Algebra 2

Monoids and groups

Cyclic monoids

A.M. Cohen, H. Cuypers, H. Sterk
A monoid that can be generated by a single element is called cyclic. Let \( k, n \in \mathbb{N} \) with \( n > 0 \). An example of a cyclic monoid with generator \( c \) is the monoid defined on the set \( \{ c^i \mid i \in \{0, \ldots, k + n - 1\} \} \) by means of the following multiplication rules.

- \( c^j \cdot c^i = c^{j+i} \) if \( j + i < k + n \);
- \( c^j \cdot c^i = c^{\text{rem}(j+i-k,n)} \), if \( j + i \geq k + n \);
- \( c^0 = 1 \) is the identity.

We refer to this monoid as \( C_{k,n} \). Clearly, \( C_{k,n} \) is cyclic with generator \( c \).
A monoid that can be generated by a single element is called *cyclic*. Let $k, n \in \mathbb{N}$ with $n > 0$. An example of a cyclic monoid with generator $c$ is the monoid defined on the set \( \{ c^i \mid i \in \{0, \ldots, k + n - 1\} \} \) by means of the following multiplication rules.

- $c^j \cdot c^i = c^{j+i}$ if $j + i < k + n$;
- $c^j \cdot c^i = c^{k + \text{rem}(j+i-k,n)}$, if $j + i \geq k + n$;
- $c^0 = 1$ is the identity.

We refer to this monoid as $C_{k,n}$. Clearly, $C_{k,n}$ is cyclic with generator $c$. 
A monoid that can be generated by a single element is called \textit{cyclic}. Let $k, n \in \mathbb{N}$ with $n > 0$. An example of a cyclic monoid with generator $c$ is the monoid defined on the set $\{c^i \mid i \in \{0, \ldots, k + n - 1\}\}$ by means of the following multiplication rules.

- $c^j \cdot c^i = c^{j+i}$ if $j + i < k + n$;
- $c^j \cdot c^i = c^{k + \text{rem}(j+i-k,n)}$, if $j + i \geq k + n$;
- $c^0 = 1$ is the identity.

We refer to this monoid as $C_{k,n}$. Clearly, $C_{k,n}$ is cyclic with generator $c$. 
A monoid that can be generated by a single element is called *cyclic*. Let $k, n \in \mathbb{N}$ with $n > 0$. An example of a cyclic monoid with generator $c$ is the monoid defined on the set \( \{ c^i \mid i \in \{0, \ldots, k + n - 1\} \} \) by means of the following multiplication rules.

- $c^j \cdot c^i = c^{j+i}$ if $j + i < k + n$;
- $c^j \cdot c^i = c^{k + \text{rem}(j+i-k,n)}$, if $j + i \geq k + n$;
- $c^0 = 1$ is the identity.

We refer to this monoid as $C_{k,n}$. Clearly, $C_{k,n}$ is cyclic with generator $c$. 
**Theorem**

*Every cyclic monoid is isomorphic with either* $[\mathbb{N}, +, 0]$ *or with* $C_{k,l}$ *for certain* $k, l \in \mathbb{N}$. 


Example

Here is a cyclic monoid. Try to do some computations. ****plaatje??***
Remark

If you think of $C_{k,n}$ in the following way, the reason for the name cyclic becomes clear. First there is the beginning piece of the monoid consisting of $e, c, c^2, \ldots, c^k$. Then comes the cyclic part, consisting of $c^k, c^{k+1}, c^{k+2}, \ldots, c^{k+n-1}, c^{k+n} = c^k$. At the end of this list we are back at the element $c^k$. After that the cyclic part repeats itself:

$$c^{k+n+1} = c^{k+1}, \quad c^{k+n+2} = c^{k+2}, \ldots$$
Remark

- $|C_{k,n}| = k + n$.
- For every $m \in \mathbb{N}$ with $m > 0$, there are precisely $m$ nonisomorphic cyclic monoids with $m$ elements, viz., $C_{m-k,k}$ for $k=1, \ldots, m$.
- If $k > 0$, then no element of $C_{k,n}$ but 1 is invertible.
- In $C_{0,n}$ every element is invertible (in other words, $C_{0,n}$ is a group, see later).
Remark

- \(|C_{k,n}| = k + n.\)
- For every \(m \in \mathbb{N}\) with \(m > 0\), there are precisely \(m\) nonisomorphic cyclic monoids with \(m\) elements, viz., \(C_{m-k,k}\) for \(k = 1, \ldots, m.\)
- If \(k > 0\), then no element of \(C_{k,n}\) but 1 is invertible.
- In \(C_{0,n}\) every element is invertible (in other words, \(C_{0,n}\) is a group, see later).
Remark

- $|C_{k,n}| = k + n$.
- For every $m \in \mathbb{N}$ with $m > 0$, there are precisely $m$ nonisomorphic cyclic monoids with $m$ elements, viz., $C_{m-k,k}$ for $k=1, \ldots, m$.
- If $k > 0$, then no element of $C_{k,n}$ but 1 is invertible.
- In $C_{0,n}$ every element is invertible (in other words, $C_{0,n}$ is a group, see later).
Remark

- $|C_{k,n}| = k + n$.
- For every $m \in \mathbb{N}$ with $m > 0$, there are precisely $m$ nonisomorphic cyclic monoids with $m$ elements, viz., $C_{m-k,k}$ for $k=1, \ldots, m$.
- If $k > 0$, then no element of $C_{k,n}$ but 1 is invertible.
- In $C_{0,n}$ every element is invertible (in other words, $C_{0,n}$ is a group, see later).
Remark

- \(|\mathbb{C}_{k,n}| = k + n\).
- For every \(m \in \mathbb{N}\) with \(m > 0\), there are precisely \(m\) nonisomorphic cyclic monoids with \(m\) elements, viz., \(\mathbb{C}_{m-k,k}\) for \(k=1, \ldots, m\).
- If \(k > 0\), then no element of \(\mathbb{C}_{k,n}\) but 1 is invertible.
- In \(\mathbb{C}_{0,n}\) every element is invertible (in other words, \(\mathbb{C}_{0,n}\) is a group, see later).