2DD52 Solutions for Midterm exam

Thursday, 26 November 2015 **TU/e**

Problem 1. We present a combined solution for both parts (a) and (b). For $q = 1, 2, 3, 4$, we denote

- by x_q the number of cars produced in the q-th quarter by regular employees;
- by y_q the number of cars produced in the q-th quarter by additionally hired workers;
- by s_q the number of cars on stock at the end of the q-th quarter, just before the quarter demand is satisfied and cars are delivered to costumers.
- by t_q the number of cars on stock at the end of the q-th quarter, after the quarter demand has been satisfied and cars have been delivered to costumers.

(1) All variables must be non-negative:

$$x_q,\ y_q,\ s_q,\ t_q \geq 0 \qquad \text{for } q = 1, 2, 3, 4$$

(2) At most 40 cars can be produced in the q-th quarter by regular employees:

$$x_q \leq 40 \qquad \text{for } q = 1, 2, 3, 4$$

(3) At the beginning of the first quarter, there are 10 cars on stock. At the beginning of the q-th quarter ($2 \leq q \leq 4$), there are t_{q-1} cars from the preceding quarter on stock. During each quarter q, there are $x_q + y_q$ new cars produced. This yields the following:

$$\begin{array}{rcccccc} s_1 & = & 10 & + & x_1 & + & y_1 \\ s_2 & = & t_1 & + & x_2 & + & y_2 \\ s_3 & = & t_2 & + & x_3 & + & y_3 \\ s_4 & = & t_3 & + & x_4 & + & y_4 \end{array}$$

(4) At the end of each quarter, the demanded cars are removed from stock. As the values t_q are non-negative, this also ensures that all demands can be satisfied:

$$\begin{array}{rcl} t_1 & = & s_1 - 40 \\ t_2 & = & s_2 - 60 \\ t_3 & = & s_3 - 75 \\ t_4 & = & s_4 - 26 \end{array}$$

(5) Finally, the overall cost of the production plan consists of: the total production cost by regular employees; the total production cost by additionally hired workers; and the total inventory cost. The resulting objective function is

$$\begin{array}{rcrl} \min Z & = & 80.000 & (x_1 + x_2 + x_3 + x_4) \\ & & +\ 100.000 & (y_1 + y_2 + y_3 + y_4) \\ & & +\ 8.000 & (t_1 + t_2 + t_3 + t_4) \end{array}$$

Problem 2.

(a) For making all variables non-negative, substitute $x_2' = -x_2$.
For getting non-negative right hand sides, multiply the first and third constraint by -1.
For getting equality constraints, introduce slack and surplus variables s_1, s_2, s_3.

This yields the following linear program in standard form:

$$\begin{array}{rrrrrrcl}
\max Z = & 5x_1 & +x_2' & & & & & \\
\text{s.t.} & -x_1 & +x_2' & -s_1 & & & = & 1 \\
 & x_1 & +x_2' & & -s_2 & & = & 3 \\
 & 2x_1 & +x_2' & & & +s_3 & = & 2 \\
\text{with} & \multicolumn{7}{l}{x_1, x_2',\ s_1, s_2, s_3 \geq 0}
\end{array}$$

(b) Introduce artificial variables a_1 and a_2 for the first two constraints. The Big-M method maximizes the auxiliary objective $Z = 5x_1 + x_2' - Ma_1 - Ma_2$.

	Z	x_1	x_2'	s_1	s_2	s_3	a_1	a_2	b
Z	1	-5	-1	0	0	0	M	M	0
a_1	0	-1	1	-1	0	0	1	0	1
a_2	0	1	1	0	-1	0	0	1	3
s_3	0	2	1	0	0	1	0	0	2

The first row is not in the right form, as there are basic variables with non-zero cost coefficient. We repair it in the following way:

	Z	x_1	x_2'	s_1	s_2	s_3	a_1	a_2	b
Z	1	-5	$-1-2M$	M	M	0	0	0	$-4M$
a_1	0	-1	$\boxed{1}$	-1	0	0	1	0	1
a_2	0	1	1	0	-1	0	0	1	3
s_3	0	2	1	0	0	1	0	0	2

Variable a_1 leaves the basis, and x_2' enters:

	Z	x_1	x_2'	s_1	s_2	s_3	a_1	a_2	b
Z	1	$-6-2M$	0	$-1-M$	M	0	$1+2M$	0	$1-2M$
x_2'	0	-1	1	-1	0	0	1	0	1
a_2	0	2	0	1	-1	0	-1	1	2
s_3	0	3	0	$\boxed{1}$	0	1	-1	0	1

Variable s_3 leaves the basis, and s_1 enters:

	Z	x_1	x_2'	s_1	s_2	s_3	a_1	a_2	b
Z	1	$-3+M$	0	0	M	$M+1$	M	0	$2-M$
x_2'	0	2	1	0	0	1	0	0	2
a_2	0	-1	0	0	-1	-1	0	1	1
s_1	0	3	0	1	0	1	-1	0	1

Simplex has terminated, as all reduced cost coefficients are non-negative. The artificial variable a_2 is in the basis, and the objective value is minus infinity. Therefore the linear program (LP.1) is infeasible.

(c) The dual of (LP.1) looks as follows:

$$\begin{array}{llllll} \min W & = & -y_1 & +3y_2 & -2y_3 & \\ \text{s.t.} & & y_1 & +y_2 & -2y_3 & \geq 5 \\ & & y_1 & -y_2 & +y_3 & \leq -1 \\ \text{with} & & \multicolumn{4}{l}{y_1 \geq 0;\ y_2, y_3 \leq 0} \end{array} \qquad \text{(LP.2)}$$

(d) As the primal (LP.1) is infeasible, its dual (LP.2) will either be infeasible as well, or otherwise must be unbounded. The point $(y_1, y_2, y_3) = (0, 0, -5/2)$ is feasible for (LP.2). Hence (LP.2) is unbounded.