). In particular, \(u \in W^{1,2}(\Omega)\).

**Proof:** Fix \(i \in \{1, \ldots, m\}\), \(v \in C^\infty_0(\Omega)\). Due to \(u_n \to u\) in \(L^2(\Omega)\) we have, by the Cauchy-Schwarz inequality,

\[
\left| \int_\Omega u_n \partial_i v \, dx - \int_\Omega u \partial_i v \, dx \right| \leq \|u_n - u\|_{L^2(\Omega)} \|\partial_i v\|_{L^2(\Omega)} \to 0
\]

and thus

\[
\int_\Omega u \partial_i v \, dx = \lim_{n \to \infty} \int_\Omega u_n \partial_i v \, dx.
\]

Analogously, from \(\partial_i u_n \to w_i\) in \(L^2(\Omega)\) we get

\[
\int_\Omega w_i v \, dx = \lim_{n \to \infty} \int_\Omega \partial_i u_n v \, dx.
\]

Therefore,

\[
\int_\Omega u \partial_i v \, dx = \lim_{n \to \infty} \int_\Omega u_n \partial_i v \, dx = \lim_{n \to \infty} \int_\Omega \partial_i u_n v \, dx = - \int_\Omega w_i v \, dx
\]

for all \(v \in C^\infty_0(\Omega)\). This means \(\partial_i u = w_i\). □

**Theorem 6.3 (The Hilbert space \(W^{1,2}(\Omega)\))**

The space \(W^{1,2}(\Omega)\) is complete with respect to the norm \(\| \cdot \|_{W^{1,2}}\).

**Proof:** Let \((u_n)\) be a Cauchy sequence in \(W^{1,2}(\Omega)\). This implies that \((u_n)\) is a Cauchy sequence in \(L^2(\Omega)\) and \(\partial_i u_n \to w_i\) is a Cauchy sequence in \(L^2(\Omega)\) for \(i = 1, \ldots, m\) (Check!). As \(L^2(\Omega)\) is complete by Theorem 3.2 there are functions \(u, w_i \in L^2(\Omega)\) such that \(u_n \to u\) and \(\partial_i u_n \to w_i\) in \(L^2(\Omega)\). By Lemma 6.2, this implies \(u \in W^{1,2}(\Omega)\) and \(\partial_i u_n \to \partial_i u\) in \(L^2(\Omega)\). Therefore \(u_n \to u\) in \(W^{1,2}(\Omega)\). (Check!). □

Differing from the situation in \(L^2(\Omega)\), the test functions are in general not dense in \(W^{1,2}(\Omega)\). This is true only for \(\Omega = \mathbb{R}^m\):

**Lemma 6.4 (Approximation by test functions in \(W^{1,2}(\mathbb{R}^m)\))**

\(C^\infty_0(\mathbb{R}^m)\) is dense in \(W^{1,2}(\mathbb{R}^m)\).
\textbf{Proof:} Let $f \in W^{1,2}(\mathbb{R}^m)$ and $\varepsilon > 0$ be given.

1. Approximate $f$ by $g_1 \in W^{1,2}(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m)$:

Let $\psi_h$ be defined as in (4.1). By Lemma 4.3, we have $f * \psi_h \in C^\infty(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$ and $f * \psi_h \xrightarrow{L^2(\mathbb{R}^m)} f$ for $h \downarrow 0$. Moreover, for $i = 1, \ldots, m$ we get $\partial_i (f * \psi_h) = f * \partial_i \psi_h$. Note that $y \mapsto \psi_h(x-y)$ is in $C^\infty(\mathbb{R}^m)$ for all $x \in \mathbb{R}^m$. Therefore,

$$
(f * \partial_i \psi_h)(x) = \int_{\mathbb{R}^m} f(y) \partial_i \psi_h(x-y) \, dy = -\int_{\mathbb{R}^m} f(y) \frac{\partial}{\partial y_i} [\psi_h(x-y)] \, dy = \int_{\mathbb{R}^m} \partial_i f(y) \psi_h(x-y) \, dy = (\partial_i f * \psi_h)(x).
$$

Hence $\partial_i (f * \psi_h) = \partial_i f * \psi_h \xrightarrow{L^2(\mathbb{R}^m)} \partial_i f$ for all $i = 1, \ldots, m$ by Lemma 4.3, and therefore $f * \psi_h \xrightarrow{W^{1,2}(\mathbb{R}^m)} f$ for $h \downarrow 0$. Consequently, if $h$ is chosen small enough and $g_1 := f * \psi_h$ then $\|f - g_1\|_{W^{1,2}} < \varepsilon/2$.

2. Approximate $g_1$ by $g \in C^\infty_0(\mathbb{R}^m)$:

Fix a function $\eta \in C^\infty_0(\mathbb{R}^m)$ such that $0 \leq \eta \leq 1$ and $\eta(0) = 1$, and define $\eta_n \in C^\infty_0(\mathbb{R}^m)$ for $n \in \mathbb{N}$ by $\eta_n(x) := \eta\left(\frac{x}{n}\right)$. Then $g_1 \eta_n \in C^\infty(\mathbb{R}^m)$ for all $n$ and $\eta_n(x) \to 1$ for all $x \in \mathbb{R}^m$ as $n \to \infty$. Therefore

$$
\|g_1 - g_1 \eta_n\|^2_2 = \int_{\mathbb{R}^m} |g_1|^2 (1 - \eta_n) \, dx \to 0
$$

as $n \to \infty$ by the dominated convergence theorem. Moreover, for $i = 1, \ldots, m$,

$$
\|\partial_i (g_1 - g_1 \eta_n)\|_2 \leq \|\partial_i g_1 - (\partial_i g_1) \eta_n\|_2 + \|g_1 \partial_i \eta_n\|_2.
$$

For the first term on the right, we find

$$
\|\partial_i g_1 - (\partial_i g_1) \eta_n\|_2 \xrightarrow{n \to \infty} 0
$$

as above, and for the second term we estimate

$$
\|g_1 \partial_i \eta_n\|_2 \leq \|g_1\|_2 \max_{x \in \mathbb{R}^m} |\partial_i \eta_n(x)| = \frac{1}{n} \|g_1\|_2 \max_{x \in \mathbb{R}^m} |\partial_i \eta(x)| \xrightarrow{n \to \infty} 0.
$$

Hence $g_1 \eta_n \xrightarrow{W^{1,2}(\mathbb{R}^m)} g_1$ as $n \to \infty$, and if $n$ is sufficiently large and $g := g_1 \eta_n$ then

$$
\|f - g\|_{W^{1,2}} \leq \|f - g_1\|_{W^{1,2}} + \|g_1 - g\|_{W^{1,2}} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
$$

\textbf{Definition:} Let $E$, $F$ be normed spaces, let $U \subset E$. A function $u : U \to F$ is called \textbf{Lipschitz continuous} iff there is a constant $L > 0$ such that

$$
\|u(x) - u(y)\|_F \leq L \|x - y\|_E, \quad x, y \in U.
$$

Let $\Omega \subset \mathbb{R}^m$ be a domain. The set $\partial \Omega := \overline{\Omega} \setminus \Omega$ is called the \textbf{boundary} of $\Omega$.

Let $m > 1$. A domain $\Omega \subset \mathbb{R}^m$ is called a \textbf{Lipschitz domain} iff $\partial \Omega$ can locally be represented as graph of a Lipschitz continuous function, i.e. iff for any $x_0 \in \partial \Omega$ there
are an orthonormal basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$, an $\varepsilon > 0$ and a Lipschitz continuous function $\Phi : B_{\mathbb{R}^m}(x_0, \varepsilon) \to \mathbb{R}$ such that

$$\partial \Omega \cap B_{\mathbb{R}^m}(x_0, \varepsilon) = \left\{ x_0 + \sum_{k=1}^{m-1} z_k e_k + \Phi(z)e_m \mid |z| < \varepsilon \right\} \cap B_{\mathbb{R}^m}(x_0, \varepsilon).$$

We state the following results without proof:

**Theorem 6.5** *(Approximation by smooth functions in $W^{1,2}(\Omega)$)*

(i) $W^{1,2}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{1,2}(\Omega)$.

(ii) Let $m = 1$ or let $\Omega$ be a Lipschitz domain. Then

$$C^\infty(\mathbb{R}^m)_{|\Omega} = \{ u|\Omega : u \in C^\infty(\mathbb{R}^m) \}$$

is a dense subset of $W^{1,2}(\Omega)$.

**Corollary:** Let $\Omega$ be a bounded interval or a bounded Lipschitz domain. Then $C^1(\overline{\Omega})$ is a dense subset of $W^{1,2}(\Omega)$.

**Proof:** Exercise!

## 7 The space $W^{1,2}_0(\Omega)$

**Definition:** The space $W^{1,2}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1,2}(\Omega)$.

As a closed subspace of the Hilbert space $W^{1,2}(\Omega)$, the space $W^{1,2}_0(\Omega)$ is a Hilbert space as well (with respect to the same inner product).

According to Lemma 6.4, $W^{1,2}_0(\mathbb{R}^m) = W^{1,2}(\mathbb{R}^m)$. In general, however, $W^{1,2}_0(\Omega)$ is strictly smaller than $W^{1,2}(\Omega)$. It contains those functions from $W^{1,2}(\Omega)$ which have “zero boundary values” in a generalized sense.

**Danger:** As functions in $W^{1,2}(\Omega)$ are defined only up to a set of measure zero, they do not have well-defined boundary values in the usual sense of continuous extension!

If $m = 1$, however, we choose the continuous representant which, by Theorem 6.1, has a continuous extension to the boundary. Then:

**Lemma 7.1** *(Vanishing boundary values in 1D)*

Let $I$ be an interval. If $x_0 \in \partial I$ and $u \in W^{1,2}_0(I)$ then $u(x_0) = 0$.

**Proof:** Fix $u \in W^{1,2}_0(I)$. By definition of $W^{1,2}_0(I)$, there is a sequence $(u_n)$ in $C^\infty_0(I)$ such that $u_n \overset{W^{1,2}}{\to} u$. From Theorem 6.1 we know that $u(x_0)$ exists and

$$|u(x_0)| = |u(x_0) - u_n(x_0)| \leq \max_{x \in I} |(u - u_n)(x)| \leq C\|u - u_n\|_{W^{1,2}} \to 0.$$  

This implies the assertion. ■

In the general case, the following result is easy to see:

**Lemma 7.2** *(Functions vanishing near $\partial \Omega$)*

Assume $u \in W^{1,2}(\Omega)$ and there is a compact set $K \subset \Omega$ such that $u = 0$ a.e. in $\Omega \setminus K$. Then $u \in W^{1,2}_0(\Omega)$.  

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Proof: Approximate $u$ by $u_h := u * \psi_h$ (cf. the proof of Lemma 6.4). The details are left as an exercise.

More generally, we have the following result which we will not prove:

Define

$$C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) \mid u|_{\partial \Omega} = 0\}.$$

**Theorem 7.3 (Continuous functions in $W^{1,2}_0(\Omega)$)**

(i) $W^{1,2}(\Omega) \cap C_0(\overline{\Omega}) \subset W^{1,2}_0(\Omega)$.

(ii) Assume $m = 1$ or $\Omega$ is a Lipschitz domain. Then

$$W^{1,2}(\Omega) \cap C_0(\overline{\Omega}) = W^{1,2}_0(\Omega) \cap C(\overline{\Omega}).$$

We are going to prove two inequalities which will be crucial for the treatment of second-order elliptic boundary value problems:

**Lemma 7.4 (Poincaré inequality¹)**

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain.

(i) There is a constant $C$ depending only on $\Omega$ such that

$$\|u\|_2 \leq C \|\nabla u\|_2$$

for all $u \in W^{1,2}_0(\Omega)$.

(ii) Assume additionally $m = 1$ or $\Omega$ is a Lipschitz domain. Then there is a constant $C$ depending only on $\Omega$ such that

$$\|u\|_2 \leq C \left(\|\nabla u\|_2 + \left|\int_\Omega u \, dx\right|\right)$$

for all $u \in W^{1,2}(\Omega)$.

**Proof of (i):** 1. We show (7.1) for $u \in C_0^\infty(\Omega)$: Extend $u$ by zero to $\mathbb{R}^m$. Write $x = (x_1, \ldots, x_m)$. As $\Omega$ is bounded there are $a, b \in \mathbb{R}$ such that $a \leq x_1 \leq b$ for $x \in \Omega$. Then $u(a, x_2, \ldots, x_m) = 0$ and therefore

$$|u(x)| = \left|\int_a^{x_1} \partial_1 u(t, x_2, \ldots, x_m) \, dt\right| \leq C \left(\int_a^{x_1} |\partial_1 u(t, x_2, \ldots, x_m)|^2 \, dt\right)^{\frac{1}{2}} \leq C \left(\int_{\mathbb{R}} |\nabla u(t, x_2, \ldots, x_m)|^2 \, dt\right)^{\frac{1}{2}}$$

for all $x \in \mathbb{R}^m$. Squaring and integrating over $\mathbb{R}$ with respect to $x_1$ gives

$$\int_{\mathbb{R}} |u(x_1, \ldots, x_m)|^2 \, dx_1 = \int_a^b |u(x_1, \ldots, x_m)|^2 \, dx_1 \leq (b - a) \max_{a \leq x_1 \leq b} |u(x_1, \ldots, x_m)|^2 \leq C \int_{\mathbb{R}} |\nabla u(t, x_2, \ldots, x_m)|^2 \, dt.$$ 

¹also known as Friedrichs' inequality

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Using this, we find
\[
\int_{\Omega} |u(x)|^2 \, dx = \int_{\mathbb{R}^m} |u(x)|^2 \, dx = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}} |u(x_1, \ldots, x_m)|^2 \, dx_1 \right) \, dx_2 \ldots dx_m
\]
\[
\leq C \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} |\nabla u(x_1, \ldots, x_m)|^2 \, dx_1 \right) \, dx_2 \ldots dx_m
\]
\[
= C \int_{\mathbb{R}^m} |\nabla u(x)|^2 \, dx = C \int_{\Omega} |\nabla u(x)|^2 \, dx.
\]
This implies (7.1) for test functions \( u \).

2. Let now \( u \in W^{1,2}_0(\Omega) \) be arbitrary. There is a sequence \( (u_n) \) in \( C_0^\infty(\Omega) \) such that \( u_n \xrightarrow{W^{1,2}} u \). This implies \( u_n \xrightarrow{L^2} u \), \( \partial_i u_n \xrightarrow{L^2} \partial_i u \) for \( i = 1, \ldots, m \), \( \|u_n\|_2 \to \|u\|_2 \), and \( \|\nabla u_n\|_2 \to \|\nabla u\|_2 \) for \( n \to \infty \) (Check!). From Step 1 we have
\[
\|u_n\|_2 \leq C \|\nabla u_n\|_2.
\]
Now (7.1) follows by taking the limit \( n \to \infty \).

**Proof of (ii) in the case \( m = 1 \):** We have \( \Omega = (a, b), a, b \in \mathbb{R} \). Without loss of generality, let \( \mathbb{K} = \mathbb{R} \). We choose the continuous representant for \( u \in W^{1,2}((a, b)) \) and estimate
\[
|u(x) - u(y)| = \left| \int_{y}^{x} u'(t) \, dt \right| \leq C \left( \int_{a}^{b} |u'(t)|^2 \, dt \right)^{1/2} = C \|u'\|_2.
\]
The result follows now from squaring this inequality and integrating with respect to \( x \) and \( y \) over \( (a, b) \). The details are left as an exercise.

## 8 Linear operators

Let \( E \) and \( F \) be normed spaces. Their norms will be denoted by \( \| \cdot \|_E \) and \( \| \cdot \|_F \), respectively.

**Definitions:** A mapping \( A : E \to F \) is called a linear operator iff
- \( A(x + y) = Ax + Ay \) for all \( x, y \in E \),
- \( A(\lambda x) = \lambda Ax \) for all \( x \in E, \lambda \in \mathbb{K} \).

A linear operator \( A \) is called bounded iff there is a \( C > 0 \) such that
\[
\|Ax\|_F \leq C\|x\|_E \quad \text{for all } x \in E,
\]
or, equivalently,
\[
\|Ax\|_F \leq C \quad \text{for all } x \in E \text{ with } \|x\| \leq 1.
\]
A linear operator \( A \) is called continuous iff for any sequence \( (x_n) \) in \( E \) holds:
\[
x_n \xrightarrow{E} x^* \implies Ax_n \xrightarrow{F} Ax^*.
\]

**Lemma 8.1** (Boundedness and continuity)

A linear operator is bounded if and only if it is continuous.
and its elements are called $A; B$

**Definition:** The (Banach) space $A$ is a Banach space and let $A$ be bounded. Then there is a sequence of elements $(x_n)$ in $E$ such that $\|x_n\|_E = 1$, $\|Ax_n\|_F \geq n$. Set $y_n := \frac{x_n}{n}$. Then

$$\|Ay_n\|_F \geq \frac{1}{n}, \|y_n\|_E = \frac{1}{n} \xrightarrow{n \to \infty} 0,$$

hence $y_n \xrightarrow{E} 0$. The continuity of $A$ implies $Ay_n \xrightarrow{E} A0 = 0$, hence $\|Ay_n\|_F \xrightarrow{n \to \infty} 0$, but this clearly contradicts (8.1). Hence $A$ is bounded.

Conversely, let $A$ be continuous. We argue by contradiction: Assume $A$ would not be bounded. Then there is a sequence of elements $(x_n)$ in $E$ such that $\|x_n\|_E = 1$, $\|Ax_n\|_F \geq n$. Set $y_n := \frac{x_n}{n}$. Then

$$\|Ay_n\|_F \geq 1, \|y_n\|_E = \frac{1}{n} \xrightarrow{n \to \infty} 0,$$

hence $y_n \xrightarrow{E} 0$. The continuity of $A$ implies $Ay_n \xrightarrow{E} A0 = 0$, hence $\|Ay_n\|_F \xrightarrow{n \to \infty} 0$, but this clearly contradicts (8.1). Hence $A$ is bounded.

The set of all bounded linear operators from $E$ to $F$ is denoted by $L(E, F)$. This set is a vector space with respect to the addition and scalar multiplication given by

$$(A + B)x := Ax + Bx, \quad (\lambda A)x := \lambda(Ax)$$

for $A, B \in L(E, F), x \in E, \lambda \in \mathbb{K}$. (Check!) For $A \in L(E, F)$ we define

$$\|A\|_{L(E, F)} := \sup_{x \in E \setminus \{0\}} \frac{\|Ax\|_F}{\|x\|_E} = \sup_{x \in E, \|x\|_E \leq 1} \|Ax\|_F = \sup_{x \in E, \|x\|_E = 1} \|Ax\|_F. \quad (8.2)$$

**Lemma 8.2 (The normed space $L(E, F)$)**

The mapping $\| \cdot \|_{L(E, F)}$ is a norm on $L(E, F)$. If $F$ is a Banach space, then $L(E, F)$ is a Banach space.

**Proof:** The first part of the lemma is easy to check. To show the second part, assume $F$ is a Banach space and let $(A_n)$ be a Cauchy sequence in $L(E, F)$. For any $x \in E$, we have

$$\|A_n x - A_m x\| \leq \|A_n - A_m\|_{L(E, F)} \|x\| \xrightarrow{n, m \to \infty} 0,$$

hence $(A_n x)$ is a Cauchy sequence in $F$. Therefore, it is convergent, and we set, for any $x \in E$,

$$A^* x := \lim_{n \to \infty} A_n x.$$

It is easy to see that $A^*$ is a linear operator. (Check!) We have to show that it is bounded and that $A_n \xrightarrow{L(E, F)} A^*$, or equivalently, $\|A_n - A^*\|_{L(E, F)} \xrightarrow{n \to \infty} 0$. Choose $\varepsilon > 0$ and $x \in E$ with $\|x\|_E \leq 1$. There is an $n_0 \in \mathbb{N}$ such that $\|A_n - A_m\|_{L(E, F)} < \varepsilon/2$ for all $m, n \geq n_0$, and there is an $m \geq n_0$ such that $\|A_m x - A^* x\|_F < \varepsilon/2$. Consequently,

$$\|(A_n - A^*) x\|_F \leq \|A_n - A_m\|_{L(E, F)} + \|A_m x - A^* x\|_F \leq \varepsilon. \quad (8.3)$$

This shows $A_n - A^* \in L(E, F)$, hence also $A^* = A_n - (A_n - A^*) \in L(E, F)$, and taking the supremum over $x \in E, \|x\|_E \leq 1$ in (8.3) gives $\|A_n - A^*\|_{L(E, F)} \xrightarrow{n \to \infty} 0$.

**Definition:** The (Banach) space $L(E, \mathbb{K})$ is called **dual space** of $E$, it is denoted by $E'$, and its elements are called **(bounded) linear functionals**.
9 Orthogonal projections in Hilbert spaces

Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)$, let $H_1$ be a closed subspace of $H$. (Note that then $(H_1, (\cdot, \cdot))$ is also a Hilbert space - check!) For fixed $x \in H$ we consider the following approximation problem:

$$
\|x - y\| \rightarrow \min! \quad y \in H_1.
$$

(M1)

**Theorem 9.1 (Orthogonal projections)**

(i) Problem (M1) has a unique solution $y^* \in H_1$.

(ii) $y^*$ is the unique solution of the variational equality

$$(x - y^*, h) = 0 \quad \forall h \in H_1$$

(V1)

(iii) For any $x \in H$, there is precisely one pair of elements $x_1, x_2$ such that

$$
x = x_1 + x_2, \quad x_1 \in H_1, \quad x_2 \perp H_1.
$$

Moreover, $x_1 = y^*$.

(iv) The mapping $P : H \rightarrow H_1$ defined by $Px := y^*$ is in $\mathcal{L}(H, H_1)$ and satisfies

$$P^2 := P \circ P = P.$$

If $H_1 \neq \{0\}$ then $\|P\|_{\mathcal{L}(H,H_1)} = 1$.

**Proof:** 1. We show that (M1) has a solution: Let

$$d := d(x, H_1) = \inf_{y \in H_1} \|x - y\|$$

be the distance from $x$ to $H_1$. There is a sequence $(y_k)$ in $H_1$ such that

$$\|x - y_k\|^2 \xrightarrow{k \to \infty} d^2.$$

Fix $k, l \in \mathbb{N}$ and set $a := x - y_k, b := x - y_l$. Then $a + b = 2(x - \frac{y_k + y_l}{2})$. As $H_1$ is a subspace, $\frac{y_k + y_l}{2} \in H_1$ and therefore $\|a + b\|^2 \geq 4d^2$. By the Parallelogram identity,

$$\|y_k - y_l\|^2 = \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2) - \|a + b\|^2 \leq 2(\|x - y_k\|^2 + \|x - y_l\|^2) - 4d^2 \xrightarrow{k,l \to \infty} 0.$$

Hence $(y_k)$ is a Cauchy sequence. As $H$ is a Hilbert space, there is an $y^* \in H$ with $y_k \xrightarrow{H} y^*$. As $H_1$ is closed, $y^* \in H_1$ by Lemma 1.2, and by Lemma 1.1

$$\|x - y^*\| = \|x - \lim_{n \to \infty} y_k\| = \|\lim_{n \to \infty} (x - y_k)\| = \lim_{n \to \infty} \|x - y_k\| = d.$$

Therefore, $y^*$ solves (M1).
2. We show that any solution $y^*$ of (M1) solves (V1): The equality in (V1) is obviously true for $h = 0$. Assume $h \in H_1 \setminus \{0\}$, $\lambda \in \mathbb{R}$. Then $y^* + \lambda h \in H_1$, and therefore

$$\|x - y^* - \lambda h\|^2 \geq \|x - y^*\|^2$$

or, after using the definition of the norm and simplifying,

$$-2\lambda \text{Re}(x - y^*, h) + \lambda^2 \|h\|^2 \geq 0$$

for all $\lambda \in \mathbb{R}$. Setting $\lambda = \frac{\text{Re}(x - y^*, h)}{\|h\|^2}$ yields

$$\text{Re}(x - y^*, h) = 0.$$

for all $h \in H_1 \setminus \{0\}$. If $\mathbb{K} = \mathbb{R}$, this is our result, if $\mathbb{K} = \mathbb{C}$ we replace $h$ by $ih$ and find

$$\text{Re}(x - y^*, ih) = \text{Re}(-i(x - y^*, h)) = \text{Im}(x - y^*, h) = 0.$$

This shows (V1) also for $\mathbb{K} = \mathbb{C}$.

3. We show that the solution of (V1) is unique: Let $y_1$, $y_2$ be two solutions of (V1). Subtracting the corresponding variational equalities yields $(y_1 - y_2, h) = 0$ for all $h \in H_1$. Setting $h := y_1 - y_2$ yields $\|y_1 - y_2\|^2 = 0$, hence $y_1 = y_2$. As any solution of (M1) is also a solution of (V1), the solution of (M1) is also unique. Hence (i) and (ii) are proved.

4. We show (iii): By (ii), the elements $x_1 := y^*$ and $x_2 := x - y^*$ have the properties demanded in (iii). To show uniqueness, assume a pair $x_1, x_2 \in H$ has the properties given in (iii). This implies

$$(x - x_1, h) = 0 \quad \forall h \in H_1.$$ 

By (ii), $x_1 = y^*$ and $x_2 = x - y^*$.

5. We show (iv): For arbitrary $x, z \in H$ we have from (iii)

$$x = Px + (x - Px), \quad Px \in H_1, \quad x - Px \perp H_1,$$

$$z = Pz + (z - Pz), \quad Pz \in H_1, \quad z - Pz \perp H_1,$$

$$x + z = P(x + z) + x + z - P(x + z), \quad P(x + z) \in H_1, \quad x + z - P(x + z) \perp H_1.$$ 

On the other hand, by adding (9.1) and (9.2) we get

$$x + z = P(x) + P(z) + x + z - (P(x) + P(z)),$$

$P(x) + P(z) \in H_1, \quad x + z - (P(x) + P(z)) \perp H_1$. Comparing this to (9.3) and using the uniqueness statement of (iii), we get

$$P(x + z) = P(x) + P(z).$$

Analogously, one shows

$$P(\lambda x) = \lambda P(x), \quad x \in H, \lambda \in \mathbb{K}.$$ 

Hence $P$ is a linear operator. By Pythagoras’ theorem,

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2,$$
hence \( \|Px\| \leq \|x\| \) for all \( x \in H \) and therefore \( P \in \mathcal{L}(H, H) \) with \( \|P\|_{\mathcal{L}(H, H)} \leq 1 \). Moreover, clearly \( Px = x \) iff \( x \in H_1 \), thus \( P^2 = P \). If \( H_1 \neq \{0\} \) then there is an \( x_0 \in H_1 \setminus \{0\} \), and \( Px_0 = x_0 \) implies \( \|P\|_{\mathcal{L}(H, H)} \geq 1 \). \( \square \)

**Definitions:** The linear operator defined in Theorem 9.1 (iv) is called the **orthogonal projection** of \( H \) onto \( H_1 \). The decomposition in (iii) is called **orthogonal decomposition** of \( x \) with respect to \( H_1 \). Any sequence \( (y_n) \) in \( H_1 \) such that

\[
\|x - y_n\| \to \inf_{y \in H_1} \|x - y\|
\]

is called a **minimizing sequence** for (M1).

## 10 The theorems of Riesz and Lax-Milgram

Let \( H \) be a Hilbert space.

Any element \( y \in H \) determines a bounded linear functional \( g_y \) on \( H \) via

\[
g_y(x) := (x, y).
\]

The following important theorem states that any bounded linear functional on \( H \) can be represented in that way.

**Theorem 10.1 (Riesz representation theorem)**

For any \( f \in H' \), there is precisely one \( y \in H \) such that

\[
f(x) = (x, y) \quad \forall x \in H.
\]

Moreover, \( \|f\|_{H'} = \|y\|_H \).

**Proof:** 1. If \( f = 0 \) then clearly (10.1) is satisfied with \( y = 0 \). Assume \( f \neq 0 \) and choose \( x_0 \in H \) such that \( f(x_0) = 1 \). The subset

\[
H_1 := \{x \in H \mid f(x) = 0\}
\]

is a closed subspace of \( H \). (Check!) Let \( P \) denote the orthogonal projection of \( H \) onto \( H_1 \) and set \( u := x_0 - Px_0 \). Then \( u \neq 0 \), \( u \perp H_1 \) by Theorem 9.1, and

\[
f(u) = f(x_0) - f(Px_0) = 1.
\]

Therefore, for any \( x \in H \),

\[
f(x - f(x)u) = f(x) - f(x)f(u) = 0,
\]

hence \( x - f(x)u \in H_1 \). As \( u \perp H_1 \) we get

\[
(x, u) = f(x)\|u\|^2,
\]

and \( f(x) = (x, y) \) with \( y := \frac{u}{\|u\|^2} \).
2. To show the uniqueness of $y$, assume

$$f(x) = (x, y_1) = (x, y_2) \quad \forall x \in H.$$  

Then $(x, y_1 - y_2) = 0$ for all $x \in H$, and choosing $x := y_1 - y_2$ yields $\|y_1 - y_2\| = 0$, hence $y_1 = y_2$.

3. We show $\|f\|_{H'} = \|y\|_H$: From (10.1) and the Cauchy-Schwarz inequality we get

$$|f(x)| \leq \|y\| \|x\|,$$

hence $\|f\|_{H'} \leq \|y\|$. If $y = 0$, this implies equality. Otherwise, from $f(y) = |f(y)| = \|y\|^2$ we get

$$\|y\| = \frac{|f(y)|}{\|y\|} \leq \|f\|_{H'},$$

and the equality is proved.

**Definitions:** A mapping $a : H \times H \rightarrow \mathbb{K}$ is called a **sesquilinear form** if:

- $a$ is linear in the first argument:
  
  $$a(\lambda x + \mu y, z) = \lambda a(x, z) + \mu a(y, z), \quad x, y, z \in H, \lambda, \mu \in \mathbb{K}$$

- $a$ is **antilinear** in the second argument:
  
  $$a(x, \lambda y + \mu z) = \overline{\lambda} a(x, y) + \overline{\mu} a(x, z), \quad x, y, z \in H, \lambda, \mu \in \mathbb{K}$$

If $\mathbb{K} = \mathbb{R}$ then $a$ is linear in both arguments and is called a **bilinear form**.

A sesquilinear form is called **bounded** iff there is a $C > 0$ such that

$$|a(x, y)| \leq C \|x\| \|y\|, \quad x, y \in H.$$  \hfill (10.2)

A sesquilinear form is called **$H$-elliptic** iff there is a $c > 0$ such that

$$\text{Re} \ a(x, x) \geq c \|x\|^2, \quad x \in H.$$  \hfill (10.3)

Let $f \in H'$ be given. We consider the variational problem

$$a(x, y) = f(x) \quad \forall x \in H.$$  \hfill (V2)

**Theorem 10.2** (*Lax-Milgram theorem*)

Assume that $a$ is a bounded, $H$-elliptic sesquilinear form. Then, for any $f \in H'$ there is precisely one $y \in H$ such that (V2) holds. It satisfies an estimate

$$\|y\|_H \leq \frac{1}{c} \|f\|_{H'},$$

with $c$ from (10.3).
Remark: The inner product of $H$ is a bounded, $H$-elliptic sesquilinear form itself. In the special case $a(\cdot, \cdot) = \langle \cdot, \cdot \rangle$, the Lax-Milgram theorem coincides with the Riesz representation theorem.

Proof of Theorem 10.2: 1. We reformulate (V2) as an operator equation in $H$: 
For any fixed $y$, the mapping $x \mapsto a(x, y)$ is a bounded linear functional on $H$, hence by the Riesz representation theorem, there is an unique $z \in H$ such that 
$$a(x, y) = \langle x, z \rangle \quad \forall x \in H.$$ 
We define a mapping $A : H \rightarrow H$ by $Ay := z$. Again by the Riesz representation theorem, there is an unique $w_f \in H$ such that 
$$f(x) = \langle x, w_f \rangle \quad \forall x \in H.$$ 
Hence (V2) is equivalent to 
$$\langle x, Ay \rangle = \langle x, w_f \rangle \quad \forall x \in H,$$
or to 
$$Ay = w_f. \quad (10.4)$$

2. We show that (10.4) has a solution for all right hand sides, i.e. that 
$$R := \{ Ay \mid y \in H \} = H.$$
2.1. $A$ is linear: For any $x, y_1, y_2 \in H$, $\lambda, \mu \in \mathbb{K}$ we have 
$$\langle x, A(\lambda y_1 + \mu y_2) \rangle = a(x, \lambda y_1 + \mu y_2) = \overline{\lambda}a(x, y_1) + \overline{\mu}a(x, y_2) = \overline{\lambda} \langle x, Ay_1 \rangle + \overline{\mu} \langle x, Ay_2 \rangle = \langle x, \lambda Ay_1 + \mu Ay_2 \rangle,$$
and therefore $A(\lambda y_1 + \mu y_2) = \lambda Ay_1 + \mu Ay_2$. (Check!) Therefore $R$ is a linear subspace of $H$. (Check!)

2.2. $A$ is bounded: For any $y \in H$, 
$$\| Ay \|^2 = \langle Ay, Ay \rangle = a(Ay, y) \leq C \| Ay \| \| y \|.$$ 
Hence $\| Ay \| \leq C \| y \|.$
2.3. $R$ is closed: Note at first that 
$$c \| y \|^2 \leq \text{Re} a(y, y) \leq \text{Re} \langle y, Ay \rangle \leq \| y \| \| Ay \|,$$
therefore 
$$\| Ay \| \geq c \| y \|. \quad (10.5)$$
Now let $(z_k)$ be a sequence in $R$ with $z_k \overset{H}{\rightarrow} z^*$. There is a sequence $(y_k)$ in $H$ such that $Ay_k = z_k$. Due to (10.5) we have 
$$\| y_k - y_l \|^2 \leq \frac{1}{c} \| Ay_k - Ay_l \|^2 = \frac{1}{c} \| z_k - z_l \|^2 \overset{k,l \rightarrow \infty}{\longrightarrow} 0,$$
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thus \((y_k)\) is a Cauchy sequence, hence it converges to some \(y^* \in H\). As \(A\) is continuous, we have

\[
z^* = \lim_{k \to \infty} z_k = \lim_{k \to \infty} Ay_k = A \lim_{k \to \infty} y_k = Ay^* \in R.
\]

Hence \(R\) is closed by Lemma 1.2.

2.4. \(R = H\): Let \(P\) denote the orthogonal projection onto \(R\). Assume there is an \(x \in H \setminus R\). Then \(x_0 := x - Px \neq 0\) and \(x_0 \perp R\). Therefore,

\[
c\|x_0\|^2 \leq \Re a(x_0, x_0) = \Re(x_0, Ax_0) = 0
\]

in contradiction to \(x_0 \neq 0\).

3. To show uniqueness for the solution of (10.4) it is sufficient to remark that \(Ay_1 = Ay_2\) implies by (10.5)

\[
c\|y_1 - y_2\| \leq \|A(y_1 - y_2)\| = 0,
\]

hence \(y_1 = y_2\).

4. The estimate on \(y\) follows from

\[
\|y\|^2 \leq \frac{1}{c} \Re a(y, y) \leq \frac{1}{c} \Re f(y) \leq \frac{1}{c} |f(y)| \leq \frac{1}{c} \|f\|_H \|y\|.
\]

Assume now additionally that

\[
a(x, y) = \overline{a(y, x)}, \quad x, y \in H. \quad (10.6)
\]

In connection with (V2) we are going to consider the minimization problem

\[
J(y) := \frac{1}{2}a(y, y) - \Re f(y) \longrightarrow \min! y \in H. \quad (M2)
\]

(Note that \(J\) is real-valued because of \(a(y, y) = \overline{a(y, y)} \in \mathbb{R}\).)

**Theorem 10.3 (Quadratic minimization problems)**

Let all the assumptions of Theorem 10.2 and (10.6) be satisfied. Then the solution of (V2) is also the unique solution of (M2).

**Proof:** Let \(y\) denote the solution of (V2). Then, for any \(x \in H\),

\[
J(x) - J(y) = \frac{1}{2}(a(x, x) - a(y, y)) - \Re f(x - y) = \frac{1}{2}(a(x, x) - a(y, y) - 2\Re a(x - y, y)).
\]

By (10.6),

\[
2 \Re a(x - y, y) = a(x - y, y) + \overline{a(x - y, y)}
\]

\[
= a(x - y, y) + a(y, x - y) = a(x, y) + a(y, x) - 2a(y, y),
\]

and hence

\[
J(x) - J(y) = \frac{1}{2}(a(x, x) - a(x, y) - a(y, y) + a(y, x))
\]

\[
= \frac{1}{2}a(x - y, x - y) \geq \frac{c_0}{2} \|x - y\|^2.
\]

Therefore, \(J(x) > J(y)\) for \(x \neq y\), i.e. \(y\) is the unique global minimum of \(J\).
11 Application to elliptic boundary value problems

11.1 Linear second order elliptic equations with homogeneous Dirichlet boundary condition

Let \( \Omega \subset \mathbb{R}^m \) be a domain.

**Definition:** Let \( L^\infty(\Omega) \) denote the space of all (equivalence classes of) **essentially bounded** measurable functions on \( \Omega \), i.e.

\[
L^\infty(\Omega) := \{ u : \Omega \rightarrow \mathbb{K} \mid |u(x)| \leq C \text{ a.e. in } \Omega \}.
\]

On \( L^\infty(\Omega) \) we introduce the norm \( \| \cdot \|_\infty \) by

\[
\|u\|_\infty := \inf\{C \geq 0 \mid |u(x)| \leq C \text{ a.e. in } \Omega \}.
\]

(Check that \((L^\infty(\Omega), \| \cdot \|_\infty)\) is indeed a normed space!)

Let \( W^{1,\infty}(\Omega) \) denote the space of all functions in \( L^\infty(\Omega) \) having weak derivatives in \( L^\infty(\Omega) \).

**Lemma 11.1 (Pointwise products in \( L^2(\Omega) \))**

Let \( a \in L^\infty(\Omega) \), \( u \in L^2(\Omega) \). Then the pointwise product \( au \) given by \((au)(x) = a(x)u(x), x \in \Omega\), is in \( L^2(\Omega) \) and

\[
\|au\|_2 \leq \|a\|_\infty \|u\|_2.
\]

**Proof:** Exercise!

From now, let \( \Omega \) be bounded, \( \mathbb{K} = \mathbb{R} \).

Consider the general second-order linear boundary value problem

\[
\begin{alignedat}{2}
-\partial_i(a_{ij}(x)\partial_j u(x)) + b_i(x)\partial_i u(x) + c(x)u(x) &= \phi(x) &\quad &\text{in } \Omega, \\
\quad u &= 0 &\quad &\text{on } \partial\Omega,
\end{alignedat}
\]

where \( i, j = 1, \ldots, m \) and we sum over all indices occurring twice in the same product. The function \( u \) is unknown, and the following data are given:

\[
a_{ij}, c \in L^\infty(\Omega), \quad b_i \in W^{1,\infty}(\Omega), \quad \phi \in L^2(\Omega).
\]

Our crucial assumption for the problem (11.1) to be **elliptic** is

\[
a_{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^m
\]

for some \( \lambda > 0 \), called the ellipticity constant. Moreover, we assume

\[
-\frac{1}{\lambda} \partial_i b_i(x) + c(x) \geq 0, \quad x \in \Omega.
\]

(In both inequalities, it is sufficient that they hold for almost all \( x \in \Omega \).)

**Examples:** The following problems fit into the general framework given above (Check!):

1. The **Poisson problem:**

\[
\begin{alignedat}{2}
-\Delta u &= \phi &\quad &\text{in } \Omega, \\
\quad u &= 0 &\quad &\text{on } \partial\Omega.
\end{alignedat}
\]
2. A general convection-diffusion-reaction equation:

\[-\Delta u + \vec{b} \cdot \nabla u + cu = \phi \quad \text{in } \Omega, \]
\[u = 0 \quad \text{on } \partial \Omega, \]

if \(-\frac{1}{2} \text{div} \vec{b} + c \geq 0\).

3. The Sturm-Liouville problem

\[-(pu')' + qu = \phi \quad \text{in } (0, 1), \]
\[u(0) = u(1) = 0, \]

with \(p \in C^1([0, 1]), q \in C([0, 1]), p(x) > 0, q(x) \geq 0 \text{ for } x \in [0, 1].\)

**Definition:** A weak solution to the problem (11.1) is a function \(u \in W^{1,2}_0(\Omega)\) such that (11.1) holds in the sense of weak derivatives.

**Danger:** The boundary condition in the second equation has to be interpreted in a generalized sense unless \(u\) has a continuous extension to \(\Omega\).

**Lemma 11.2 (Strong solutions are weak solutions)**

Assume \(a_{ij} \in C^1(\Omega), u \in C^2(\Omega) \cap C^1(\overline{\Omega})\) solves (11.1). Then \(u\) is a weak solution of (11.1).

**Proof:** By our assumptions, all derivatives in (11.1) exist as classical derivatives, therefore also as weak ones. It remains to show that \(u \in W^{1,2}_0(\Omega)\). As \(\Omega\) is bounded, we get \(u \in W^{1,2}(\Omega)\) from \(u \in C^1(\overline{\Omega})\) and finally \(u \in W^{1,2}_0(\Omega)\) from (11.1) and Theorem 7.3 (i).

**Lemma 11.3 (Weak or variational formulation)**

A function \(u \in W^{1,2}_0(\Omega)\) is a weak solution to (11.1) iff

\[a(u, v) = f(v) \quad \forall v \in W^{1,2}_0(\Omega), \quad \text{(11.5)}\]

where

\[a(u, v) := \int_{\Omega} (a_{ij} \partial_j u \partial_i v + b_i \partial_i u v + cuv) \, dx, \]
\[f(v) := \int_{\Omega} \phi v \, dx.\]

**Definition:** The variational equation (11.5) is called the weak or variational formulation of (11.1).

To prove Lemma 11.3 and later, we will need the following simple result:

**Lemma 11.4 (Boundedness of \(a\) and \(f\))**

(i) \(a\) is a bounded bilinear form on \(W^{1,2}(\Omega)\).

(ii) \(f\) is a bounded linear form on \(W^{1,2}(\Omega)\) with \(\|f\|_{(W^{1,2}(\Omega))'} \leq \|\phi\|_2\).
Proof: Exercise!

Proof of Lemma 11.3: ”⇒” Let $u$ be a weak solution to (11.1). By our definitions,
\begin{equation}
    a(u, v) = f(v) \quad \forall v \in C_0^\infty(\Omega).
\end{equation}
Now for $v \in W_0^{1,2}(\Omega)$ arbitrary, choose a sequence $(v_n)$ of test functions such that $v_n \stackrel{W^{1,2}}{\to} v$. By the continuity of $a$ and $f$ and by (11.6),
\begin{equation}
    a(u, v) = \lim_{n \to \infty} a(u, v_n) = \lim_{n \to \infty} f(v_n) = f(v).
\end{equation}

”⇐”: Let $u \in W_0^{1,2}(\Omega)$ solve (11.5). Then, in particular, (11.6) holds, and therefore $u$ is a weak solution to (11.1).

Lemma 11.5 (Smooth weak solutions are strong solutions)

Let $m = 1$ or let $\Omega$ be a Lipschitz domain. Assume $a_{ij} \in C^1(\Omega)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a weak solution to (11.1). Then (11.1)$_1$ is satisfied in the sense of classical derivatives, and (11.1)$_2$ holds.

Proof: The function $a_{ij}\partial_j u$ is continuously differentiable for $i = 1, \ldots, m$. Hence, for arbitrary $v \in C_0^\infty(\Omega)$, we have by partial integration on supp $v$
\begin{equation}
    \int_\Omega a_{ij}\partial_j u \partial_i v \, dx = - \int_\Omega \partial_i(a_{ij}\partial_j u)v \, dx.
\end{equation}
Consequently, writing
\begin{equation}
    z := -\partial_i(a_{ij}\partial_j u) + b_i \partial_i u + cu - \phi,
\end{equation}
we get
\begin{equation}
    \int_\Omega z v \, dx = (z, v) = 0 \quad \forall v \in C_0^\infty(\Omega)
\end{equation}
and therefore, by Lemma 4.2, also for all $v \in L^2(\Omega)$. Hence $z = 0$. i.e. (11.1)$_1$ holds (almost everywhere in $\Omega$). The validity of (11.1)$_2$ follows from Theorem 7.3 (ii).

According to these results, the analysis of (11.1) proceeds in two steps:

1. Show existence and uniqueness of a weak solution.
2. Under the assumption of higher regularity for the data, prove higher regularity of the solution.

We will concentrate here on the first step.

Theorem 11.6 (Existence and uniqueness of a weak solution to (11.1))

Problem (11.5) has precisely one solution $u \in W_0^{1,2}(\Omega)$. It satisfies an estimate
\begin{equation}
    \|u\|_{W^{1,2}} \leq C\|\phi\|_2
\end{equation}
with $C$ independent of $\phi$. 

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Proof: We set \( H := W^{1,2}_0(\Omega) \) and want to apply the Lax-Milgram theorem 10.2. By Lemma 11.4, both \( a \) and \( f \) are bounded, thus it remains to check that \( a \) is \( W^{1,2}_0(\Omega) \)-elliptic. Assume first that \( u \in C^1_0(\Omega) \). We have

\[
\int_{\Omega} b_i \partial_i uu \, dx = \frac{1}{2} \int_{\Omega} b_i \partial_i (u^2) \, dx = -\frac{1}{2} \int_{\Omega} \partial_i b_i u^2 \, dx.
\]

Consequently, by (11.3), (11.4) and the Poincaré inequality (7.1),

\[
a(u, u) = \int_{\Omega} (a_{ij} \partial_i u \partial_j u + (-\frac{1}{2} \partial_i b_i + c) u^2) \, dx \\
\geq \lambda \int_{\Omega} |\nabla u|^2 \, dx \geq \frac{\lambda}{2} \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \geq \alpha \|u\|^2_{W^{1,2}}
\]

with \( \alpha = \min(\frac{\lambda}{2}, \frac{\lambda}{C}) > 0 \).

Taking now \( u \in W^{1,2}_0(\Omega) \) arbitrary and approximating it by a sequence \((u_n)\) of test functions with \( u_n \xrightarrow{W^{1,2}} u \) we find

\[
a(u, u) = \lim_{n \to \infty} a(u_n, u_n) \geq \lim_{n \to \infty} \alpha \|u_n\|^2_{W^{1,2}} = \alpha \|u\|^2_{W^{1,2}}
\]

(Check!) Now our result follows from Theorem 10.2 and Lemma 11.4.

11.2 Linear second order elliptic equations with homogeneous natural boundary condition

Assume \( K = R \).

If \( m > 1 \), assume that \( \Omega \) is a Lipschitz domain and assume additionally that \( \Omega \) has a piecewise \( C^1 \) boundary, i.e. \( \partial \Omega \) consists of a finite number of \( C^1 \)-hypersurfaces, i.e. zero level sets of \( C^1 \) functions with nonvanishing gradient. A boundary point \( x \in \partial \Omega \) is called regular iff there is an \( \varepsilon > 0 \) such that \( \partial \Omega \cap B(x, \varepsilon) \) is a \( C^1 \)-hypersurface.

On the set \((\partial \Omega)^*\) of regular points of \( \partial \Omega \), we introduce the outer normal vector field \( n : (\partial \Omega)^* \to \mathbb{R}^m \) by the following demands: For all \( x \in (\partial \Omega)^* \):

- \( n(x) \) is orthogonal to \( \partial \Omega \) in \( x \)
- \( |n(x)| = 1 \)
- \( n(x) \) is directed outwards, i.e. \( x + \tau n(x) \notin \Omega \) for all sufficiently small \( \tau > 0 \).

It can be shown that such an \( n \) exists, is uniquely determined by these demands, and is continuous. It is possible to introduce a Lebesgue measure and a corresponding integral on \( \partial \Omega \) such that the integral of continuous functions on \((\partial \Omega)^*\) can be calculated in the usual way. Moreover, it can be shown that the set of the boundary points that are not regular is a set of measure zero and therefore does not play a role when integrals are calculated.

For the domain \( \Omega \), Gauss’ integral theorem is valid, and therefore we find

\[
\int_{\Omega} \partial_i uv \, dx = -\int_{\Omega} u \partial_i v \, dx + \int_{\partial \Omega} u v n_i \, dS, \quad i = 1, \ldots, m, \quad u, v \in C^1(\Omega) \cap C(\overline{\Omega}). \quad (11.7)
\]

(Check! What is the corresponding formula for \( m = 1 \)?)
We consider the general second-order linear boundary value problem

\[
\begin{aligned}
-\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu &= \phi \quad \text{in } \Omega, \\
 a_{ij}\partial_j u n_i &= 0 \quad \text{on } (\partial\Omega)^*. \\
\end{aligned}
\] (11.8)

Here we assume again (11.2) for the data.

**Danger:** For the boundary condition (11.8) to make sense, we have to demand that the functions \(a_{ij}\) have well-defined values almost everywhere on \(\partial\Omega\). This is the case if, for example, \(a_{ij}\) is piecewise continuous on \(\Omega\). However, as we will see below, this is not necessary if we restrict our attention to weak solutions.

Again we assume (11.3) and replace (11.4) by the stronger demands

\[-\frac{1}{2}\partial_i b_i + c \geq \mu > 0 \quad \text{for almost all } x \in \Omega \quad \text{and} \quad b_i n_i \geq 0 \text{ on } (\partial\Omega)^*. \quad (11.9)
\]

**Examples:**

1. A general convection-diffusion-reaction equation:

\[
\begin{aligned}
-\Delta u + \tilde{\nabla} u + cu &= \phi \quad \text{in } \Omega, \\
\nabla u \cdot n &=: \frac{\partial u}{\partial n} = 0 \quad \text{on } (\partial\Omega)^*, \\
\end{aligned}
\]

if \(-\frac{1}{2} \text{div } \tilde{b} + c \geq \mu > 0\). (Check!)

The directional derivative \(\frac{\partial u}{\partial n}\) is called **outer normal derivative** of \(u\).

2. The Sturm-Liouville problem

\[
\begin{aligned}
-(pu')' + qu &= \phi \quad \text{in } (0, 1), \\
u'(0) = u'(1) &= 0, \\
\end{aligned}
\]

with \(p \in C^1([0,1]), q \in C([0,1]), p(x) > 0, q(x) > 0\) for \(x \in [0,1]\).

Recall the definitions of \(a\) and \(f\) from Lemma 11.3.

**Definition:** A **weak solution** to (11.8) is a function \(u \in W^{1,2}(\Omega)\) such that

\[
a(u,v) = f(v) \quad \forall v \in W^{1,2}(\Omega). \quad (11.11)
\]

The variational equation (11.11) is called **weak or variational formulation** of our boundary value problem (11.8).

The relations between the strong and weak formulations are as follows:

**Lemma 11.7**

(i) (**Weak solutions**) Let \(u\) be a weak solution to (11.8). Then \(u\) solves (11.8) in the sense of weak derivatives.
(ii) **(Strong solutions are weak solutions)**

Assume \( a_{ij} \in C^1(\Omega) \cap C(\overline{\Omega}) \) and let \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) be a (strong) solution of (11.8). Then \( u \) is a weak solution of (11.8).

(iii) **(Smooth weak solutions are strong solutions)**

Assume \( a_{ij} \in C^1(\Omega) \cap C(\overline{\Omega}) \) and let \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) be a weak solution of (11.8). Then \( u \) solves (11.8)_1 in the sense of strong derivatives, and (11.8)_2 holds.

**Proof:**

(i): Eq. (11.11) implies in particular

\[
a(u, v) = f(v) \quad \forall v \in C_0^{\infty}(\Omega),
\]

and this is by definition (11.8)_1 in the sense of weak derivatives.

(ii): Let \( v \in C^1(\overline{\Omega}) \). Multiplying (11.8)_1 by \( v \) and integration over \( \Omega \) yields

\[
- \int_\Omega \partial_i(a_{ij}\partial_j u) v \, dx + \int_\Omega (b_i \partial_i u v + cu v) \, dx = \int_\Omega \phi v \, dx = f(v).
\]

Applying (11.7) in the first integral yields

\[
a(u, v) - \int_{\partial \Omega} a_{ij} \partial_j u n_i v \, dS = f(v), \tag{11.12}
\]

and the boundary integral term vanishes due to (11.8)_2. Therefore

\[
a(u, v) = f(v) \quad \forall v \in C^1(\overline{\Omega}).
\]

The general case \( v \in W^{1,2}(\Omega) \) follows now from the fact (Corollary of Theorem 6.5) that \( C^1(\overline{\Omega}) \) is dense in \( W^{1,2}(\Omega) \). (The details are left as an exercise.)

(iii): We know from (i) that (11.8)_1 is satisfied in the weak sense. By assumption, the functions \( a_{ij} \partial_j u \) are in \( C^1(\Omega) \) for \( i = 1, \ldots, m \), and therefore the strong derivatives exist and coincide with the weak ones. This shows that (11.8)_1 is satisfied in the sense of strong derivatives.

To show (11.8)_2, assume that \( z(x_0) := a_{ij}(x_0) \partial_j u(x_0) n_i(x_0) \neq 0 \) for some \( x_0 \in (\partial \Omega)^* \). We assume without loss of generality \( z(x_0) > 0 \). As \( z \) is continuous, there is an \( \varepsilon > 0 \) such that \( z(x) > 0 \) for all \( x \in \partial \Omega \cap B(x_0, \varepsilon) \). Now choose \( v \in C^1(\overline{\Omega}) \) such that \( v \geq 0 \), \( v(x_0) > 0 \) and \( \text{supp} \, v \subset B(x_0, \varepsilon) \). (Give an example!) then

\[
\int_{\partial \Omega} a_{ij} \partial_j u n_i v \, dS > 0. \tag{11.13}
\]

(Check!) On the other hand, we derive (11.12) as in (ii). Therefore, as \( v \in W^{1,2}(\Omega) \) and \( u \) solves (11.11),

\[
\int_{\partial \Omega} a_{ij} \partial_j u n_i v \, dS = 0,
\]

contradicting (11.13). Hence (11.8)_2 is satisfied.

The differential operator in the boundary \( a_{ij} \partial_j u n_i \) appears “naturally” by integration by parts, cf. (11.12). Therefore, (11.8)_2 is called (homogeneous) **natural boundary condition** for the equation (11.8)_1.

Existence and uniqueness of the weak solution can be proved now by similar means as in the case of Dirichlet boundary conditions. Note, however, that we will work in the larger space \( W^{1,2}(\Omega) \) here and cannot use the Poincaré inequality for this reason.
Theorem 11.8 (Existence and uniqueness of a weak solution to (11.8))

Problem (11.11) has precisely one solution \( u \in W^{1,2}(\Omega) \). It satisfies an estimate

\[ \|u\|_{W^{1,2}} \leq C\|\phi\|_2 \]

with \( C \) independent of \( \phi \).

Proof: We set \( H = W^{1,2}(\Omega) \) and want to apply the Lax-Milgram theorem 10.2. In view of Lemma 11.4, it only remains to check that \( a \) is \( W^{1,2}(\Omega) \)-elliptic. As in the proof of Theorem 11.6 we get

\[ a(u, u) \geq \int_{\Omega} |\nabla u|^2 \, dx + \mu \int_{\Omega} u^2 \, dx \geq \min(\lambda, \mu)\|u\|_{W^{1,2}}^2 \]

for any \( u \in C^1(\overline{\Omega}) \). By an approximation argument we conclude

\[ a(u, u) \geq \min(\lambda, \mu)\|u\|_{W^{1,2}}^2 \quad \forall u \in W^{1,2}(\Omega). \]

(Exercise!) Now our result follows from Theorem 10.2 and Lemma 11.4.

11.3 The Neumann problem for the Poisson equation

As in the previous subsection, let \( \mathbb{K} = \mathbb{R} \) and let \( \Omega \) be a bounded Lipschitz domain with piecewise \( C^1 \)-boundary. We consider the problem

\[
\begin{aligned}
-\Delta u &= \phi \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } (\partial \Omega)^*. \\
\end{aligned}
\]

(11.14)

The boundary condition \((11.14)_2\) is called Neumann boundary condition, and the problem is called Neumann problem for the Poisson equation.

Although this boundary condition is natural for \((11.14)_1\), this problem is not covered by the general one in the previous subsection because \((11.9)\) is not satisfied. The weak formulation is

\[ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} \phi v \, dx = f(v) \quad \forall v \in W^{1,2}(\Omega) \]  

(cf. (11.11)). We have the same relations between weak and strong solutions as in Lemma 11.7.

However, the situation concerning the solvability of (11.15) is more complicated. In general, there will be no solution for arbitrary \( \phi \in L^2(\Omega) \), and if there is a solution, it is not unique. Indeed, setting \( v \equiv 1 \) in (11.15) yields

\[ \int_{\Omega} \phi \, dx = 0 \]

as a necessary solvability condition, and if a function \( u \in W^{1,2}(\Omega) \) solves (11.15) then the same is true for all functions \( \tilde{u} := u + c \) with arbitrary \( c \in \mathbb{R} \). (Check!)

On the other hand, all (weak) solutions can be obtained in that way:
Lemma 11.9 (Weak solutions to (11.14) differ by a constant)

Let \( u_1, u_2 \in W^{1,2}(\Omega) \) be two solutions of (11.15). Then there is a constant \( c \in \mathbb{R} \) such that \( u_2 = u_1 + c \).

**Proof:** Set \( w := u_2 - u_1 \). Then
\[
\int_{\Omega} \nabla w \cdot \nabla v \, dx = 0 \quad \forall v \in W^{1,2}(\Omega),
\]
and setting \( v := w \) yields \( \| \nabla w \|_2^2 = 0 \), i.e. \( \partial_i w = 0 \) in \( L^2(\Omega) \) for all \( i = 1, \ldots, m \). By Lemma 5.2 this means \( w = c \) a.e. for some constant \( c \in \mathbb{R} \), and the lemma is proved.

In view of these results, it is sensible to demand
\[
\int_{\Omega} u \, dx = 0 \quad (11.17)
\]
as an auxiliary condition to enforce uniqueness of the solution.

**Theorem 11.10** (Conditional existence and uniqueness for weak solutions of (11.14))

Assume (11.16). There is precisely one \( u \in W^{1,2}(\Omega) \) satisfying (11.15) and (11.17). It satisfies an estimate
\[
\| u \|_{W^{1,2}} \leq C \| \phi \|_2
\]
with \( C \) independent of \( \phi \).

**Proof:** We introduce the subspace
\[
H := \left\{ v \in W^{1,2}(\Omega) \mid \int_{\Omega} v \, dx = 0 \right\}.
\]
As \( H \) is closed in \( W^{1,2}(\Omega) \) (Check!), it is a Hilbert space with respect to the inner product of \( W^{1,2}(\Omega) \). We are going to apply the Lax-Milgram theorem 10.2. It follows from (11.4) that \( a \) and \( f \) are bounded on \( H \) as well. We will show that \( a \) is \( H \)-elliptic. By the Poincaré inequality (7.2), we have
\[
a(u, u) = \int_{\Omega} |\nabla u|^2 \, dx \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + c\| u \|_2^2 \geq c'\| u \|_{W^{1,2}}^2
\]
for any \( u \) in \( H \) with positive constants \( c, c' \).

Therefore, the Lax-Milgram theorem yields that there is precisely one \( u \in H \) such that
\[
a(u, v) = f(v) \quad \forall v \in H. \quad (11.18)
\]
Let now \( v \in W^{1,2}(\Omega) \) be arbitrary. We set
\[
\overline{v} := \frac{\int_{\Omega} v \, dx}{\int_{\Omega} dx}, \quad v_1 = v - \overline{v}.
\]
Then \( v_1 \in H \) and \( f(\overline{v}) = 0 \) due to (11.16) (Check!), and therefore
\[
a(u, v) = a(u, v_1) \overset{(11.18)}{=} f(v_1) = f(v) - f(\overline{v}) = f(v).
\]
This proves our theorem.
11.4 The Stokes equations with homogeneous Dirichlet boundary conditions

The Stokes equations describe the stationary motion of a viscous incompressible liquid with negligible inertial forces. The motion is driven by an external force field $\phi = (\phi_1, \ldots, \phi_m)^T$. We describe the motion by the velocity field $u = (u_1, \ldots, u_m)^T$ of the liquid and its (scalar) pressure field $p$. We assume that the liquid moves inside a given fixed domain $\Omega \subset \mathbb{R}^m$ and is at rest at its boundary $\partial \Omega$ (so-called no-slip boundary condition).

In this case, the Stokes equations read

\[
\begin{aligned}
-\Delta u + \nabla p &= \phi \quad \text{in } \Omega, \\
\text{div} u &= 0 \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (11.19)

where

\[
\text{div} u := \sum_{i=1}^m \partial_i u_i. \tag{11.20}
\]

Note that (11.19) is a vector-valued equation. In coordinates it reads

\[-\Delta u_i + \partial_i p = \phi_i, \quad i = 1, \ldots, m.\]

To discuss the solvability of (11.19) we first have to introduce function spaces of vector-valued functions:

**Definition:** Let

\((L^2(\Omega))^m := \{ u : \Omega \to \mathbb{R}^m \mid u_i \in L^2(\Omega), i = 1, \ldots, m \}.\)

The space \((L^2(\Omega))^m\) is a Hilbert space with respect to the inner product \((\cdot, \cdot)_{(L^2(\Omega))^m}\) given by

\[(u, v)_{(L^2(\Omega))^m} := \sum_{i=1}^m (u_i, v_i)_{L^2(\Omega)} = \sum_{i=1}^m \int_{\Omega} u_i v_i \, dx.\]

(Check!)

**Definition:** Let

\((W^{1,2}(\Omega))^m := \{ u \in (L^2(\Omega))^m \mid u_i \in W^{1,2}(\Omega), \}i = 1, \ldots, m \}.\)

The space \((W^{1,2}(\Omega))^m\) is a Hilbert space with respect to the inner product \((\cdot, \cdot)_{(W^{1,2}(\Omega))^m}\) given by

\[(u, v)_{(W^{1,2}(\Omega))^m} := \sum_{i=1}^m (u_i, v_i)_{W^{1,2}} = \sum_{i=1}^m \int_{\Omega} (u_i v_i + \nabla u_i \nabla v_i) \, dx.\]

(Check!)

**Definition:** Let

\((C_0^\infty(\Omega))^m := \{ u : \Omega \to \mathbb{R}^m \mid u_i \in C_0^\infty(\Omega), i = 1, \ldots, m \}\)

and let \((W^{1,2}_0(\Omega))^m\) be the closure of \((C_0^\infty(\Omega))^m\) in \((W^{1,2}(\Omega))^m\). Then

\((W^{1,2}_0(\Omega))^m := \{ u \in (W^{1,2}(\Omega))^m \mid u_i \in W^{1,2}_0(\Omega), i = 1, \ldots, m \}.\)
Finally, note that for the operator given by (11.20) (in the sense of weak derivatives) we have
\[ \text{div} \in L\left((W^{1,2}(\Omega))^m, L^2(\Omega)\right). \] (11.21)

Consequently, the space
\[ W^{1,2}_{0,\sigma}(\Omega) := \left\{ u \in (W^{1,2}_0(\Omega))^m \mid \text{div} u = 0 \right\} \]
is a closed subspace of \((W^{1,2}(\Omega))^m\), hence a Hilbert space with respect to \((\cdot, \cdot)_{(W^{1,2}_0(\Omega))^m}\).

For the weak formulation of the Stokes equations we introduce on \((W^{1,2}_0(\Omega))^m\) the bilinear form
\[ a(u, v) := \int_{\Omega} \nabla u_i \nabla v_i \, dx, \quad u, v \in (W^{1,2}_0(\Omega))^m, \]
and the linear form \(f\) given by
\[ f(v) := \int_{\Omega} \phi_i v_i \, dx, \quad v \in (W^{1,2}_0(\Omega))^m. \]

Again, we assume that \(\Omega\) is bounded.

**Lemma 11.11** (The (bi-)linear forms \(a\) and \(f\))

(i) \(a\) is a bounded bilinear form on \((W^{1,2}_0(\Omega))^m\).

(ii) \(f\) is a bounded linear form on \((W^{1,2}_0(\Omega))^m\).

(iii) \(a\) is \((W^{1,2}_0(\Omega))^m\)-elliptic.

**Proof:** Exercise!

**Definition:** A vector field \(u \in W^{1,2}_{0,\sigma}(\Omega)\) satisfying the variational equality
\[ a(u, v) = f(v) \quad \forall v \in W^{1,2}_{0,\sigma}(\Omega) \] (11.22)
is called a weak solution of the Stokes equations.

**Note:** The weak solution does not contain the pressure field \(p\). If one is interested in the pressure as well, a different weak formulation has to be used.

**Lemma 11.12** (Strong solutions are weak solutions)

Let \(u \in (C^2(\Omega) \cap C^1(\Omega))^m\), \(p \in C^1(\Omega) \cap C(\Omega)\) be a solution to (11.19). Then \(u\) solves (11.22).

**Proof:** Due to our assumptions and Theorem 7.3 we have that \(u \in W^{1,2}_{0,\sigma}(\Omega)\). Let \(v \in (C_0^{\infty}(\Omega))^m\). Then, by integration by parts,
\[ f(v) = \int_{\Omega} \phi_i v_i \, dx = \int_{\Omega} (-\Delta u_i v_i + \partial_i pv_i) \, dx = a(u, v) - \int_{\Omega} p \text{div} v \, dx. \]

\[ \text{div} \] a vector field \(u\) that satisfies \(\text{div} u = 0\) is called **solenoidal**. This motivates the index \(\sigma\) in the notation \(W^{1,2}_{0,\sigma}(\Omega)\).
Now let \( v \in W^{1,2}_{0,\sigma}(\Omega) \). Then there is a sequence \((v_n)\) in \((C^\infty_0(\Omega))^m\) such that \( v_n \to v \) in \((W^{1,2}(\Omega))^m\). Note that due to (11.21) this implies \( \text{div} v_n \to 0 \) in \( L^2(\Omega) \). Consequently, by Lemma 11.11 (i) and (ii),

\[
 f(v) = \lim_{n \to \infty} f(v_n) = \lim_{n \to \infty} \left( a(u, v_n) - \int_\Omega p \text{div} v_n \, dx \right) = a(u, v).
\]

(Check!) This shows the assertion.

\( \square \)

**Theorem 11.13** (Existence and uniqueness of weak solutions for the Stokes equations)

For any \( \phi \in (L^2(\Omega(0))^m \) there is precisely one \( u \in W^{1,2}_{0,\sigma}(\Omega) \) satisfying (11.22). It satisfies an estimate

\[
 \|u\|_{(W^{1,2}(\Omega))^m} \leq C\|\phi\|_{(L^2(\Omega))^m}
\]

with \( C \) independent of \( \phi \).

**Proof:** Exercise!
A Appendix: Lebesgue integration

A.1 Measure theory

Let $X$ be a set and $\mathcal{A}$ a set of subsets of $X$. Then $\mathcal{A}$ is called a $\sigma$-algebra iff

(i) $X \in \mathcal{A}$,

(ii) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$,

(iii) if $A_1, A_2, \ldots \in \mathcal{A}$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

If $\mathcal{A}$ is a $\sigma$-algebra then $(X, \mathcal{A})$ is called a measurable space. The elements of $\mathcal{A}$ are called measurable sets.

Let $\Sigma$ be a set of subsets of $X$. Let $\mathcal{A}$ be the set of all $\sigma$-algebras $\mathcal{A}$ such that $\Sigma \subseteq \mathcal{A}$.

Define

\[ \mathcal{A}(\Sigma) := \bigcap_{\mathcal{A} \in \Omega} \mathcal{A}. \]

Then $\mathcal{A}(\Sigma)$ is a $\sigma$-algebra (check!). This is the smallest $\sigma$-algebra containing $\Sigma$ and is called the $\sigma$-algebra generated by $\Sigma$.

Let $\tau$ be a set of subsets of $X$. Then $\tau$ is called a topology on $X$ iff

(i) $\emptyset, X \in \tau$,

(ii) if $U, V \in \tau$ then $U \cap V \in \tau$,

(iii) if $I$ is any index set and $U_i \in \tau$ for all $i \in I$ then $\bigcup_{i \in I} U_i \in \tau$.

The elements of a topology are called open sets.

If $\tau$ is a topology on $X$ is then $\mathcal{B}(X) := \mathcal{A}(\tau)$ is called the Borel $\sigma$-algebra on $X$. The elements of $\mathcal{B}(X)$ are called Borel sets. For all $m \in \mathbb{N}$ we write $\mathcal{B}^m = \mathcal{B}(\mathbb{R}^m)$ en $\mathcal{B} = \mathcal{B}^1$, where $\tau$ is the usual topology of open sets on $\mathbb{R}^m$.

Define $[0, \infty] = [0, \infty) \cup \{\infty\}$. The order relation $\leq$ on $[0, \infty)$ can be extended to an order relation $\leq$ on $[0, \infty]$ by defining $x \leq \infty$ for all $x \in [0, \infty]$. Then any nonempty subset of $[0, \infty]$ has a supremum. The concept of limit can also be extended to $[0, \infty]$. Then any increasing sequence in $[0, \infty]$ is convergent. Addition and multiplication on $[0, \infty)$ can be extended to $[0, \infty]$ by defining $x + \infty = \infty + x = \infty$ for all $x \in [0, \infty]$ and $x \cdot \infty = \infty \cdot x = \infty$ for all $x \in (0, \infty]$. Moreover, we define $0 \cdot \infty = \infty \cdot 0 = 0$. The set $[0, \infty]$ is given the smallest topology containing all sets of the form $(a, b), [0, b)$ and $(a, \infty]$ with $0 \leq a < b \leq \infty$.

Let $(X, \mathcal{A})$ be a measurable space. A measure on $\mathcal{A}$ is a function $\mu: \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$,

(ii) for all pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$ we have $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

(This property is called $\sigma$-additivity of $\mu$.)
Then the triple \((X, \mathcal{A}, \mu)\) is called a **measure space**. For all \(A \in \mathcal{A}\), \(\mu(A)\) is called the **measure** of \(A\). If \(A \in \mathcal{A}\) and \(\mu(A) = 0\), then any subset \(B \subset A\) is called **\(\mu\)-negligible** subset. Note that not always \(B \in \mathcal{A}\). If \(P(x)\) is a statement for all \(x \in X\), then we say that \(P\) holds **almost everywhere** (a.e.) if \(\{x \in X : P(x)\text{ is false}\}\) is negligible. If \((X, \mathcal{A}, \mu)\) is a measure space, then the \(\sigma\)-algebra generated by \(\mathcal{A}\) and all \(\mu\)-negligible subsets is called the **completion** of \(\mathcal{A}\).

**Theorem A.1** For all \(m \in \mathbb{N}\) there is a unique measure \(\mu: \mathcal{B}^m \to [0, \infty]\) such that

\[
\mu([a_1, b_1) \times \cdots \times [a_m, b_m)) = (b_1 - a_1) \cdots (b_m - a_m)
\]

everywhere, for all \(a_1 < b_1, \ldots, a_m < b_m\). Let \(\Lambda_m\) be the completion of \(\mathcal{B}^m\) in the measure space \((\mathbb{R}^m, \mathcal{B}^m, \mu)\). Then there is a unique measure \(\lambda_m: \Lambda_m \to [0, \infty]\) such that \(\lambda_m(A) = \mu(A)\) for all \(A \in \mathcal{B}^m\).

This measure \(\lambda_m\) is called the **Borel–Lebesgue measure**. We write \(\lambda = \lambda_1\).

The Borel–Lebesgue measure has the following properties:

(i) \(\lambda_m(K) < \infty\) for all compact \(K \subset \mathbb{R}^m\),

(ii) \(\lambda_m(A) = \sup\{\mu(K) : K \subset A, \text{ K compact}\}\) for all \(A \in \Lambda_m\),

(iii) \(\lambda_m(A) = \inf\{\mu(U) : U \supset A, \text{ U open}\}\) for all \(A \in \Lambda_m\).

**A.2 Integration theory**

Let \((X, \mathcal{A}_1)\) and \((Y, \mathcal{A}_2)\) be two measurable spaces. A mapping \(f: X \to Y\) is called **measurable mapping** if \(f^{-1}(B) \in \mathcal{A}_1\) for all \(B \in \mathcal{A}_2\). (Recall: \(f^{-1}(B) := \{x \in X : f(x) \in B\}\).)

Let \((X, \mathcal{A})\) be a measurable space. A function \(f: X \to \mathbb{R}\) is called **measurable function** if \(f\) is a measurable map from \((X, \mathcal{A})\) to \((\mathbb{R}, \mathcal{B})\). Analogously, a function \(f: X \to \mathbb{C}\) or \(f: X \to [0, \infty]\) is called a **measurable function** if \(f\) is a measurable map from \((X, \mathcal{A})\) to \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) or \(([0, \infty], \mathcal{B}([0, \infty]))\), respectively. Any continuous function \(f: \mathbb{R}^m \to \mathbb{C}\) is measurable from \((\mathbb{R}^m, \mathcal{B}^m)\) to \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\). A **step function** is a measurable function \(f: X \to \mathbb{R}\) such that \(f(X)\) is finite.

Let \((X, \mathcal{A}, \mu)\) be a measure space. For a step function \(f: X \to [0, \infty]\) define \(\int f \in [0, \infty]\) by

\[
\int f = \sum_{k=1}^{n} a_k \mu(A_k)
\]

as \(f = \sum_{k=1}^{n} a_k I_{A_k}\) and \(A_1, \ldots, A_n\) are pairwise disjoint. Let now \(f: X \to [0, \infty]\) be a measurable function. Define \(\int f \in [0, \infty]\) by

\[
\int f = \sup\{ \int g : 0 \leq g \leq f \text{ en g is a step function}\}.
\]

The following theorems are of fundamental importance.

**Theorem A.2** (Monotone convergence theorem, Beppo Levi theorem) Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(f, f_1, f_2, \ldots : X \to [0, \infty]\) be measurable functions and assume \(f_n \uparrow f\) pointwise. Then also \(\int f_n \uparrow \int f\).
Lemma A.3 (Fatou's Lemma) Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(f, f_1, f_2, \ldots : X \to [0, \infty]\) be measurable functions. Then

\[
\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n.
\]

For a function \(f: X \to \mathbb{R}\) define \(f^+, f^- : X \to [0, \infty)\) by \(f^+ = \max(f, 0)\) and \(f^- = \max(-f, 0)\). Then \(f = f^+ - f^-\).

A measurable function \(f: X \to \mathbb{R}\) is called integrable if \(\int |f| < \infty\). Note that then also \(\int f^+ < \infty\) and \(\int f^- < \infty\). If \(f\) is integrable, define \(\int f := \int f^+ - \int f^-\).

A measurable function \(f: X \to \mathbb{C}\) is called integrable if \(\int |f| < \infty\). If \(f\) is integrable, define \(\int f := \int \text{Re} f + i \int \text{Im} f\).

Remark: If necessary, we also write \(\int f(x) \, dx = \int f(x) \, d\mu(x) = \int f \, d\mu\) voor \(\int f\).

The following theorem provides a practical tool for interchanging limit and integral:

Theorem A.4 (Lebesgue's dominated convergence theorem) Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(f, f_1, f_2, \ldots : X \to \mathbb{C}\) be measurable functions and assume \(\lim_{n \to \infty} f_n = f\) b.o.. Assume there is an integrable function \(g\) such that for all \(n \in \mathbb{N}\) we have \(|f_n - f| \leq g\) a.e.. Then \(\lim_{n \to \infty} \int |f_n - f| = 0\) and \(\lim_{n \to \infty} \int f_n = \int f\).

Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(E \in \mathcal{A}\). Define

\[\mathcal{A}|_E = \{A \cap E : A \in \mathcal{A}\}.
\]

Let \(\mu_E\) be the restriction of \(\mu\) to \(\mathcal{A}|_E\). For a function \(f: E \to \mathbb{C}\) define

\[\int_E f \, d\mu := \int f \, d\mu_E\]

if the right hand side exists.

If \(f: [a, b] \to \mathbb{R}\) is a Riemann-integrable function we denote the Riemann integral by \(\int_a^b f(x) \, dx\).

Lemma A.5 Assume \(a, b \in \mathbb{R}\), \(a < b\) and let \(f: [a, b] \to \mathbb{R}\) be a (properly) Riemann integrable function. Then \(f\) is measurable from \((\mathbb{R}, \Lambda_1)\) to \((\mathbb{R}, \mathcal{B})\) and

\[\int_a^b f(x) \, dx = \int_{[a, b]} f \, d\lambda.
\]

Here \(\lambda\) is the Borel–Lebesgue measure on \(\Lambda_1\).

Remark: For all \(m \in \mathbb{N}\) and measurable \(f: (\mathbb{R}^m, \Lambda_m) \to \mathbb{R}\) there exists a measurable function \(g: (\mathbb{R}^m, \mathcal{B}^m) \to \mathbb{R}\) such that \(f = g\) a.e..

The following theorem holds for the \(\sigma\)-algebras \(\mathcal{B}^m, \mathcal{B}^n\) en \(\mathcal{B}^{m+n}\) on \(\mathbb{R}^m\), \(\mathbb{R}^n\), and \(\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n\), respectively, and also with the \(\sigma\)-algebras \(\Lambda_m, \Lambda_n\) en \(\Lambda_{m+n}\).

Theorem A.6 (Fubini) Assume \(m, n \in \mathbb{N}\).
I. Let \( f: \mathbb{R}^m \times \mathbb{R}^n \to [0, \infty] \) be a measurable function. For all \( x \in \mathbb{R}^m \), define \( f_x: \mathbb{R}^n \to [0, \infty] \) by \( f_x(y) = f(x, y) \). Then \( f_x \) is a measurable function for almost all \( x \). The mapping \( x \mapsto \int f(x, y) \, dy \) is an (almost everywhere defined) measurable function from \( \mathbb{R}^m \) to \([0, \infty]\). Moreover,

\[
\int f = \int f_x \, dx.
\]

Thus

\[
\int f = \int \left( \int f(x, y) \, dy \right) \, dx.
\]

Analogous results hold if one integrates over \( x \) first.

II. Let \( f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C} \) be an integrable function. For all \( x \in \mathbb{R}^m \) define \( f_x: \mathbb{R}^n \to \mathbb{C} \) by \( f_x(y) = f(x, y) \). Then \( f_x \) is integrable for almost all \( x \in \mathbb{R}^m \). Define \( \tilde{F}: \mathbb{R}^m \to \mathbb{C} \) by

\[
\tilde{F}(x) = \begin{cases} 
\int f_x & \text{if } f_x \text{ is integrable}, \\
0 & \text{else}.
\end{cases}
\]

Then

\[
\int f = \int \tilde{F}(x) \, dx.
\]

Thus

\[
\int f = \int \left( \int f(x, y) \, dy \right) \, dx.
\]

Analogous results hold if one integrates over \( x \) first.

**Theorem A.7 (Sets of Lebesgue measure zero)** A set \( M \subset \mathbb{R}^n \) has Lebesgue measure zero if and only if for any \( \varepsilon > 0 \) there is a sequence of (open) balls \( B(x_1, r_1), B(x_2, r_2), \ldots \) in \( \mathbb{R}^n \) such that \( \sum_{k=1}^{\infty} r_k^n < \varepsilon \) and \( M \subset \bigcup_{k=1}^{\infty} B(x_k, r_k) \).

**References**