Solutions for

Tussentoets Wiskunde 2 (2DD52) Donderdag 27 November 2014, 11:00–12:30

Problem 1. We present a combined solution for both parts (a) and (b).

- We denote the four plant locations by their initial letters G, L, M, N, and we denote the three Soda types Blue, Green, Red by their initial letters B, G, R.
- For $i \in \{G, L, M, N\}$ and $j \in \{B, G, R\}$, we introduce a variable x_{ij} that indicates the amount of soda of type j (measured in m^3) to be produced at plant location i.

(1) The overall expected profit for soda of type j produced at plant i then equals the amount x_{ij} multiplied by the expected profit per m^3 in € for soda type j. Therefore the objective function is

$$\begin{aligned} \max\ Z \quad = \quad & 550\,(x_{GB} + x_{LB} + x_{MB} + x_{NB}) \\ & + 400\,(x_{GG} + x_{LG} + x_{MG} + x_{NG}) \\ & + 350\,(x_{GR} + x_{LR} + x_{MR} + x_{NR}) \end{aligned}$$

(2) At every plant i, the processed amount of water must not exceed the available quantity:

$$\begin{array}{llllll} x_{GB} & +x_{GG} & +x_{GR} & \leq & 400 \\ x_{LB} & +x_{LG} & +x_{LR} & \leq & 900 \\ x_{MB} & +x_{MG} & +x_{MR} & \leq & 550 \\ x_{NB} & +x_{NG} & +x_{NR} & \leq & 350 \end{array}$$

(3) At every plant i, the amount of worked hours must not exceed the available amount:

$$\begin{array}{lllll} 2x_{GB} & +4x_{GG} & +3x_{GR} & \leq & 1800 \\ 2x_{LB} & +4x_{LG} & +3x_{LR} & \leq & 3200 \\ 2x_{MB} & +4x_{MG} & +3x_{MR} & \leq & 2300 \\ 2x_{NB} & +4x_{NG} & +3x_{NR} & \leq & 1200 \end{array}$$

(4) For every soda type j, the total produced amount must not exceed the production ceiling:

$$\begin{array}{llllll} x_{GB} & +x_{LB} & +x_{MB} & +x_{NB} & \leq & 700 \\ x_{GG} & +x_{LG} & +x_{MG} & +x_{NG} & \leq & 800 \\ x_{GR} & +x_{LR} & +x_{MR} & +x_{NR} & \leq & 300 \end{array}$$

(5) Finally, all quantities x_{ij} must be non-negative:

$$x_{ij} \geq 0 \qquad \text{for all } i \in \{G, L, M, N\} \text{ and } j \in \{B, G, R\}.$$

Problem 2.

(a) For making all variables non-negative, substitute $x_2' = -x_2$ and $x_3' = -x_3$.
For getting a maximization problem, multiply the objective function by -1.
For getting non-negative right hand sides, multiply the inequalities by -1.
For getting equality constraints, introduce slack and surplus variables s_1, s_2, s_3.

This yields the following linear program in standard form:

$$\begin{array}{llllllllll} \max W & = & 3x_1 & +x_2' & +x_3' & & & & & \\ \text{s.t.} & & -x_1 & +x_2' & +x_3' & -s_1 & & & = & 1 \\ & & x_1 & +x_2' & & & -s_2 & & = & 3 \\ & & 2x_1 & -x_2' & -x_3' & & & +s_3 & = & 2 \\ \text{with} & & \multicolumn{8}{l}{x_1, x_2', x_3',\ s_1, s_2, s_3 \geq 0} \end{array}$$

(b) For the first two constraints, the 2-phase method introduces artificial variables a_1 and a_2. The objective in the first phase is to maximize $-a_1 - a_2$.

	W	x_1	x_2'	x_3'	s_1	s_2	s_3	a_1	a_2	b
W	1	0	0	0	0	0	0	1	1	0
a_1	0	-1	1	1	-1	0	0	1	0	1
a_2	0	1	1	0	0	-1	0	0	1	3
s_3	0	2	-1	-1	0	0	1	0	0	2

The first row is not in the right shape, as some basic variables have non-zero cost coefficients. Hence we repair it in the following way:

	W	x_1	x_2'	x_3'	s_1	s_2	s_3	a_1	a_2	b
W	1	0	-2	-1	1	1	0	0	0	-4
a_1	0	-1	$\boxed{1}$	1	-1	0	0	1	0	1
a_2	0	1	1	0	0	-1	0	0	1	3
s_3	0	2	-1	-1	0	0	1	0	0	2

Variable a_1 leaves the basis, and x_2' enters:

	W	x_1	x_2'	x_3'	s_1	s_2	s_3	a_1	a_2	b
W	1	-2	0	1	-1	1	0	2	0	-2
x_2'	0	-1	1	1	-1	0	0	1	0	1
a_2	0	2	0	-1	$\boxed{1}$	-1	0	-1	1	2
s_3	0	1	0	0	-1	0	1	1	0	3

Variable a_2 leaves the basis, and s_1 enters:

	W	x_1	x'_2	x'_3	s_1	s_2	s_3	a_1	a_2	b
W	1	0	0	0	0	0	0	1	1	0
x'_2	0	1	1	0	0	-1	0	0	1	3
s_1	0	2	0	-1	1	-1	0	-1	1	2
s_3	0	3	0	-1	0	-1	1	0	1	5

The first phase terminates with $a_1 = a_2 = 0$. We remove the columns for the artificial variables and start the second phase (with the original objective function):

	W	x_1	x'_2	x'_3	s_1	s_2	s_3	b
W	1	-3	-1	-1	0	0	0	0
x'_2	0	1	1	0	0	-1	0	3
s_1	0	2	0	-1	1	-1	0	2
s_3	0	3	0	-1	0	-1	1	5

The first row is not in the right shape, as some basic variables have non-zero cost coefficients. Hence we repair it in the following way:

	W	x_1	x'_2	x'_3	s_1	s_2	s_3	b
W	1	-2	0	-1	0	-1	0	3
x'_2	0	1	1	0	0	-1	0	3
s_1	0	2	0	-1	1	-1	0	2
s_3	0	3	0	-1	0	-1	1	5

Now we see that the reduced cost coefficient of s_2 is negative, while none of the other entries in the corresponding column is positive. We conclude that the LP is unbounded.

(c) The dual of (LP.1) looks as follows:

$$
\begin{array}{llrrrcr}
\max V & = & -y_1 & -3y_2 & -2y_3 & & \\
\text{s.t.} & & y_1 & -y_2 & -2y_3 & \leq & -3 \\
 & & y_1 & +y_2 & -y_3 & \geq & 1 \\
 & & y_1 & & -y_3 & \geq & 1 \\
\text{with} & & \multicolumn{5}{l}{y_1, y_2 \leq 0;\ y_3 \geq 0}
\end{array}
\qquad \text{(LP.2)}
$$

(d) Since the primal (LP.1) is unbounded, its dual (LP.2) must be infeasible.