Special linear groups generated by transvections and embedded projective spaces

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Abstract

We give a characterization of the ‘special linear groups’ \( T(\Psi, W) \) as linear groups generated by a non-degenerate class \( \Sigma \) of abstract root groups such that the elements of \( A \in \Sigma \) are transvections.

1 Introduction

1.1 Transvections. Suppose \( W \) is a left vector space of arbitrary dimension defined over some skew field \( L \). For any linear map \( t : W \rightarrow W \) (acting from the right) and vector \( w \in W \), we define the commutator \( [w, t] \) to equal \( wt - w \). Here \( wt \) is the image of \( w \) under the linear map \( t \). If \( U \) is a subspace of \( W \) and \( S \) a set of linear maps of \( W \), then \( [U, S] \) is the subspace of \( W \) spanned by \( \{[u, s] \mid u \in U, s \in S\} \).

An invertible linear map \( t : W \rightarrow W \) is called a transvection if

(a) \( [W, t] \) is 1-dimensional and
(b) \( [W, t] \subseteq C_W(t) = \{w \in W \mid wt = w\} \).

Suppose \( t : W \rightarrow W \) is a transvection. From the definition it is clear that \( C_W(t) \) is a hyperplane of \( W \), it is called the axis of \( t \). The 1-dimensional subspace \( [W, t] \) is called the center of \( t \).

Let \( w_t \) be a vector spanning the center \( [W, t] \) of \( t \). Then there is an element \( \psi \in W^* \), the dual of \( W \) (a right vector space over \( L \)), with kernel \( C_W(t) \) such that the action of \( t \) on \( W \) can be described as follows:

\[
t : w \mapsto w + (w\psi)w_t \text{ for } w \in W.
\]

The study of linear groups generated by transvections has been initiated by the work of McLaughlin \([9, 10]\) who classified all irreducible subgroups of \( \text{GL}(V) \), \( V \) finite-dimensional over a field \( k \), generated by full linear transvection subgroups. (A full linear transvection subgroup of \( \text{GL}(V) \) consists of all transvections to a fixed center
and axis in \( \text{GL}(V) \). Later many others have worked on linear groups generated by transvections, see for example [2, 4, 6, 5, 11, 12, 7, 18, 8, 13, 19].

### 1.2 Transvection subgroups.

Let \( \psi \) be a point in \( W^* \) (i.e., a 1-dimensional subspace) and choose \( 0 \neq \psi \in \psi \). We define \( \ker \psi := \ker \psi \). Let \( p \) be a point in \( \ker \psi \) and choose \( 0 \neq p \in p \). For \( c \in L \), we define

\[
t_c : w \mapsto w + (w\psi)cp \quad \text{for} \quad w \in W.
\]

We call \( T_{p,\psi} := \{ t_c \mid c \in L \} \) the linear transvection subgroup associated to the points \( p \) in \( W \) and \( \psi \) in \( W^* \).

The set of all transvection subgroups of \( \text{GL}(W) \) form a conjugacy class \( \Sigma \) of abelian subgroups in the group \( \text{SL}(W) \) they generate. In fact they form a class of abstract root subgroups in the sense of Timmesfeld [17], i.e., \( \Sigma \) is a conjugacy class in \( \langle \Sigma \rangle \), and for any two \( A \) and \( B \) in \( \Sigma \) we have one of the following (for details see Section 3):

(a) \( [A, B] = 1 \);
(b) \( [A, B] = [a, B] = [A, b] \in \Sigma \) for any \( a \in A^# \) and \( b \in B^# \); moreover \( [A, B] \) commutes with \( A \) and with \( B \).
(c) For each \( a \in A^# \) there is a \( b \in B^# \) with \( A^a = B^b \).

A class \( \Sigma \) of abstract root subgroups is called degenerate if there are no \( A, B \in \Sigma \) as in case (b). In [5] we characterized the linear classical groups related to polarities as groups generated by a degenerate class of abstract root subgroups such that each non-trivial element in a root subgroup is a transvection.

In this paper we are concerned with the non-degenerate case (where all three cases (a), (b), (c) occur). All transvection subgroups of \( \text{GL}(W) \) form a non-degenerate class of abstract root subgroups in \( \text{SL}(W) \), provided \( \dim W \geq 3 \). The group \( \text{SL}(W) \) is a special case of the following class of groups generated by transvection subgroups as first described by Cameron and Hall [2]. Let \( \Psi \) be a subspace of \( W^* \). We define \( \Sigma(\Psi, W) := \{ T_{p,\psi} \mid \psi \subseteq \Psi, p \subseteq \ker \psi \setminus \text{Ann}_W(\Psi) \} \) and \( T(\Psi, W) := \langle \Sigma(\Psi, W) \rangle \leq \text{GL}(W) \), for details see Section 3. The set \( \Sigma(\Psi, W) \) of transvection subgroups is a class of abstract root subgroups of \( T(\Psi, W) \). We notice that the groups \( T(\Psi, W) \) centralize the subspace \( \text{Ann}_W(\Psi) \) of \( W \). If \( \text{Ann}_W(\Psi) = 0 \), then \( T(\Psi, W) \) acts irreducibly on \( W \).

The main goal of this paper is to give a characterization of these 'special linear groups' \( T(\Psi, W) \) as groups generated by their linear transvection subgroups. We now describe the exact setting we will work in:

### 1.3 Setting.

Let \( K \) be a skew field and \( V \) a vector space over \( K \). Assume that \( 1 \neq G \leq \text{GL}(V) \) such that:
(1) $G$ is generated by a non-degenerate class $\Sigma$ of abstract root subgroups of $G$, in the sense of Timmesfeld [17]. (This means that that all three possibilities (a), (b), (c) above occur.)

(2) For $A \in \Sigma$, every $a \in A^\#$ is a transvection on $V$.

(3) $V = [V, G]$.

Notice that we do not assume that for each $A \in \Sigma$ the commutator space $[V, A]$ is 1-dimensional nor that $C_V(A)$ is a hyperplane in $V$, but see (4.1)(a).

For a group $G$ satisfying (1) and (2) but not necessarily (3), we have that $[V, G] = [[V, G], G]$ and the non-trivial elements of root subgroups in $\Sigma$ induce transvections on $[V, G]$. However, the action of $G$ on $[V, G]$ may have a non-trivial kernel, as one can see by considering the action of a suitable $G := T(\Psi, W)$ on the dual space $W^*$. Indeed $[W^*, G] = \Psi \neq W^*$ and any product $t_{r, x} = t_{x, \psi}t_{r-x, \psi}$, where $0 \neq r \in \text{Ann}_W(\Psi)$, is in the kernel of the action.

In this setting we can prove:

**1.4 Main Theorem.** Assume $G$ is a subgroup of $\text{GL}(V)$ generated by a class $\Sigma$ of abstract root groups as in the setting of (1.3) above.

Then there is a vector space $W$ over a skew field $L$ and a subspace $\Psi$ of $W^*$, an injective linear map $\varphi : W \to V$ (with respect to an embedding $\alpha : L \to K$) and an isomorphism $\delta : T(\Psi, W) \to G$ such that

$$(ws)\varphi = (w\varphi)(s\delta) \quad \text{for all } w \in W, s \in T(\Psi, W).$$

Moreover, $\delta$ induces a bijection between $\Sigma(\Psi, W)$ and $\Sigma$.

If we fix a complement $W_0$ of $W$ to $\text{Ann}_W(\Psi)$, and denote by $\Psi_0$ the subspace $\{\psi|_{W_0} \mid \psi \in \Psi\}$ of $W_0^*$, then the group $G \simeq T(\Psi, W)$ is isomorphic to a semi-direct product of $\dim \text{Ann}_W(\Psi)$ copies of $\Psi$ by the quasi-simple group $T(\Psi_0, W_0)$ (see Section 3).

The image in $G$ of this quasi-simple subgroup plays a crucial role in the proof of our main result. This subgroup of $G$ is found in the following way. In the proof of Theorem 1.4 we first construct a point-line geometry $\mathcal{G}$ on the set of centers of elements in $\Sigma$. This is done in Section 4. This point-line geometry contains a maximal linear subspace $G_0$ isomorphic to a classical projective space, $\mathbb{P}(W_0)$ (over the skew field $L$) say, weakly embedded in $\mathbb{P}(V)$. The action of the group $G_0$ generated by the elements in $\Sigma$ having the center in $G_0$ is induced by a semi-linear mapping $\varphi_0 : W_0 \to V$, and we can identify $G_0$ inside $\text{GL}(W_0)$ with $T(\Psi_0, W_0)$. In particular, if $C_V(G) = 0$ the group $G$ equals $G_0$. So, in Section 5, we prove the following result:
1.5 Theorem. Assume $G$ is a subgroup of $\text{GL}(V)$ generated by a class $\Sigma$ of abstract root groups as in the setting of (1.3) above.

If $C_V(G) = 0$ (e.g., $V$ is irreducible), then $G$ is quasi-simple and there is a vector space $W$ over a skew field $L$ and a subspace $\Psi$ of $W^*$ with $\text{Ann}_W(\Psi) = 0$, an injective linear map $\varphi : W \to V$ (with respect to an embedding $\alpha : L \to K$) and an isomorphism $\delta : T(\Psi, W) \to G$ such that

$$(ws)\varphi = (w\varphi)(s\delta) \quad \text{for all } w \in W, s \in T(\Psi, W).$$

Moreover, $\delta$ induces a bijection between $\Sigma(\Psi, W)$ and $\Sigma$.

To finish the proof of Theorem 1.4, it remains to determine the kernel of $G$ in the action on $V/C_V(G)$. This is done by studying the action of $G$ on $G$ and is subject of Section 6.

Finally we notice that our results are, although inspired by the monumental work of Timmesfeld ([15, 16, 17]), independent of these results. Under the assumption that the group $G$ under investigation is a subgroup of $\text{GL}(V)$, we determine $G$ (also when $G$ is not quasi-simple) and its embedding in $\text{GL}(V)$. Besides elementary group theory, we use linear algebra and geometry (in particular, classical results on projective spaces and their embeddings) to obtain our theorems.

2 Projective spaces

In this section we discuss some basic results on projective spaces. First, we give some definitions.

2.1 Definitions. A point-line geometry (or a partial linear space) is a pair $\Gamma = (P, L)$ consisting of a set $P$ of points and a set $L$ of subsets (of cardinality $\geq 2$) of $P$ called lines such that there is at most one line on any pair of points. Two points are said to be collinear if there is a line containing them. A partial linear space is connected if its collinearity graph (i.e., the graph with as vertices the points of $\Gamma$ and two vertices being adjacent whenever they are collinear) is connected.

A subspace of $\Gamma$ is a subset $X$ of the point set closed under lines, i.e., any line meeting $X$ in at least two points is contained in $X$. As the intersection of any collection of subspaces is again a subspace, we can define for each subset $X$ of $P$ the subspace $(X)$ to be the intersection of all subspaces containing $X$. A subspace $X$ is often identified with the partial linear space with as points the elements in $X$ and as lines the lines of $\Gamma$ contained in $X$. A hyperplane of $\Gamma$ is a subspace which meets any line of $\Gamma$ non-trivially.

A point-line geometry is called a projective space if any two points are collinear and the Axiom of Pasch holds, i.e., if $l$, $m$ are distinct lines intersecting in a point $x$, then any two lines meeting both $l$ and $m$ in points distinct from $x$ intersect in a point.
Any two intersecting lines of a projective space generate a subspace called \(\text{projective plane}\), which is a subspace with the property that any two lines contained in this subspace intersect. Conversely, if all subspaces of a connected partial linear space generated by two intersecting lines are projective planes, the partial linear space is projective.

We say, a projective space is \(\text{classical}\) if it is the geometry \(\mathbb{P}(V)\) of 1- and 2-dimensional subspaces of a vector space \(V\) over a skew field.

2.2 Weak embeddings. Let \(\Gamma = (\mathcal{P}, \mathcal{L})\) be a projective space, \(K\) a skew field, \(V\) a vector space over \(K\) and \(\mathbb{P}(V)\) the associated projective space. A \textit{weak embedding} \(\pi : \Gamma \to \mathbb{P}(V)\) is an injective map from \(\mathcal{P}\) to the point set of \(\mathbb{P}(V)\) such that for each \(\ell \in \mathcal{L}\), the subspace of \(\mathbb{P}(V)\) spanned by \(\pi(\ell)\) is a line which intersects \(\pi(\mathcal{P})\) exactly in \(\pi(\ell)\).

We say that \(\Gamma\) is \textit{weakly embedded} in \(\mathbb{P}(V)\).

For completeness (and convenience of the reader) we include the following generalization of a classical result in projective geometry.

2.3 Weak Embedding Theorem for projective spaces. Let \(\Gamma = (\mathcal{P}, \mathcal{L})\) be a projective space containing a plane, \(K\) a skew field, \(V\) a vector space over \(K\) and \(\mathbb{P}(V)\) the associated projective space. Assume that \(\pi : \Gamma \to \mathbb{P}(V)\) is a weak embedding.

Then \(\Gamma = \mathbb{P}(W)\) with \(W\) a vector space (of dimension \(\geq 3\)) over the skew field \(L\) and there exists an embedding \(\alpha : L \to K\) and an injective semi-linear mapping \(\varphi : W \to V\) (with respect to \(\alpha\)) such that

\[
\pi(Lx) = K(x\varphi), \quad \text{for all } 0 \neq x \in W
\]

(i.e., the weak embedding is induced by an injective semi-linear mapping).

**Proof.** The proof follows Baer [1]. First, we check that Desargues’ Theorem holds. For this take 10 points of a plane of \(\Gamma\) which form a Desargues configuration, \(C\) say. We have to show that a special triple of points of \(C\) is collinear, call them \(p, q, r\). The set \(\pi(C)\) is a Desargues configuration in the projective space \(\mathbb{P}(V)\). Since Desargues’ Theorem holds in \(\mathbb{P}(V)\), we see that \(\pi(p), \pi(q), \pi(r)\) are collinear in \(\mathbb{P}\). By the weak embedding axiom, \(p, q, r\) are collinear in \(\Gamma\).

We have obtained that the projective space \(\Gamma\) is classical. Let \(\Gamma = \mathbb{P}(W)\). For \(0 \neq x \in W\), we also write \(\pi(x)\) for \(\pi((x))\).

Let \(x, y \in W\) be linearly independent and choose \(x' \in V\) with \(\pi(x) = \langle x'\rangle\). Since \(\pi\) is a weak embedding, there exists a unique \(y' \in V\) such that \(\pi(y) = \langle y'\rangle\) and \(\pi(x - y) = \langle x' - y'\rangle\). We denote \(y'\) by \(h(x, x', y)\).

For \(x, y, z \in W\) linearly independent, \(\langle y - z\rangle = \langle y, z\rangle \cap \langle x - y, x - z\rangle\). Applying \(\pi\), we obtain:
(* ) If \( x, y, z \in W \) are linearly independent and \( h(x, x', y) = y' \) and \( h(x, x', z) = z' \), then \( h(y, y', z) = z' \).

(Note that \( x', y', z' \) are linearly independent, since \( \pi \) is a weak embedding.)

Since \( \dim W \geq 3 \), there are \( u, v, w \in W \) linearly independent. Choose \( u' \in V \) with \( \pi(u) = \langle u' \rangle \) and set \( v' := h(u, u', v) \) and \( w' := h(u, u', w) \).

Let \( 0 \neq t \in W \). We show that two of the vectors \( h(u, u', t) \), \( h(v, v', t) \) and \( h(w, w', t) \) are defined and equal. We then denote this vector by \( t\varphi \). (The third vector possibly is undefined.) Namely, \( \langle u, v \rangle \cap \langle u, w \rangle \cap \langle v, w \rangle = 0 \). Hence without loss \( t \not\in \langle u, v \rangle \). Then \( h(u, u', v) = v' \) and \( h(u, u', t) := t' \). By (*) we obtain \( h(v, v', t) = t' \).

Setting \( 0\varphi = 0 \), we have defined a map \( \varphi : W \to V \) with \( \pi(Lt) = K(t\varphi) \) for \( 0 \neq t \in W \).

Next, we show that \( \varphi \) respects addition (and hence is injective). Let \( a, b \in W \). It is sufficient to deal with \( a, b \) linearly independent. Then \( a + b \neq 0 \) and there exist \( x, y \in \{ u, v, w \} = \{ x, y, z \} \), \( a + b \not\in \langle x, y \rangle \) such that \( (a + b)\varphi = h(x, x', a + b) = h(y, y', a + b) \). Without loss \( x \not\in \langle a, b \rangle \). (Otherwise rename \( x \) and \( y \).) Hence \( x, a, b \) are linearly independent. Set \( h(x, x', a) =: a', h(x, x', b) =: b' \). Since \( \langle x - a - b \rangle = \langle x - a, b \rangle \cap \langle x - b, a \rangle \), we obtain

\[
\pi(x - a - b) = \langle x' - a', b' \rangle \cap \langle x' - b', a' \rangle = \langle x' - a' - b' \rangle.
\]

Now \( \langle a + b \rangle = \langle a, b \rangle \cap \langle x - a - b, x \rangle \) yields that

\[
\pi(a + b) = \langle a', b' \rangle \cap \langle x' - a' - b', x' \rangle = \langle a' + b' \rangle.
\]

Thus \( (a + b)\varphi = h(x, x', a + b) = a' + b' = h(x, x', a) + h(x, x', b) = a\varphi + b\varphi \). (For the last equation, use that \( a \not\in \langle x, y \rangle \) or \( a \not\in \langle x, z \rangle \), and similarly for \( b \).)

For \( 0 \neq t \in W \), \( 0 \neq c \in L \), we have \( \langle t\varphi \rangle = \pi(t) = \pi(\alpha t) = \langle (\alpha t)\varphi \rangle \). Hence there exists \( \alpha(c, t) \in K \) with \( \pi(c) = \alpha(c, t)(t\varphi) \). A standard calculation shows that \( \alpha(c, t) \) is independent of \( t \). For fixed \( 0 \neq t \in W \), we define \( c^\alpha := \alpha(c, t) \), for \( 0 \neq c \in L \), and \( 0^\alpha := 0 \). Then \( \alpha : L \to K \) is a field embedding and \( \varphi : W \to V \) is an injective semi-linear mapping (with respect to \( \alpha \)) which induces \( \pi \). This proves the theorem.

2.4 Elations. Let \( \Gamma \) be a projective space, \( H \) a hyperplane of \( \Gamma \) and \( p \) a point of \( H \). An elation with center \( p \) and axis \( H \) is a (non-trivial) automorphism of \( \Gamma \) which fixes all points in \( H \) and leaves any line on \( p \) invariant.

2.5 Let \( \Gamma = \mathbb{P}(W) \) be a classical projective space, \( \dim W \geq 3 \). Then any elation with center \( p \) and axis \( H \) is induced by a linear transvection.

Proof. Let \( \theta \) be an elation with center \( p \) and axis \( H \). There exists a bijective semi-linear mapping \( \varphi : W \to W \) (with respect to an automorphism \( \alpha : L \to L \)) which induces \( \theta \), see (2.3) for example. Since \( \theta \) fixes all points in \( H \), there exists \( \lambda \in L \) such that \( x\varphi = \lambda x \) for \( x \in H \) and \( d^\alpha = ad\lambda^{-1} \) for \( d \in L \). Now \( t : W \to W \) defined by \( wt := \lambda^{-1}(w\varphi) \) for \( w \in W \) is a transvection with center \( p \) and axis \( H \) inducing \( \theta \). □
3 Linear transvection subgroups

Let $W$ be a vector space over the skew field $L$. In this section we give some preliminaries on transvections and transvection subgroups in $\text{GL}(W)$.

3.1 Let $t_1$ and $t_2$ be two transvections in $\text{GL}(W)$. Then the following holds:

(a) $[t_1, t_2] = 1$ if and only if $[W, t_1] \subseteq C_W(t_2)$ and $[W, t_2] \subseteq C_W(t_1)$.

(b) Assume that $t_1 t_2 \neq 1$. Then $t_1 t_2$ is a transvection if and only if $[W, t_1] = [W, t_2]$ or $C_W(t_1) = C_W(t_2)$.

(c) Assume that $[t_1, t_2] \neq 1$. Then $[t_1, t_2]$ is a transvection with center $p$ and axis $H$ if and only if $p = [W, t_1] \subseteq C_W(t_2) = H$ (but $[W, t_2] \not\subseteq C_W(t_1)$) or $p = [W, t_2] \subseteq C_W(t_1) = H$ (but $[W, t_1] \not\subseteq C_W(t_2)$).

Proof. Straightforward. \(\square\)

3.2 Definitions. Let $\psi$ be a point in $W^*$ and choose $0 \neq \psi \in \psi$. We define $\ker \psi := \ker \psi$. Let $p$ be a point in $\ker \psi$ and choose $0 \neq p \in p$. We define

$$t_{p, \psi} : w \mapsto w + (w \psi)p$$

for $w \in W$.

We call $T_{p, \psi} := \{ t \in \text{GL}(W) \mid [W, t] \subseteq p, \ker \psi \subseteq C_W(t) \}$ the linear transvection subgroup associated to the points $p$ in $W$ and $\psi$ in $W^*$. Notice that $T_{p, \psi}$ equals $\{ t_c \mid c \in L \}$, where $t_c = t_{c, \psi}$ for $c \in L$. So, $T_{p, \psi}$ is a subgroup of $\text{GL}(W)$, isomorphic to $(L, +)$.

Let $\Psi$ be a subspace of $W^*$. Denote by $\text{Ann}_W(\Psi)$ the subspace $\{ w \in W \mid w \psi = 0 \text{ for all } \psi \in \Psi \}$ of $W$. We define

$$\Sigma(\Psi, W) := \{ T_{p, \psi} \mid \psi \subseteq \Psi, p \subseteq \ker \psi \setminus \text{Ann}_W(\Psi) \}$$

and $T(\Psi, W) := (\Sigma(\Psi, W)) \leq \text{GL}(W)$.

For two linear transvection subgroups the following is known, see Timmesfeld [17, (11.2)] for example.

3.3 For $p, q$ points of $W$ and $\varphi, \psi$ points of $W^*$ with $p \subseteq \ker \varphi$, $q \subseteq \ker \psi$, the following hold:

(a) If $p \subseteq \ker \varphi$ and $q \subseteq \ker \varphi$, then $[T_{p, \varphi}, T_{q, \psi}] = 1$.

(b) If $p \not\subseteq \ker \psi$ and $q \not\subseteq \ker \varphi$, then $\langle T_{p, \varphi}, T_{q, \psi} \rangle \simeq \text{SL}_2(L)$ acting naturally on $p + q$ and centralizing $\ker \varphi \cap \ker \psi$ In particular, for each $1 \neq t_1 \in T_{p, \varphi}$ there is a $t_2 \in T_{q, \psi}$ with $T_{q, \psi}^t = T_{p, \varphi}^t$. 

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Proof. Let $x, y \in W \setminus \Ann_W(\Psi)$. It suffices to prove the lemma for $x, y$ linearly independent and $(x, y) \cap \Ann_W(\Psi) = 0$. But then there exists a $\psi \in \Psi$ with $x \psi = 1$ and $y \psi = 1$. So, $-x + y \in \ker \psi$ and $t_{-x+y,\psi}$ is a transvection in $T(\Psi, W)$ with $xt_{-x+y,\psi} = x + (x\psi)(-x + y) = y$. \hfill $\Box$

3.5 Suppose $\Ann_W(\Psi)$ has codimension at least 2 in $W$. Then $\Sigma(\Psi, W)$ is a class of abstract root subgroups in $T(\Psi, W)$. If $\Ann_W(\Psi)$ has codimension at least 3 in $W$, this class is non-degenerate.

Proof. By (3.3) we only have to show that $\Sigma(\Psi, W)$ is a conjugacy class in $T(\Psi, W)$. Let $T_1$ and $T_2$ be two elements in $\Sigma := \Sigma(\Psi, W)$. Then by (3.4) we can assume that they have the same center. If they also have the same axis, they coincide. So assume $C_W(T_1) \neq C_W(T_2)$ and let $x \in C_W(T_1) \setminus C_W(T_2)$ and $y \in C_W(T_2) \setminus C_W(T_1)$ with $-x + y \notin \Ann_W(\Psi)$. Let $\psi_1 \in \Psi$ with $\ker \psi_1 = C_W(T_1)$. Then, without loss of generality, we can assume that $x \psi_1 = y \psi_1 = 1$. So, $t := t_{-x+y,\psi_1+\psi_2} \in T(\Psi, W)$, it centralizes the center of $T_1$ and $T_2$ and maps $x$ to $y$. In particular, $T^t_1 = T_2$. \hfill $\Box$

Finally, we analyze the structure of $T(\Psi, W)$.

3.6 Let $W_0$ be a complement of $\Ann_W(\Psi)$ in $W$ and $\Psi_0 = \{\psi|_{W_0} \mid \psi \in \Psi\}$. The group $T(\Psi, W)$ is isomorphic to a split extension of $\Ann_W(\Psi) \otimes \Psi$ by $T(\Psi_0, W_0)$. If $W_0$ has dimension at least 3, the group $T(\Psi_0, W_0)$ is quasi-simple.

Proof. For $T_{x,\psi} \in \Sigma(\Psi, W)$ and $0 \neq r \in \Ann_W(\Psi)$, not only $t_{x,\psi}$ and $t_{-x,\psi}$ are transvections of $G := T(\Psi, W)$ but also $t_{r,\psi} = t_{x,\psi} t_{-x,\psi}$.

Hence $N := \langle t_{r,\psi} \mid r \in \Ann_W(\Psi), \psi \in \Psi \rangle$ is an abelian subgroup of $G$. As $\Ann_W(\Psi)$ is centralized by $G$ and $\Psi$ is $G$-invariant, we see that $N$ is a normal subgroup of $G$. Moreover, $G_0 := T(\Psi_0, W_0)$ is a complement to $N$ in $G$. Namely, for $t_{p,\psi}$ a generator of $G$, we have $p = w + r$ with $w \in W_0$, $r \in \Ann_W(\Psi)$ and $t_{p,\psi} = t_{w,\psi} t_{r,\psi} \in G_0 N$.

As each transvection $t_{w,\psi}$ induces a homomorphism $v \in W \mapsto [v, t] = (v \psi) w$ from $W$ to $W$, which we can identify with $w \otimes \psi \in W \otimes W^*$, we find that $N$ is isomorphic to $\Ann_W(\Psi) \otimes \Psi$.

Finally, we prove that $G_0$ is quasi-simple when $W_0$ has dimension at least 3. For a finite-dimensional subspace $E$ of $W_0$, we have $\Psi_0|_E = E^*$.

Assume $\dim(W_0) \geq 3$ and let $\Sigma_0 := \Sigma(\Psi_0, W_0)$. Then, by (3.3), the group $G_0$ is perfect. If $M$ is a non-central normal subgroup of $G_0$, then there is a point $a$ of $W_0$ and an $m \in M$ with $b := a m \neq a$. Let $A \in \Sigma_0$ with $[W, A] = a$, and $b$ not in $C_W(A)$.\hfill 8
Then \([A, A^m] \neq 1\). First assume that \(C := [A, A^m] \in \Sigma_0\). But then \(C \leq [A, AM] \leq M\) and \(G_0 = \langle \Sigma_0 \rangle \leq M\). We are thus left with the case that \(\langle A, A^m \rangle \cong \text{SL}_2(L)\). Choose \(B \in \Sigma_0\) with \([W, B] = [W, A]\) and \([W, A^m] \in C_W(B)\). Then \(D := [B, A^m] \in \Sigma_0\) and \(D \leq [B, AM] \leq M\). Thus \(G_0 = M\) as above. This proves that \(G_0\) is quasi-simple. \(\Box\)

4 The geometry of \(\Sigma\)

Suppose \(G\) is a subgroup of \(\text{GL}(V)\), where \(V\) is a vector space over the skew field \(K\), generated by a non-degenerate class \(\Sigma\) of abstract root groups as in the setting of (1.3).

For an element \(A \in \Sigma\), we denote by \(C_\Sigma(A)\) the set of all \(B \in \Sigma\) with \([A, B] = 1\). This set \(C_\Sigma(A)\) can be partitioned into the set \(\Lambda_A\) consisting of those \(B \in C_\Sigma(A)\) for which there is a \(C \in \Sigma\) with \(B = [A, C]\) and its complement. The set of all elements \(B \in \Sigma\) with \([A, B] \in \Sigma\) is denoted by \(\Psi_A\) and the set of all \(B \in \Sigma\) generating a rank 1 group with \(A\) is denoted by \(\Omega_A\). (Here \(\langle A, B \rangle\) is a rank 1 group, provided that \(A \neq B\) and for any \(a \in A^\#\), there exists some \(b \in B\) with \(B^a = A^b\) (and similarly for any \(b\) there is an \(a\)).

Note that the definition of \(\Lambda_A\) differs from Timmesfeld’s in [17]. (The definitions are easily shown to be equivalent in special linear groups \(T(\Psi, W)\).) A priori it is not clear that \(A \in \Lambda_B\) if \(B \in \Lambda_A\); but see (4.4) below. If \(B \in \Lambda_A\) then \([V, A] = [V, B]\) or \(C_V(A) = C_V(B)\). In (4.6) below we will show that in the first case the converse holds.

4.1 The following holds for \(A, B \in \Sigma\):

(a) \([V, A]\) is 1-dimensional and \(C_V(A)\) is a hyperplane.

(b) If \([A, B] = 1\), then \([V, A] \subseteq C_V(B)\) and \([V, B] \subseteq C_V(A)\).

(c) If \([A, B] = C \in \Sigma\), then either \([V, C] = [V, A] \subseteq C_V(B) = C_V(C)\) or \([V, C] = [V, B] \subseteq C_V(A) = C_V(C)\).

(d) If \(\langle A, B \rangle\) is a rank 1 group, then \([V, A] \not\subseteq C_V(B)\) and \([V, B] \not\subseteq C_V(A)\).

(e) If \([V, A] = [V, B]\) and \(C_V(A) = C_V(B)\), then \(A = B\).

Proof. Since we assume \(\Sigma\) to be non-degenerate, there are \(C, D \in \Sigma\) such that \(A = [c, D] = [C, d]\) for any \(c \in C^\#\) and \(d \in D^\#\). Fix such \(c\) and \(d\). Then, by (3.1), either all transvections in \(A = [c, D] = [C, d]\) have axis \(C_V(c)\) and center \([V, d]\) or they all have axis \(C_V(d)\) and center \([V, c]\). This proves (a).

Moreover, (a) implies that each element from \(\Sigma\) is contained in a linear transvection group. So, (b), (c) and (d) follow immediately from (3.1) and (3.3).

Finally suppose \([V, A] = [V, B]\) and \(C_V(A) = C_V(B)\). Let \(C \in \Sigma\) such that \(\langle A, C \rangle\) is a rank 1 group. By (b) - (d), also \(\langle B, C \rangle\) is a rank 1 group. Now, fix \(c \in C^\#\) and
Let \( A, B \in \Sigma \) with \( C = [A, B] \in \Sigma \). Then each element of \( AC \cap \Sigma \) is contained in an element of \( AC \cap \Sigma = \{ C \} \cup A B \). Moreover, if \( D, E \) are two distinct elements in \( AC \cap \Sigma \), then \( AC = DE \).

**Proof.** (See Timmesfeld [17, (3.11)].) Let \( d = a c \) with \( a \in A^* \), \( c \in C \). Then \( c = [a, b] \) for some \( b \in B \) and \( d = ac = ab \) is contained in \( A^* \).

For the second part of the statement notice that \( AA^* = A[A, b] = AC \) for any \( b \in B^* \) and that \( AC \) is invariant under \( B \). □

**4.3** For \( A \in \Sigma \), there exist \( B, C \in \Lambda_A \) such that \( [V, B] = [V, A] \) and \( C_V(C) = C_V(A) \).

**Proof.** By assumption there exist \( T_1, T_2 \in \Sigma \) such that \( [T_1, T_2] = T_3 \in \Sigma \) and \( [V, T_1] \subseteq C_V(T_2) \). Choose \( g, h \in G \) such that \( A = T_1^g, A = T_2^h \). Then we may set \( B := T_3^g \) and \( C := T_3^h \). □

**4.4** Let \( A, B, C \in \Sigma \) with \( B \in \Lambda_A \).

(a) If \( C_V(A) = C_V(B) \), then there exists \( D \in AB \cap \Sigma \) such that \( [V, D] \subseteq C_V(C) \).

(b) If \( [V, A] = [V, B] \), then there exists \( D \in AB \cap \Sigma \) with \( [V, C] \subseteq C_V(D) \).

(c) For \( C \in \Omega_A \), there is a unique element \( F \) in \( AB \cap \Psi_C \). Moreover \([A, [C, F]] = F \).

(d) \( G_A(B) \) is 2-transitive on \( AB \cap \Sigma \). In particular \( B' \in \Lambda_{A'} \) for any two distinct \( A' \) and \( B' \) in \( AB \cap \Sigma \).

**Proof.** (a) Without loss of generality \([V, A], [V, B] \not\subseteq C_V(C)\).

First, assume that \([V, C] \subseteq C_V(A) = C_V(B)\). Then \([A, C] = [B, C]\), since they both have center \([V, C]\) and axis \( C_V(A) = C_V(B)\), see (4.1)(e). Fix \( a \in A^* \), \( c \in C^* \). Then there exists \( b \in B^* \) such that \([a, c] = [b, c]\). Hence \( c^{-a} = c^{-b} \in C^a \cap C^b \) and \( C^a = C^b \). Denote by \( D \) the element in \( AB \cap \Sigma \) which contains \( ab^{-1} \), see (4.2).

Since \( C_V(C) \) is invariant under \( ab^{-1} \in D \), we obtain \([V, D] \subseteq C_V(C)\) (namely, if \( C_V(C) \subseteq C_V(D) \), then \([V, A] \subseteq C_V(A) = C_V(D) = C_V(C)\), a contradiction).

Hence we are left with the case that \( A \) and \( B \) generate a rank 1 group with \( C \).

Fix \( c \in C^* \) and choose \( a \in A \) and \( b \in B \) such that \( A^c = C^a \) and \( B^c = C^b \). Then \( C_V(C)ab^{-1} = C_V(A)cb^{-1} = C_V(B)bc^{-1} = C_V(C) \) and (a) follows as in the previous case.

(b) is similar and (c) is straightforward.
The above lemma shows that $A \subseteq \Lambda_B$ whenever $B \in \Lambda_A$.

For the next lemma without the assumption that $G$ is a subgroup of $GL(V)$ compare Timmesfeld [17, (9.3)] (with different definition of $\Lambda_A$).

4.5 Let $A, B \in \Sigma$ with $\langle A, B \rangle$ a rank 1 group. Then $G = \langle \Lambda_A, B \rangle$ and $C_V(A) = \langle [V, C] \mid C \in \Lambda_A \rangle = \langle [V, T] \mid T \in C_\Sigma(A) \rangle$.

Proof. Set $R := (\Lambda_A, B)$ and $\Delta := B^R$. Since $A \leq \langle \Lambda_A \rangle$ (by (4.4)(d)) and $A = B^{ab^{-1}}$ for suitable $a \in A^\#$, $b \in B^\#$, we see $A \subseteq \Delta$. Next, we show $\Lambda_A \subseteq \Delta$. Let $C \in \Lambda_A$. Then $[B, C] \neq 1$. If $\langle B, C \rangle$ is a rank 1 group, then $C = B^{ab^{-1}}$ for suitable $c \in C^\#$, $b \in B^\#$, and $C \subseteq \Delta$. Thus we are left with $C \in \Psi_B$. Let $D \in AC \cap \Sigma$, $D \neq A, C$. Then $\langle B, D \rangle$ is a rank 1 group and $D \subseteq \Delta$. By (4.4)(d) there exists $T \in \Sigma$ with $[D, T] = A$. Since $C \in AD \cap \Sigma \setminus \{A\} = D^T$, and $T \in \Lambda_A$, we see $C \subseteq \Delta$. This proves $\Lambda_A \cup \{A\} \subseteq \Delta$.

For $r \in R$, $\Lambda_{A^r} = (\Lambda_A)^r \subseteq \Delta^r = \Delta$. This implies that $\Delta$ contains the connected component containing $A$ of the graph $\mathcal{E}(\Sigma)$ with vertex set $\Sigma$ and edges the pairs $(A', B')$, where $B' \in \Lambda_{A'}$. But $\mathcal{E}(\Sigma)$ is connected. Indeed, since $\Sigma$ is a conjugacy class generating $G$, it suffices to show that $A, B$ are connected in $\mathcal{E}(\Sigma)$, whenever $A, B \in \Sigma$ do not commute. (Note that $\Sigma \not\subseteq C_\Sigma(A) \cup C_\Sigma(B)$.) If $B \in \Psi_A$, then clearly $(A, [A, B], B)$ is a path connecting $A$ and $B$. If $B \in \Omega_A$, choose $C \in \Lambda_A$ and $D \in AC \cap \Sigma$ with $[D, B] \in \Sigma$, see (4.4). Then $(A, D, [B, D], B)$ is a path we are looking for. We conclude that $\Delta = \Sigma$ and $R = G$.

The second statement follows immediately (with $V = [V, G]$).

4.6 Let $A, C \in \Sigma$, $A \neq C$. If $[V, A] = [V, C]$, then $C \in \Lambda_A$.

Proof. If there exists a $B \in \Omega_A$ but not in $\Omega_C$, then $[V, A] = [V, C] \not\subseteq C_V(B)$, so we have $B \in \Psi_C$ and $D := [B, C] \in \Sigma$. Moreover, $[V, A] \subseteq C_V(C) = C_V(D)$ and $[V, D] = [V, B] \not\subseteq C_V(A)$. But then $D \in \Psi_A$ and $[A, D] = C$, hence the lemma.

By symmetry of the argument we can assume that $\Omega_A = \Omega_C$. Since $A \neq C$, we have $C_V(A) \neq C_V(C)$. Since $C_V(A) = \langle [V, B] \mid B \in \Lambda_A \rangle$, there is a $B \in \Lambda_A$ with $[V, B] \not\subseteq C_V(C)$. In particular, $B \in \Psi_C$. Let $D := [B, C]$ and $E \in \Omega_D$. By (4.4) there are elements $b_1, b_2 \in B$ with $E \in \Psi_{C^{b_1}}$ and $E \in \Omega_{C^{b_2}}$. But, as $\Omega_C = \Omega_A$ is $B$-invariant, $\Omega_{C^{b_1}} = \Omega_C = \Omega_{C^{b_2}}$ and we find a contradiction.

4.7 Let $A, B \in \Sigma$. If $[V, B] \not\subseteq C_V(A)$, then there exist $A', B' \in \Sigma$ with $B' \in \Lambda_A$, $C_V(A') = C_V(B')$ such that

$$
\{[V, A], [V, B] a \mid a \in A\} = \{[V, T] \mid T \in A'B' \cap \Sigma\}.
$$
Proof. First, assume that $[V, A] \subseteq C_V(B)$. Set $A' := [A, B]$. By (4.2), we have $A'B \cap \Sigma = \{A'\} \cup B^A$. Since $[V, A] = [V, A']$, the lemma follows with $B' := B$.

Next, assume that $[V, A] \not\subseteq C_V(B)$. Fix $A \neq C \in \Sigma$ such that $[V, A] = [V, C]$, see (4.3). Then $C \in \Lambda_A$ by (4.6). Thus by (4.4)(b), there exists $D \in AC \cap \Sigma$ such that $[V, B] \subseteq C_V(D)$. Since $\{A\} \cup B^A = \{B\} \cup A^B$ and $[V, A] = [V, D]$, we obtain

$$\{[V, A], [V, B]|a|a \in A\} = \{[V, B], [V, D]|b|b \in B\}$$

and we can apply the above first paragraph to $B$ and $D$ instead of $A$ and $B$. \hfill $\Box$

The above lemma does not apply to the case where $C_V(A) = C_V(C)$, since possibly $[V, A]$ and $[V, C]$ coincide modulo $C_V(G)$.

4.8 Let $A, B \in \Sigma$, $B \in \Lambda_A$, with $C_V(A) = C_V(B)$. Then there are $A', B' \in \Sigma$ with $\langle A', B' \rangle$ a rank 1 group such that

$$\{[V, T]|T \in AB \cap \Sigma\} = \{[V, A'], [V, B']|a'|a' \in A'\}.$$

Proof. Let $C \in \Sigma$ such that $\langle A, C \rangle$ is a rank 1 group. Denote by $D$ the unique element in $AB \cap \Sigma$ such that $A' := [D, C] \in \Sigma$, see (4.4). Then $D \neq A$.

Choose $E \in \Sigma$ such that $\langle D, E \rangle$ is a rank 1 group. Denote by $F$ the unique element in $AB \cap \Sigma$ such that $B' := [E, F] \in \Sigma$, see (4.4). Then $F \neq D$.

Necessarily, $\langle A', B' \rangle$ is a rank 1 group. (Namely, $[V, A'] = [V, D] \not\subseteq C_V(E) = C_V(B')$ and $[V, B'] = [V, F] \not\subseteq C_V(C) = C_V(A')$, since $F \neq D$.)

Since $[A', F] = D$ (namely, $[V, A'] = [V, D] \subseteq C_V(D) = C_V(F)$ and $[V, F] \not\subseteq C_V(C) = C_V(A')$, since $F \neq D$), we obtain

$$\{[V, A'], [V, B']|a'|a' \in A'\} = \{[V, D], [V, F]|a'|a' \in A'\} = \{[V, T]|T \in FD \cap \Sigma\}$$

with (4.2). But $FD \cap \Sigma = AB \cap \Sigma$, see (4.2), which proves the lemma. \hfill $\Box$

4.9 Definition. Let $\mathcal{P}$ be the set $\{[V, T]|T \in \Sigma\}$. The elements in $\mathcal{P}$ will be called points. A line is a set $\{[V, T]|T \in AB \cap \Sigma\}$, where $A, B \in \Sigma$, $B \in \Lambda_A$, with $C_V(A) = C_V(B)$. By (4.3), lines exist. They have at least three points. Denote by $\mathcal{L}$ the set of all lines and set $\mathcal{G} = (\mathcal{P}, \mathcal{L})$.

Let $A, B \in \Sigma$ with $[V, A] \neq [V, B]$. We notice the following: If $A \in \Lambda_B$, then, by definition, $[V, A]$ and $[V, B]$ are collinear in $\mathcal{G}$. Also, if $[V, A] \not\subseteq C_V(B)$, then, by (4.7) $[V, A]$ and $[V, B]$ are collinear. See (4.13) below for a criterion when two points are (non-)collinear.

4.10 Let $l$ be a line of $\mathcal{G}$. Then the subspace of $\mathbb{P}(V)$ generated by $l$ is a projective line meeting $\mathcal{P}$ in exactly $l$. In particular, $\mathcal{G}$ is a partial linear space.
\textbf{Proof.} Let $A, B \in \Sigma$ such that $(A, B)$ is a rank 1 group with $[V, A], [V, B]$ on $l$ (see (4.8) for the existence of $A$ and $B$). Suppose that $C \in \Sigma$ with $[V, C]$ in the projective line generated by $[V, A], [V, B]$, but distinct from $[V, A]$. Then $\langle A, C \rangle \neq 1$. (Otherwise $\langle A, C \rangle \subseteq C_V(A)$, a contradiction.) Hence $[V, A]$ and $[V, C]$ are collinear by some line $m$, see (4.7). Choose $A', C' \in \Sigma$ such that $C' \in \Lambda_A$, $[V, A'] = [V, A]$, $[V, C'] = [V, C]$ and $m = \{[V, X] \mid X \in A'C' \cap \Sigma \}$. For any $B' \in \langle A, B \rangle \cap \Sigma$, there exists an $E \in A'C' \cap \Sigma$ with $[V, E] \subseteq C_V(B')$, see (4.4). Hence $[V, E] \subseteq ([V, A'] + [V, C']) \cap C_V(B') = \langle l \rangle \cap C_V(B') = [V, B']$. In particular, $l \subseteq m$.

Now choose two elements $A_1$ and $B_1$ with $A_1 \in \Omega_{B_1}$, $[V, A] = [V, A_1]$, $[V, B] = [V, B_1]$ and $m = \{[V, X] \mid X \in \langle A_1, B_1 \rangle \cap \Sigma \}$. Then $(A_1, B_1)$ is a rank 1 group and $m = \{[V, X] \mid X \in \langle A_1, B_1 \rangle \cap \Sigma \}$, as $[V, B] = [V, B_1]$. But the latter is $\{[V, X] \mid X \in \langle A, B \rangle \cap \Sigma \} = l$, as $[V, A] = [V, A_1]$. This proves that $l$ contains all points of $P$ in $[V, A] + [V, B]$.

The second part of the lemma follows immediately. \hfill $\square$

The above lemma implies that two collinear points $a$ and $b$ are on a unique line which we also denote by $ab$.

\textbf{4.11} Let $l$ be a line and $a$ a point not on $l$. If $a$ is collinear to all points of $l$, then $a$ and $l$ generate a subspace of $G$ isomorphic to a projective plane.

\textbf{Proof.} Suppose $a$ is collinear to all points of $l$. By $\pi$ we denote the set of points on any line through $a$ meeting $l$. We will prove that $\pi$ is a subspace of $G$ isomorphic to a projective plane.

Let $b$ be a point of $l$. Since $a$ and $b$ are collinear by assumption, there are $A, B \in \Sigma$ such that $[V, A] = a$, $[V, B] = b$ and $\langle A, B \rangle$ a rank 1 group, see (4.8).

Let $[V, T_1] \neq [V, T_2]$ on $l$ with $T_2 \in \Lambda_{T_1}$ and $C_V(T_1) = C_V(T_2)$. By (4.4), there exists $T_3 \in T_1T_2 \cap \Sigma$ with $[V, T_3] \subseteq C_V(A)$. Note that $c := [V, T_3]$ is on $l$, see (4.10).

Choose $A', C \in \Sigma$ such that $[V, A'] = a$, $[V, C] = c$ and $\langle A', C \rangle$ a rank 1 group. Then $A' \neq A$, since $[V, C] \not\subseteq C_V(A')$, but $[V, C] \subseteq C_V(A)$. By (4.6), $A' \in \Lambda_A$. We obtain that $\langle A, C \rangle \in \Sigma$ and $A$ is the unique element in $AA' \cap \Psi_C$. Denote by $A_0$ the unique element in $AA' \cap \Psi_B$, see (4.4). Then $A_0 \neq A$ and necessarily $A_0 \in \Omega_C$. Renaming $A' := A_0$, we obtain that $\langle A, B \rangle$ and $\langle A', C \rangle$ are rank 1 groups with

\begin{equation}
(V, C] \subseteq C_V(A), \quad [V, B] \subseteq C_V(A').
\end{equation}

We claim that each point on $l$ is centralized by a unique element of $AA' \cap \Sigma$. Namely, let $p$ be a point on $l$. Then there exists $A_1 \in AA' \cap \Sigma$ such that $p \subseteq C_V(A_1)$ by (4.4)(b). Assume there is also $A_2 \in AA' \cap \Sigma$, $A_2 \neq A_1$, such that $p \subseteq C_V(A_2)$. Then $A_1A_2 \cap \Sigma = AA' \cap \Sigma$ (see (4.2)) and $p$ is centralized by $A$ and $A'$. But $p$ is on $l = bc$, a contradiction by (*).

Next, let $d$ and $e$ be two points of $\pi$ different from $a$ such that the intersection points $f_1$ and $f_2$ of $ad$ and $ae$ with $l$ are distinct.
Choose $F_i \in \Sigma$ such that $f_i = [V, F_i]$ and $a \not\in C_V(F_i)$ $(i = 1, 2)$. Denote by $A_i$ the unique element in $AA' \cap \Sigma$ with $f_i \subseteq C_V(A_i)$ $(i = 1, 2)$. As before we find $A_1 \neq A_2$. We obtain that both $\langle F_1, A_2 \rangle$ and $\langle F_2, A_1 \rangle$ are rank 1 groups.

Write $d = f_1a_2$ with $a_2 \in A_2$ and $e = f_2a_1$ with $a_1 \in A_1$. Then $f_1a_1a_2 = d$ and $f_2a_1a_2 = e$, as $f_i$ is centralized by $A_i$. Hence $l a_1a_2 = de$ and $d$ and $e$ are collinear. (This implies that any two points of $\pi$ are collinear.)

Any point $p$ of $de$ is of the form $f_1a_2$ with $f$ on $l$. Hence $p$ is contained in $\langle a, f \rangle \cap \mathcal{P} = af$, see (4.10). This yields $p$ is in $\pi$ and $\pi$ is a subspace of $\mathcal{G}$.

By (4.4), there exists a point $p$ on $l$ with $p \subseteq C_V(a_1a_2)$. Hence $p = pa_1a_2$ is contained in $l$ and $de$ and the lines $l$ and $de$ meet non-trivially. (This implies that any two lines in $\pi$ intersect.) Thus $\pi$ is a projective plane.

4.12  (a) Every line is contained in a (projective) plane.

(b) $\mathcal{G}$ is connected of diameter at most 2.

(c) A point is collinear to at least all but one of the points of a line.

Proof. Let $l$ be a line. There exists a plane on $l$. Namely, choose $[V, A]$ and $[V, B]$ on $l$ with $B \in A_3$. $C_V(A) = C_V(B)$. Let $C \in \Sigma$ with $\langle A, C \rangle$ a rank 1 group. For $T \in AB \cap \Sigma$, $[V, C] \not\subseteq C_V(A) = C_V(T)$. Hence $[V, C]$ is collinear with all points on $l$, see (4.7). By (4.11), $l$ and $[V, C]$ generate a projective plane. This proves (a).

Let $a, b$ be collinear points and set $l = ab$. Let $c$ be a point not on $l$. First assume $c$ is collinear to $a$. Choose $A, C \in \Sigma$ such that $[V, A] = a$, $[V, C] = c$ and $\langle A, C \rangle$ is a rank 1 group, see (4.8). As the intersection of $C_V(C)$ and $l$ contains at most one point, (4.7) implies that $c$ is collinear to at least all but one of the points of $l$.

Since, by assumption, $\Sigma$ is a conjugacy class generating $G$, and it follows from (4.7) that $\mathcal{G}$ is connected. (Note that $\Sigma \not\subseteq C_\Sigma(A) \cup C_\Sigma(B)$, for $A, B \in \Sigma$.) The diameter of $\mathcal{G}$ is at most 2 (see Cuypers [3]). Indeed, otherwise there are points $p$ and $s$ at distance 3. Choose a shortest path $(p, q, r, s)$ from $p$ to $s$ in $\mathcal{G}$. Let $z$ be a third point on $qr$. By the preceding paragraph, $p$ and $s$ are both collinear to $z$, a contradiction. This proves (b).

Now assume $c$ is not collinear to any point on $l = ab$. Denote by $\pi$ a plane on $l$. If there is a point $d$ in $\pi$ collinear with $c$, the above implies that $c$ is collinear to all points of $\pi \setminus l$. Denote by $e$ a third point on the line $cd$. If $e$ is collinear to all points on $da$, then $da$ and $e$ generate a projective plane by (4.11) (which contains $a$, $c$, and $d$) and hence $a$ and $c$ are collinear, a contradiction. Thus $e$ is not collinear to a point $a'$ on $ad$ and similarly to a point $b'$ on $bd$, $a' \neq b'$. The lines $a'b'$ and $ab$ meet in a point $f$ of $\pi$. Necessarily, $f$ is not collinear to $c$ and $e$. But $f$ is collinear to a third point, namely $d$, on the line $cd$, a contradiction to the above.

So to finish the proof of (c), it suffices to show that $c$ is collinear to a point of $\pi$. But, as the diameter of $\mathcal{G}$ is at most 2, there is a point $p$ collinear with $c$ and collinear
with at least all but one of the points of $\pi$. In particular, this point $p$ is in a plane $\pi'$ meeting $\pi$ in a line. But then $c$ is collinear with at least all but one of the points on $\pi'$ and therefore certainly with a point of $\pi$. 

4.13 Let $a$ and $b$ be distinct points. Then the following statements are equivalent.

(a) The projective line on $a$ and $b$ meets $C_{V}(G)$ non-trivially (in a projective point).

(b) $a$ and $b$ are not collinear.

(c) For each $B \in \Sigma$ with $b = [V, B]$, there is an $A \in \Sigma$ with $a = [V, A]$ and $C_{\Sigma}(A) = C_{\Sigma}(B)$.

Proof.

(a)$\Rightarrow$(b) If $a$ and $b$ are collinear, then there are $A', B' \in \Sigma$ with $a = [V, A']$ and $b = [V, B']$ and $\langle A', B' \rangle$ a rank 1 group, see (4.8). In particular, $C_{V}(G) \subseteq C_{V}(A') \cap C_{V}(B')$ meets the projective line on $a$ and $b$ trivially.

(b)$\Rightarrow$(a) Let $a, b$ be non-collinear points. Choose $C \in \Sigma$ and set $[V, C] = c$. Assume $a \not\subseteq C_{V}(C)$, but $b \subseteq C_{V}(C)$. Then $a$ and $c$ are collinear and $ac$ consists of $c$ and the points $ac$, $c \in C$, see (4.7). Since $a, b$ are non-collinear, $b$ is collinear to all points $ac$, $c \in C'$ of the line $ac$, see (4.12)(c).

Hence $b$ is collinear to $ac$ and $bc$ is not collinear to $ac$, but $bc = b$, a contradiction.

By symmetry of the argument we have $a \subseteq C_{V}(C)$ if and only if $b \subseteq C_{V}(C)$. Let $D \in \Omega_{A}$, and denote by $r$ the projective point on the projective line on $a$ and $b$, which is in $C_{V}(D)$. Then $r$ is centralized by $\langle \Lambda_{A}, D \rangle = G$, see (4.5).

(a)$\Rightarrow$(c) Let $A, B$ with $A, B \in \Sigma$ with $[V, A] = a$ and $[V, B] = b$. First assume that $C_{V}(A) = C_{V}(B)$. Then for $C \in C_{\Sigma}(A)$, we have $[V, B] \subseteq [V, A] + C_{V}(G) \subseteq C_{V}(C)$ and $[V, C] \subseteq C_{V}(A) = C_{V}(B)$, see (4.1). Thus $C \in C_{\Sigma}(B)$. By symmetry of the argument we obtain $C_{\Lambda}(A) = C_{\Sigma}(B)$.

We are thus left with the case that there is $C \subseteq \Lambda_{B}$ with $[V, C] \in C_{V}(B)$ but $[V, C] \not\subseteq C_{V}(A)$. (Recall that $C_{V}(B)$ is spanned by the points $[V, C]$ with $C \in \Lambda_{B}$, see (4.5).) Then $C \in \Psi_{A}$, and we can replace $A$ by $A' = [A, C]$ with $[V, A'] = [V, A]$ and $C_{V}(A') = C_{V}(C) = C_{V}(B)$.

(c)$\Rightarrow$(a) Let $C \in \Omega_{A}$. Then $C_{V}(C)$ meets the projective line on $a$ and $b$ in a projective point.

Since $G = (C_{\Sigma}(A), C)$, see (4.5), this point is in $C_{V}(G)$. □

5 A quasi-simple subgroup of $G$

Consider the situation as studied in the previous section. With Zorn’s Lemma any linear subspace of $G$ is contained in a maximal linear subspace. Fix such a maximal
linear subspace \( G_0 \) of \( G \). Then \( G_0 \) is a projective space as follows by (4.11) and (2.1). By \( V_0 \) we denote the subspace of \( V \) generated by \( G_0 \). Let \( \Sigma_0 := \{ A \in \Sigma \mid [V,A] \subseteq G_0 \} \) and \( G_0 := \langle \Sigma_0 \rangle \).

5.1 For \( A \in \Sigma \), there exists \( A_0 \in \Sigma_0 \) with \( C_V(A) = C_V(A_0) \).

Proof. We may assume \( A \not\in \Sigma_0 \). Since \( G_0 \) is a maximal linear subspace of \( G \), there is \( b \in G_0 \) such that \([V,A]\) and \( b \) are distinct and non-collinear. By (4.13), there is \( B \in \Sigma_0 \) with \( C_{\Sigma}(A) = C_{\Sigma}(B) \). Now (4.5) yields \( C_V(A) = C_V(B) \).

5.2 Fix \( A \in \Sigma_0 \). Then \( A \) induces the full elation subgroup of \( \text{Aut}(G_0) \) with center \([V,A]\) and axis \( C_{V_0}(A) \cap G_0 \).

Proof. Any \( 1 \neq a \in A \) induces an elation of \( G_0 \) with center \([V,A]\) and axis \( C_V(A) \cap G_0 \). Choose \( B \in \Sigma_0 \) such that \( \langle A, B \rangle \) is a rank 1 group. (Such \( B \) exists by (4.8).)

Denote by \( \ell \) the line on \([V, A] \) and \([V, B] \). Then \( A \) acts transitively on the points of \( \ell \setminus \{[V, A]\} \). But the full elation subgroup acts regularly. The Frattini argument implies that \( A \) induces the full elation subgroup.

5.3 Denote by \( \mathcal{H} \) the set of all hyperplanes of \( G_0 \) appearing as axis of an element of \( \Sigma_0 \). Then the following hold:

(a) If \( A \) is in \( \mathcal{H} \) and \( b \) is a point of \( A \), then there is an element in \( \Sigma_0 \) with center \( b \) and axis \( A \).

(b) If \( A, B \) are in \( \mathcal{H} \) and \( C \) is a hyperplane of \( G_0 \) containing \( A \cap B \), then \( C \in \mathcal{H} \).

(c) If \( b \) is a point of \( G_0 \), then there is some element in \( \mathcal{H} \) which does not contain \( b \).

Proof. (a),(c) Choose \( A \in \Sigma_0 \) with axis \( A \). By (4.8), we may write \( b = [V, B] \) and \([V, A] = [V, A_1]\) with \( \langle A_1, B \rangle \) a rank 1 group. Then \([B, A]\) is an element of \( \Sigma_0 \) with center \( b \) and axis \( A \). Moreover, \( b \) is not contained in the axis of \( A_1 \).

(b) Without loss \( A \neq B \). Let \( a \) be a point in \( A \cap B \) and choose \( A, B \in \Sigma \) with center \( a \) and axis \( A \) and \( B \), respectively, see (a). By (4.6) \( A \in \Lambda B \). Let \( d \) be a point of \( C \) not in \( A \cap B \). Then, by (4.4), there is an element \( C \in AB \cap \Sigma_0 \) centralizing \( d \). But then the axis of \( C \) contains \( A \cap B \) and \( d \) and thus equals \( C \).

By (4.11) and (2.1), \( G_0 \) is a projective space, but not a projective line. Moreover, \( G_0 \) is weakly embedded in \( \mathbb{P}(V_0) \) by (4.10). We apply the weak embedding theorem for projective spaces (2.3). Let \( G_0 \) be isomorphic to the classical projective space \( \mathbb{P}(W) \), where \( W \) is a vector space over the skew field \( L \). The weak embedding is induced by an injective semi-linear mapping \( \varphi : W \to V_0 \) (with respect to an embedding \( \alpha : L \to K \)). Note that \( \{ \langle w\varphi \rangle \mid 0 \neq w \in W \} \) is the point set of \( G_0 \) (bijective correspondence with the points in \( \mathbb{P}(W) \)).

Since \( \mathbb{P}(W) \) and \( G_0 \) are isomorphic projective spaces, we get an isomorphism of the automorphism groups which induces a bijection between the sets of elation subgroups.
5.4 **Notation.** Any \( A \in \Sigma_0 \) corresponds to a (unique) linear transvection subgroup \( T_{p,\psi} \) in \( \text{GL}(W) \). (This means they coincide projectively, when \( \mathbb{P}(W) \) is identified with \( \mathcal{G}_0 \).) The center of \( A \) is \( V, A = \langle p\varphi \rangle \) and the axis is \( C_{\psi_0}(A) \cap \mathcal{G}_0 = \{ \langle w\varphi \rangle \mid 0 \neq w \in \ker \psi \} \). Taking the vector space subspaces spanned in \( V_0 \), the last equation yields that \( \langle (\ker \psi)\varphi \rangle = C_{\psi_0}(A) \).

5.5 **There is a subspace \( \Psi \) of the dual space \( W^\ast \) (with \( \text{Ann}_W(\Psi) = 0 \)) such that the point-hyperplane pairs \( (p, \ker \psi) \) for which there is an \( A \in \Sigma_0 \) corresponding to \( T_{p,\psi} \) are exactly the pairs \( (p, \ker \psi) \) with \( p \) a point of \( W \) and \( \psi \) a point of \( \Psi \) with \( p \subseteq \ker \psi \) (bijective correspondence).

**Proof.** In (5.3) we showed that \( \mathcal{H} \) is a subspace of the dual of \( \mathcal{G}_0 \). Identify \( \mathcal{G}_0 = \mathbb{P}(W) \). Call \( \Psi \) the subspace of \( W^\ast \) arising from \( \mathcal{H} \). Then \( \text{Ann}_W(\Psi) = 0 \) by (5.3)(c) and we have proved the lemma.

5.6 **Let** \( w_1, \ldots, w_n \in W \) **be linearly independent over** \( L \). **Then** \( w_1\varphi, \ldots, w_n\varphi \) **are linearly independent over** \( K \).

**Proof.** There exists \( \psi \in \Psi \) such that \( \langle w_1, \ldots, w_n \rangle \not\subseteq \ker \psi \). (Otherwise \( \langle w_1, \ldots, w_n \rangle \) is mapped to 0 by the whole of \( \Psi \), a contradiction.) Let \( A \in \Sigma_0 \) with axis \( (\ker \psi)\varphi \).

Write \( \langle w_1, \ldots, w_n \rangle = \langle u_1, \ldots, u_n \rangle \) with \( u_1, \ldots, u_{n-1} \in \ker \psi \). Then

\[ X := \langle w_1\varphi, \ldots, w_n\varphi \rangle_K = \langle u_1\varphi, \ldots, u_n\varphi \rangle_K \not\subseteq C_V(A) \]

by (5.4). But \( u_1\varphi, \ldots, u_{n-1}\varphi \) are contained in \( X \cap C_V(A) \) and are linearly independent over \( K \) by induction. This shows \( n - 1 \leq \dim X \cap C_V(A) \leq n - 1 \) and \( \dim X = n \). \( \square \)

5.7 **Proposition.** **Set** \( S_0 \) **be the quasi-simple group** \( T(\Psi, W) \). **There exists an isomorphism** \( \delta : S_0 \to G_0 \) **such that**

\[ (ws)\varphi = (w\varphi)(s\delta), \quad \text{for all} \ w \in W, s \in S_0. \]

Moreover, \( \delta \) induces a bijection between \( \Sigma(\Psi, W) \) and \( \Sigma_0 \).

**Proof.** First, we show that for \( s \in S_0 \), there exists a unique linear mapping \( s\delta : V_0 \to V_0 \) such that \( (ws)\varphi = (w\varphi)(s\delta) \) for all \( w \in W \).

Namely, let \( B = \{ w_i \mid i \in I \} \) be a basis of \( W \) over \( L \). Then \( B\varphi \) is linearly independent over \( K \) by (5.6) and \( \langle B\varphi \rangle_K = V_0 \). Fix \( s \in S_0 \). Define \( (w_i\varphi)(s\delta) := (w_is)\varphi \) and extend linearly to \( V_0 \). We obtain a linear mapping \( s\delta : V_0 \to V_0 \) such that \( (ws)\varphi = (w\varphi)(s\delta) \) for all \( w \in W \). Since \( W\varphi \) contains a basis of \( V_0 \) over \( K \), necessarily \( s\delta \) is unique.

For \( s, t \in S_0 \), we have \( (st)\delta = (s\delta)(t\delta) \) and \( 1\delta = 1 \). Hence \( (s^{-1}\delta) = (s\delta)^{-1} \) and \( s\delta \in \text{GL}(V) \). We obtain an injective homomorphism \( \delta : S_0 \to \text{GL}(V) \).
To prove (5.7), we have to show that $S_0 \delta = G_0$.

First, let $A \in \Sigma_0$. By (5.5) there are $p \in \mathbb{P}(W)$ and $\psi \in \mathbb{P}(\Psi)$ such that $A$ corresponds to $T_p, \psi$. Fix $1 \neq a \in A$. Define $t : \mathbb{P}(W) \to \mathbb{P}(W)$ by $(qt)v = (q, \varphi)a$ for $q$ a point of $\mathbb{P}(W)$.

Then $t \in \text{Aut}(\mathbb{P}(W))$ is an elation with center $p$ and axis ker $\psi$. Moreover $t$ is induced by a linear mapping, also denoted by $t$, see (5.2). By definition $t \delta = a$. We have shown $A \leq T_p, \psi \delta$ and $G_0 \leq S_0 \delta$.

Next, let $p \in \mathbb{P}(W)$ and $\psi \in \mathbb{P}(\Psi)$ with $p \in \text{ker} \psi$. By (5.5) there is $A \in \Sigma_0$ which corresponds to $T_p, \psi$. Let $1 \neq t \in T_p, \psi$. Then $t \delta$ is a transvection in $\text{GL}(V)$ with $[V, t\delta] = [V, A]$ and $C_V(t\delta) = C_V(A)$. Since $A$ induces the full elation subgroup belonging to this pair, there is an $a \in A$ with $a = t \delta$. This yields $T_p, \psi \delta \leq A$ and $S_0 \delta \leq G_0$. Thus $S_0 \delta = G_0$ and the proposition holds.

\[ 5.8 \] $V = V_0 \oplus C_V(G)$

Proof. By (4.13), $V = V_0 + C_V(G)$. Let $0 \neq v \in V_0 \cap C_V(G)$. Then $v = \lambda_1(w_1)\varphi + \cdots + \lambda_n(w_n)\varphi$ for some independent $w_i \in W$ and $\lambda_i \in K$. By the previous result (5.7), there is a $\psi \in \Psi$ mapping $w_1, \ldots, w_{n-1}$ to 0 but not $w_n$. (For $E := \langle w_1, \ldots, w_n \rangle$, we have $\Psi|_E = E^*$.). However, for $A \in \Sigma_0$ corresponding to $t_{w_1, \psi}$, we see $[v, A] \neq 0$, a contradiction.

\section{Identification of the group $G$}

We assume the hypotheses of Theorem 1.4 and use the notation of the previous sections.

Recall that the points of $\mathcal{G}$ are the $[V, T]$ where $T \in \Sigma$. Any pair of non-collinear points spans a projective line which meets $C_V(G)$ in a projective point, see (4.13). Denote by $\mathcal{R}$ the set of all these points.

Fix $A$ in $\Sigma$ with $[V, A] \subseteq V_0$. Choose $w_A \in W$ with $[V, A] = \langle w_A \varphi \rangle$ and set $v_A := w_A \varphi$.

Let $0 \neq c \in C_V(G)$ such that $\langle v_A + c \rangle$ is a point of $\mathcal{G}$. By (4.13) there exists $A_1 \in \Sigma$ with $(v_A + c) = [V, A_1]$ and $C_{\Sigma}(A) = C_{\Sigma}(A_1)$.

\subsection{Assume that $c_1, c_2 \in C_V(G)$ such that $\langle v_A + c_1 \rangle$ and $\langle v_A + c_2 \rangle$ are points of $\mathcal{G}$. Then also $\langle v_A + c_1 + c_2 \rangle$ and $\langle v_A + \lambda^c c_1 \rangle$ are points of $\mathcal{G}$, for $\lambda \in L$.}

Proof. Let $0 \neq \lambda \in L$. There exists $g \in G_0 \simeq T(\Psi, W)$ with $w_A g = \lambda^{-1} w_A$, see (3.4). Hence $\langle v_A + \lambda^c c_1 \rangle = \langle v_A + c_1 \rangle g$ is a point of $\mathcal{G}$, which proves the second claim.

Let $B \in \Sigma$ with $[V, A] \neq [V, B] \subseteq V_0$. Choose $w_B \in W$ with $[V, B] = \langle w_B \varphi \rangle$ and set $v_B := w_B \varphi$. By (4.12) we may choose $C \in \Sigma_0$ such that $[V, A], [V, B]$ and $[V, C]$ span a projective plane in $G_0$. There exists $g \in G_0 \simeq T(\Psi, W)$ with $w_A g = w_B$, see (3.4).
Hence \(\langle v_B + c_2 \rangle \) is a point of \( \mathcal{G} \) \( (i = 1, 2) \). Also \(\langle v_A + c_1 \rangle, \langle v_B + c_2 \rangle \) and \( v_C \) span a projective plane, see (4.11) and (4.13). By (3.4), there exist \( g_1, g_2 \in G_0 \) such that \( w_{A}g_1 = w_{A} + w_C \) and \( w_{A}g_2 = w_B - w_C \). Hence \( \langle v_A + v_C + c_1 \rangle = \langle v_A + c_1 \rangle g_1 \) and \( \langle v_B - v_C + c_2 \rangle = \langle v_A + c_2 \rangle g_2 \) are points of \( \mathcal{G} \). By the Axiom of Pasch, the intersection point \( \langle v_A + v_B + c_1 + c_2 \rangle \) of the lines spanned by \( \langle v_A + c_1 \rangle \) and \( \langle v_B + c_2 \rangle \) and by \( \langle v_A + v_C + c_1 \rangle \) and \( \langle v_B - v_C + c_2 \rangle \) is in \( \mathcal{G} \). Using an element \( g \) of \( G_0 \) with \( (w_{A} + w_B)g = w_A \), see (3.4), we obtain that \( \langle v_A + c_1 + c_2 \rangle \) is a point of \( \mathcal{G} \).

6.2 Notation. Let \( r \) be a point in \( \mathcal{R} \) and choose \( v_r \in r \) such that \( \langle v_A + v_r \rangle \) is a point of \( \mathcal{G} \). Define \( M_r := \{ m \in K \mid \langle v_A + mv_r \rangle = \langle V, T \rangle \text{ for some } T \in \Sigma \} \). Note that \( v_r \in C_V(G) \). Hence (6.1) yields that \( L^a \subseteq M_r \), and \( M_r \) is a vector space over \( L^a \) with the usual scalar multiplication. The points of \( \mathcal{G} \) on \( \langle [V, A], r \rangle \) are indexed by \( M_r \).

Let \( \{ m_i \mid i \in I_r \} \) be a basis of \( M_r \) over \( L^a \). Denote by \( R_r := \oplus_{i \in I_r} Lr_i \) a formal direct sum of 1-dimensional spaces \( Lr_i \) over \( L \). Define \( \hat{W} := W \oplus R \), where \( R := \oplus_{r \in \mathcal{R}} R_r \). We extend \( \varphi : W \to V \) semi-linearly to \( \hat{\varphi} : \hat{W} \to V \) by \( r_i \varphi := m_i v_r \) for \( i \in I_r, \quad r \in \mathcal{R} \).

6.3 The linear mapping \( \hat{\varphi} : \hat{W} \to V \) induces a surjective map from the set of points of \( \hat{W} \), not contained in \( R \), to the set of points of \( \mathcal{G} \).

Proof. Let \( 0 \neq w \in W \) and set

\[
\hat{w} := w + \sum_{r \in \mathcal{R}} \left( \sum_{i \in I_r} \lambda_{i,r} r_i \right),
\]

where only finitely many of the scalars \( \lambda_{i,r} \) are non-zero. First, we show that \( \langle \hat{w} \hat{\varphi} \rangle \) is a point of \( \mathcal{G} \). For \( r \in \mathcal{R} \), set \( m_r := \sum_{i \in I_r} \lambda_{i,r} m_i \in M_r \). Then

\[
\langle \hat{w} \hat{\varphi} \rangle = \langle w \varphi + \sum_{r \in \mathcal{R}} m_r v_r \rangle,
\]

with only finitely many of the \( m_r \) non-zero. Choose \( g \in G_0 \simeq T(\Psi, W) \) with \( w = w_{A}g \). Then \( \langle \hat{w} \hat{\varphi} \rangle = \langle v_A + \sum_{r \in \mathcal{R}} m_r v_r \rangle g \) is a point of \( \mathcal{G} \) by (6.1) with induction.

It remains to shows that the mapping induced by \( \hat{\varphi} \) is surjective. Let \( a \) be a point of \( \mathcal{G} \), without loss \( a \not\in G_0 \). Then there exist \( A_0 \in \Sigma_0, A_1 \in \Sigma \) such that \( a = \langle [V, A_0] \rangle \) and \( [V, A_1] \) are not collinear. Denote by \( r \) the point of \( \mathcal{R} \) on this line. Let \( g \in G_0 \) with \( A_0 = A^0 \). Then \( a = \langle v_A + mv_r \rangle g \) with \( m \in M_r \). Write \( m = \sum_{i \in I_r} \lambda_i m_i \) with \( \lambda_i \in L \). Then \( a = \langle (w_{A}g + \sum_{i \in I_r} \lambda_i r_i) \hat{\varphi} \rangle \) and the mapping induced by \( \hat{\varphi} \) is surjective.

6.4 Notation. Define \( R_0 := R \cap \ker \hat{\varphi} \) and set \( \tilde{W} := \hat{W}/R_0 \) with induced semi-linear mapping \( \tilde{\varphi} \). Note that \( \tilde{W} \simeq W \oplus \tilde{R} \), where \( \tilde{R} := R/R_0 \).

6.5 The semi-linear mapping \( \tilde{\varphi} : \tilde{W} \to V \) is injective and induces a bijective map from the set of points of \( \tilde{W} \), not contained in \( \tilde{R} \), to the set of points of \( \mathcal{G} \).
Proof. For the second statement, by (6.3) we only have to show that the induced mapping is injective on points of $\tilde{W}$, not contained in $\tilde{R}$. Let $0 \neq w_1, w_2 \in W$, $s_1, s_2 \in R$ such that $\langle (w_1 + s_1 + R_0)\tilde{\varphi} \rangle = \langle (w_2 + s_2 + R_0)\tilde{\varphi} \rangle$. Then $\langle w_1\varphi + s_1\tilde{\varphi} \rangle = \langle w_2\varphi + s_2\tilde{\varphi} \rangle$. Hence $\langle w_1\varphi \rangle = \langle w_2\varphi \rangle$. (Namely, otherwise $\langle w_1\varphi \rangle$ and $\langle w_2\varphi \rangle$ are distinct collinear points of $G_0$. We find $T_1, T_2 \in \Sigma_0$ such that $\langle T_1, T_2 \rangle$ is a rank 1 group and $[V, T_1] = \langle w_1\varphi \rangle$, $[V, T_2] = \langle w_2\varphi \rangle$. But $[V, T_1] \subseteq [V, T_2] + C_V(G) \subseteq C_V(T_2)$, a contradiction.) We obtain $\langle w_1 \rangle = \langle w_2 \rangle$, since the weak embedding is injective on points.

Let $0 \neq \lambda \in L$ such that $w_2 = \lambda w_1$. Then $\langle w_1\varphi + s_1\tilde{\varphi} \rangle = \langle w_1\varphi + (\lambda^{-1}s_2)\tilde{\varphi} \rangle$. Since $\langle w_1\varphi \rangle \cap C_V(G) = 0$, we obtain $s_1\tilde{\varphi} = (\lambda^{-1}s_2)\tilde{\varphi}$ and $s_1 - \lambda^{-1}s_2 \in \ker \tilde{\varphi} \cap R = R_0$. Thus $\lambda s_1 + R_0 = s_2 + R_0$ and $\langle w_1 + s_1 + R_0 \rangle = \langle w_2 + s_2 + R_0 \rangle$.

Finally we prove that $\tilde{\varphi}$ is injective. Let $w \in W$, $r \in R$ such that $(w + r + R_0)\tilde{\varphi} = w\varphi + r\tilde{\varphi} = 0$. Then $w\varphi = -r\tilde{\varphi} \in \langle w\varphi \rangle \cap C_V(G) = 0$. Since $\varphi : W \to V$ is injective, we obtain $w = 0$. Thus $r \in R \cap \ker \tilde{\varphi} = R_0$. This yields that $\ker \tilde{\varphi} = 0$.

6.6 Proposition. Set $\tilde{\Psi} := \{\psi \in \tilde{W}^* | \psi|_W \in \Psi, \psi|_R = 0\}$ and $S := T(\tilde{\Psi}, \tilde{W}) \leq GL(\tilde{W})$. Then there exists an isomorphism $\delta : S \to G$ such that $$(w\delta)\tilde{\varphi} = (w\varphi)(s\delta)$$ for all $w \in \tilde{W}$, $s \in S$.

Moreover, $\delta$ induces a bijection between $\Sigma(\tilde{\Psi}, \tilde{W})$ and $\Sigma$.

Proof. First, we show that for $s \in S$, there exists a unique linear mapping $s\delta : V \to V$ such that $(w\delta)\tilde{\varphi} = (w\varphi)(s\delta)$ for all $w \in \tilde{W}$.

Namely, let $B = \{w_i | i \in I\}$ be a basis of $W$ over $L$. Recall that $\tilde{\varphi}$ and $\varphi$ coincide on $W$. Then $B\tilde{\varphi}$ is linearly independent over $K$ by (5.6) and $\langle B\tilde{\varphi} \rangle_K = V_0$. Recall $\tilde{W} = W \oplus \tilde{R}$ and $V = V_0 \oplus C_V(G)$. Fix $s \in S$. Define $(w_i\tilde{\varphi})(s\delta) := (w_i\varphi)\tilde{\varphi}$ and extend linearly to $V_0$. Set $c(s\delta) := c$ for $c \in C_V(G)$ and obtain a linear mapping $s\delta : V \to V$ such that $(w\delta)\tilde{\varphi} = (w\varphi)(s\delta)$ for all $w \in \tilde{W}$. Since $\tilde{W}\tilde{\varphi}$ contains a basis of $V$ over $K$, necessarily $s\delta$ is unique.

As in (5.7) we obtain an injective homomorphism $\delta : S \to GL(V)$. To prove (6.6), we have to show that $\delta G = G$.

Let $A \in \Sigma$. Then there is a point $p$ of $\tilde{W}$, not contained in $\tilde{R}$, such that $\langle p\tilde{\varphi} \rangle = [V, A]$, see (6.5). Choose $A_0 \in \Sigma_0$ with $C_V(A) = C_V(A_0)$, see (5.1). Then there exists a point $\psi$ in $\Psi$ with $\langle (\ker \psi)\varphi \rangle = C_{V_0}(A_0)$. Extend $\psi$ trivially to $\tilde{R}$. We obtain $\langle (\ker \psi)\tilde{\varphi} \rangle = C_{V_0}(A_0) \oplus C_V(G) = C_V(A)$. We show that $A = T_p\psi \delta$.

Fix $1 \neq a \in A$. Define $t : F(\tilde{W}) \to F(\tilde{W})$ by $\langle wt\tilde{\varphi} \rangle = \langle w\varphi a \rangle$ for $w$ a point of $\tilde{W}$, not contained in $\tilde{R}$, and $rt = r$ for $r$ a point in $\tilde{R}$. Then $t \in \Aut(F(\tilde{W}))$ is an elation with center $p$ and axis $\ker \psi$. Moreover $t$ is induced by a linear mapping, also denoted by $t$. By definition $t\delta = a$. We have shown $A \leq T_p\psi \delta$ and $G \leq S\delta$.

For $T_0 \in \Sigma_0$, there is a linear transvection subgroup $T$ in $T(\Psi, W)$ with $T\delta = T_0$, see (5.7). Let $g \in G$ with $A = T_0^g$ and write $g = s\delta$ with $s \in S$. Then $(T^s)\delta T_0^g = A \leq (T_p\psi)\delta$. This yields $T^s = T_p\psi$ and $A = (T_p\psi)\delta$. 

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As \( A \) runs over \( \Sigma \), \( T_{p,\psi} \) runs over \( \Sigma(\tilde{\Psi}, \tilde{W}) \). (Let \( T_{p,\psi} \in \Sigma(\tilde{\Psi}, \tilde{W}) \) with \( p = \langle w + r \rangle \), \( w \in W \), \( r \in \tilde{R} \). Then there is \( A \in \Sigma_0 \) which corresponds to \( T_{(w),\psi} \). Choose \( B \in \Sigma \) such that \([V, B] = \langle p\tilde{\varphi} \rangle \) and \( C_{\Sigma}(A) = C_{\Sigma}(B) \), see (4.13). Then \( B \) corresponds to \( T_{p,\psi} \).) Hence \( G = S\delta \).

This proves Main Theorem (1.4).

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