The geometry of hyperbolic lines in polar spaces

Hans Cuypers
Department of Mathematics
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

April 2, 2009

Abstract

In this paper we consider partial linear spaces induced on the point set of a polar space, but with as lines the hyperbolic lines of this polar space. We give some geometric characterizations of these and related spaces. These results can be viewed as the geometric counterpart of the theory of $k$-transvection groups as developed by Timmesfeld and others.

1 Introduction

Classical polar spaces arise from symplectic and pseudo-quadratic forms on a vector space $V$. In particular, they consist of those points and lines of the corresponding projective space $\mathbb{P}(V)$ on which the form vanishes. Actually, the results of Tits [26], Veldkamp [27], Buekenhout-Shult [4] and Johnson [19] imply that under some weak additional conditions (on non-degeneracy and rank) all these polar spaces can be characterized by the famous ‘one-or-all axiom’. In this paper we provide a characterization of the geometries that have the same point set as a polar space, but whose lines are the so-called hyperbolic lines of the polar space. Our results are inspired by and can be viewed as a geometric counterpart of Timmesfeld’s theory of transvection groups [24]. Before stating our results, we introduce some notation and terminology.

A partial linear space is a pair $(P,L)$ consisting of a set $P$ of points and a set $L$ of subsets of $P$ of size at least 2 called lines, such that two points are in at most one line. A partial linear space is called thick if all lines contain at least 3 points. If two points $p$ and $q$ are on a line, then they are called collinear. This is denoted by $p \sim q$, while the unique line containing them is denoted by $pq$. By $x \perp y$ we denote that $x$ and $y$ are equal or non-collinear. (As we are interested in the geometry of hyperbolic lines, this notation differs from the common notation that $\perp$ denotes collinearity.)
By definition, $x$ is not collinear to itself. The set $x^\perp$, respectively $x^\sim$, denotes the set of all points non-collinear, respectively collinear to $x$. By $(P, \perp)$ and $(P, \sim)$ we denote the graphs with point set $P$ and two points $x, y$ being adjacent if and only if $x \perp y$ or $x \sim y$, respectively. The space $\Pi$ is called connected if $(P, \sim)$ is connected and co-connected if $(P, \perp)$ is connected.

A subspace $X$ of $\Pi$ is a subset of $P$ with the property that any line intersecting it in at least two points is completely contained in $X$. Since the intersection of subspaces is again a subspace, we can, for each subset $Y$ of $P$, define the subspace $\langle Y \rangle$ generated by $Y$ to be the intersection of all subspaces containing $Y$. A plane is a subspace generated by two intersecting lines.

The following result is the main theorem of this paper.

**1.1 Main Theorem.** Let $\Pi = (P, L)$ be a connected and co-connected partial linear space satisfying the following:

(a) all lines in $L$ contain at least 4 points;

(b) every triple of points $x, y$ and $z$ with $x \sim y \sim z$ and $x \perp z$ is contained in a subspace isomorphic to the dual of an affine plane;

(c) if $x$ and $y$ are points with $x^\perp \subseteq y^\perp$ then $x = y$;

(d) if $X$ is a subspace of $\Pi$ isomorphic to a dual affine plane and $p$ a point not collinear with two non-collinear points $q$ and $r$ of $X$, then $p$ is not collinear to all points of $q^\perp \cap X$.

Then $(P, \perp)$ is a non-degenerate polar graph of rank at least 2.

Moreover, if

(e) for all $h \in L$ we have $h = h^\perp \perp$.

Then $(P, L)$ is the geometry of points and hyperbolic lines of the polar graph $(P, \perp)$.

The classical polar spaces over finite fields (but with possibly infinite rank) with thick hyperbolic lines are the symplectic and unitary spaces. In these spaces any two intersecting lines are contained in a subspace forming either a dual affine plane or a classical unital. The following result characterizes the geometry of hyperbolic lines of these spaces by this property. To state the result we need the following definitions. A partial linear space $\Pi = (P, L)$ is called planar, if and only if every pair of intersecting lines is contained in a unique plane. If all lines in $L$ have the same cardinality $q + 1$, then $q$ is called the order of $\Pi$.

**1.2 Theorem.** Let $\Pi = (P, L)$ be a planar and connected partial linear space of finite order $q \geq 3$. Suppose the following hold in $\Pi$:

(a) all lines in $L$ contain at least 4 points;

(b) every triple of points $x, y$ and $z$ with $x \sim y \sim z$ and $x \perp z$ is contained in a subspace isomorphic to the dual of an affine plane;

(c) if $x$ and $y$ are points with $x^\perp \subseteq y^\perp$ then $x = y$;

(d) if $X$ is a subspace of $\Pi$ isomorphic to a dual affine plane and $p$ a point not collinear with two non-collinear points $q$ and $r$ of $X$, then $p$ is not collinear to all points of $q^\perp \cap X$.

Then $(P, \perp)$ is a non-degenerate polar graph of rank at least 2.

Moreover, if

(e) for all $h \in L$ we have $h = h^\perp \perp$.

Then $(P, L)$ is the geometry of points and hyperbolic lines of the polar graph $(P, \perp)$. 

2
(a) all planes are finite and either isomorphic to a dual affine plane or linear;

(b) in a linear plane no 4 lines intersect in 6 points;

(c) for all points \( x \) and \( y \) we have that \( x^\perp \subseteq y^\perp \) implies \( x = y \);

(d) there exists a pair of non-collinear points;

(e) if \( \pi \) is a linear plane and \( x \) a point, then \( x^\perp \cap \pi \neq \emptyset \).

Then \( q \) is a prime power and \( \Pi \) is isomorphic to the geometry of hyperbolic lines in a non-degenerate symplectic or unitary polar space over the field \( \mathbb{F}_q \) or \( \mathbb{F}_{q^2} \), respectively.

Notice that the classical unitals do satisfy (b) of the above theorem, as was observed by O’Nan, see for example [21]. The restriction that \( q \) is at least 3 is necessary, as the Fischer spaces of 3-transposition groups with trivial \( O_2 \) and in which no affine 3-subspaces exist do also satisfy the other conditions, see [11, 10]. In particular, the Fischer space of the sporadic group \( \text{Fi}_{22} \) satisfies the conditions of the theorem for \( q = 2 \).

Combining the above result with [11, 10] we obtain the following result.

1.3 Theorem. Let \( \Pi = (P, L) \) be a planar and connected partial linear space of finite order \( q \geq 2 \). Suppose the following hold in \( \Pi \):

(a) all planes are finite and either isomorphic to a dual affine plane or a classical unital;

(b) there exists a pair of non-collinear points;

(c) for all \( x, y \in P \) we have that \( x^\perp \subseteq y^\perp \) or \( x^\circ \cup \{x\} \subseteq y^\circ \cup \{y\} \) implies \( x = y \);

(d) if \( \pi \) is a linear plane of \( \Pi \) and \( x \) a point, then \( x^\perp \cap \pi \neq \emptyset \).

Then we have one of the following:

(a) \( q \) is a prime power and \( \Pi \) is isomorphic to the geometry of hyperbolic lines in a non-degenerate symplectic or unitary polar space over the field \( \mathbb{F}_q \) or \( \mathbb{F}_{q^2} \), respectively.

(b) \( \Pi \) is a symmetric Fischer space.

(c) \( \Pi \) is an orthogonal Fischer space over \( \mathbb{F}_2 \).

(d) \( \Pi \) is an orthogonal Fischer space over \( \mathbb{F}_3 \) of dimension at most 7.

(e) \( \Pi \) is a sporadic Fischer space related to \( \Omega^+(8, 2) : S_3 \) or \( \text{Fi}_{22} \).
The remainder of this paper is organized as follows. In the next section we discuss various examples appearing in the theorems. Then in Section 3 we start with the proof of Theorem 1.1. Here we prove Reduction Theorem 3.12 in which we show how a geometry satisfying some weaker conditions than as in the hypothesis of Theorem 1.1 reduces to a geometry satisfying the hypothesis of (1.1). The proof of Theorem 1.1 is then finished in Section 4. Section 5 is devoted to a proof of Theorem 1.2, while Theorem 1.3 will be proven in Section 6. In the final section of this paper we briefly discuss some applications of our results in groups theory, graph theory and in the theory of Lie algebras.

2 Hyperbolic lines in classical polar spaces

In this section we discuss some basic results on polar spaces and their hyperbolic lines and describe various examples of geometries occurring in the theorems from the first section.

2.1 Definitions. A partial linear space $S = (P, L)$ is called a polar space if it satisfies the so-called one-or-all axiom:

For each point $p$ and line $l$ of $S$, the point $p$ is collinear with either one or all points of $l$.

The radical $\text{Rad}(S)$ of a polar space $S$ is the set of points which are collinear to all points. A polar space is non-degenerate, if its radical is empty. A non-degenerate polar space in which a point $p$ is either on a line $l$ or just collinear with a unique point of $l$ is also called a generalized quadrangle.

If $S = (P, L)$ is a polar space and $p \in P$, then by $p^\perp$ we denote the set of all points equal or collinear to $p$. A non-degenerate polar space and $p \neq q$ denotes that $q \in p^\perp$. (Notice the difference with the definition of $\perp$ as given in the introduction!) If $X$ is a subset of $P$, then $X^\perp = \bigcap_{x \in X} x^\perp$.

The following three results are well known to experts in the field, but for the sake of completeness we include them in this paper.

2.2 Let $S$ be a thick non-degenerate polar space containing lines. If $p$ and $q$ are points with $p \not\perp q$, then $p^\perp$ and $q^\perp$ generate $S$.

Proof. Suppose $p \not\perp q$ and let $r$ be point not in $p^\perp$. If $q$ and $r$ are collinear, then the line $qr$ meets $p^\perp$ is a point. But then $r$ is in the subspace of $S$ generated by $q$ and $p^\perp$. Thus to prove the lemma is suffices to prove that the induced subgraph of the collinearity graph on the complement of $p^\perp$ is connected. This is done below.

So suppose $r$ and $q$ are not collinear. If they have a common neighbor outside $p^\perp$, we are done. Thus assume that $q^\perp \cap r^\perp \subseteq p^\perp$. Fix two non-collinear points $s, t$
in \( q^\perp \cap r^\perp \). On the line \( qs \) we can find a point \( x \) different from \( r \) and \( t \). This point is collinear with a point \( y \in qt \) different from \( q \) and \( t \). But then \( r, x, y, q \) is a path from \( r \) to \( t \) completely outside \( p^\perp \), proving connectedness. \( \square \)

2.3 If \( S = (\mathcal{P}, \mathcal{L}) \) is a non-degenerate polar space and \( x, y \in \mathcal{P} \) are points with \( x^\perp \subseteq y^\perp \), then \( x = y \).

Proof. Suppose \( x \neq y \) are points with \( x^\perp \subseteq y^\perp \). Let \( u \) be a point not collinear with \( y \). Then \( u \) is collinear to a point \( z \) on the line \( xy \) through \( x \) and \( y \). Let \( v \) be a point not collinear to \( z \). Then \( v \) is collinear to a point \( w \) on \( xy \) different from \( x \) and \( y \) and \( z \). The point \( u \) is collinear to a point on \( vw \), which we may assume to be \( v \). Now \( y^\perp \) meets \( uv \) in a point \( r \). This point is then also collinear with \( x \) and hence with \( z \) and \( w \). But \( u \) is the only point on \( uv \) collinear with \( z \) and \( v \) the only point collinear with \( w \) and we find \( u = r = v \) contradicting \( u \neq v \). \( \square \)

A hyperbolic line of a polar space \( S \) is a set of points of the form \( \{ (x, y)^\perp \} \) where \( x, y \) are two non-collinear points.

2.4 Suppose that \( S \) is a thick non-degenerate polar space. Then for any two points \( z_1, z_2 \) of a hyperbolic line \( H \) we have \( z_1 \not\in z_2 \) and \( H = \{ z_1, z_2 \} \perp \perp \).

Proof. Suppose \( x, y \in \mathcal{P} \) are non-collinear points and \( H = \{ x, y \} \perp \perp \) is the hyperbolic line on \( x \) and \( y \). For any point \( z \in H \) different from \( x \) we have \( x^\perp \cap y^\perp \subseteq z^\perp \). Thus \( x^\perp \cap y^\perp \subseteq x^\perp \cap z^\perp \). We will prove the sets to be equal. Suppose the contrary and let \( u \in x^\perp \cap z^\perp \) but not in \( y^\perp \). We can assume \( u \) to be different from \( x \). Indeed, if \( u = x \), then replace \( u \) by another point on \( xz \). Then the line on \( x \) and \( u \) contains a point \( v \neq u \) in \( y^\perp \). But then \( v \in x^\perp \cap y^\perp \) and hence also in \( z^\perp \). Now every line on \( x \) contains a point \( x' \in y^\perp \) and then also in \( z^\perp \), which implies that \( x^\perp \subseteq z^\perp \), contradicting \( S \) to be non-degenerate; see (2.3). So indeed, \( x^\perp \cap y^\perp = x^\perp \cap z^\perp \). Moreover, we also find \( z \not\in x \). Hence \( H \) also equals \( \{ x, z \} \perp \perp \) is the hyperbolic line on \( x \) and \( z \).

Now suppose \( z_1, z_2 \) are two points in \( H \), then applying the above twice we find \( H = \{ x, z_1 \} \perp \perp = \{ z_1, z_2 \} \perp \perp \) and \( x \not\in z_1 \not\perp z_2 \). \( \square \)

If we denote by \( \mathcal{H} \) the set of all hyperbolic lines of a non-degenerate polar space \( S = (\mathcal{P}, \mathcal{L}) \), then \( (\mathcal{P}, \mathcal{H}) \) is a partial linear space, called the geometry of hyperbolic lines of \( S \).

2.5 Classical polar spaces. Let \( \mathbb{K} \) be a skew field with anti-automorphism \( \sigma \), where \( \sigma^2 = id \) and \( \epsilon \in \mathbb{K} \). Assume that \( W \) is a right vector space over \( \mathbb{K} \). As in Tits [26, (8.2.1), (8.1.2)], a \( (\sigma, \epsilon) \)-hermitian form \( f \) on \( W \) is a mapping \( f : W \times W \to \mathbb{K} \) such that for all \( v, w, u \in W \) and \( \lambda, \mu \in \mathbb{K} \) we have

(a) \( f(v + w, u) = f(v, u) + f(w, u) \) and \( f(u, v + w) = f(u, v) + f(u, w) \);
(b) \( f(v\lambda, w\mu) = \lambda^\sigma f(v, w)\mu; \)

(c) \( f(v, w) = f(w, v)^\sigma. \)

The form \( f \) is called trace-valued if and only if \( f(v, v) \in \{ t + t^\sigma \epsilon \mid t \in K \} \) for all \( v \in V \).

We set

\[
\mathbb{K}_{\sigma, \epsilon} := \{ c - ce^\sigma \mid c \in K \}, \quad \mathbb{K}^{\sigma, \epsilon} := \{ c \in K \mid ce^\sigma = -c \}.
\]

A pseudo-quadratic form (or \((\sigma, \epsilon)\)-quadratic form) \( q \) with associated trace-valued \((\sigma, \epsilon)\)-hermitian form \( f : W \times W \rightarrow K \) is a map \( q : W \rightarrow K/\mathbb{K}_{\sigma, \epsilon}, \) with the property that

\[
q(v\lambda) = \lambda^\sigma q(v)\lambda, \quad \text{and} \quad q(v + w) = q(v) + q(w) + f(v, w) + \mathbb{K}_{\sigma, \epsilon}
\]

for all \( v, w \in W \) and \( \lambda \in K \).

Suppose \( W \) is endowed with a \((\sigma, \epsilon)\)-hermitian form \( f \) or pseudo-quadratic form \( q \) with associated \((\sigma, \epsilon)\)-hermitian form \( f \). For any subset \( U \) of \( W \), we set \( U^\perp = \{ w \in W \mid f(w, u) = 0 \) for all \( u \in U \}. \) The radical of \( f \) is the space \( \text{Rad}(W, f) = W^\perp \). For any \( U, V \subseteq W \) we denote by \( U \perp V \) that \( f(u, v) = 0 \) for all \( u \in U \) and \( v \in V \).

A vector \( w \in W \) with \( f(w, w) = 0 \) or \( q(w) = 0 \), respectively, is called isotropic. (Note that such a vector is also called singular in the literature.) We assume that there exists an isotropic vector, not contained in \( \text{Rad}(W, f) \). A subspace \( U \) of \( W \) is called isotropic if \( U \perp U \). In the case where \( W \) is endowed with a pseudo-quadratic form, we define the isotropic radical of \((W, q)\) to be the subspace \( R = \{ r \in \text{Rad}(W, f) \mid q(r) = 0 + \mathbb{K}_{\sigma, \epsilon} \} \).

We call \( f \) non-degenerate if \( \text{Rad}(W, f) = 0 \) and \( q \) non-degenerate if \( R = \{ 0 \} \). Fix a complement \( W_0 \) to \( R \) in \( W \). Then \( q|_{W_0} \) is non-degenerate. We set

\[
\Phi_q := \{ c \in K \mid \text{there exists } r_c \in \text{Rad}(W_0, f) \text{ with } q(r_c) = c + \mathbb{K}_{\sigma, \epsilon} \}.
\]

Then \( \Phi_q \) is an additive subgroup of \( K \). For each \( c \in \Phi_q \), there is a unique \( r_c \in \text{Rad}(W_0, f) \) with \( q(r_c) = c + \mathbb{K}_{\sigma, \epsilon} \). Further, \( \ell^\sigma \Phi_q d = \Phi_q \) for \( \ell \neq d \in K \) and \( \mathbb{K}_{\sigma, \epsilon} \subseteq \Phi_q \subseteq \mathbb{K}^{\sigma, \epsilon} \). If \( \text{Rad}(W_0, f) = 0 \) (in particular if \( \text{char } K \neq 2 \)), then \( \Phi_q = \mathbb{K}_{\sigma, \epsilon} \).

By \( S = S_f \) (or \( S_q \), respectively) we denote the polar space consisting of the point set \( \mathcal{P} \) of isotropic 1-spaces and line set \( \mathcal{L} \) of all isotropic 2-spaces of \( W \) with respect to \( f \) (or \( q \), respectively). Any polar space arising in this way is called classical.

By the results of Tits [26, (8.1) and (8.2)], any non-degenerate classical polar space with non-empty line set can be obtained (up to isomorphism) as \( S_f \) or \( S_q \) with \( f \) or \( q \) as below.

(I) \( f : W \times W \rightarrow K \) is a quadratic form, i.e. a \((id, 1)\)-quadratic form and \( 1 \in \Phi_q \) or \( \Phi_q = \{ 0 \} \).

(II) \( q : W \rightarrow K \) is a quadratic form, i.e. a \((id, 1)\)-quadratic form and \( 1 \in \Phi_q \) or \( \Phi_q = \{ 0 \} \).
[(III)] \( q : W \to \mathbb{K}/\mathbb{K}_{\sigma,-1} \) is a pseudo-quadratic form with associated (trace-valued) \((\sigma,-1)\)-hermitian form \( f : W \times W \to \mathbb{K} \). Moreover, \( 1 \in \Phi_q \).

### 2.6 Hyperbolic lines in classical polar spaces

Let \( S \) be a non-degenerate classical polar space \( S_f \) or \( S_q \) of some non-degenerate \((\sigma,\epsilon)\)-hermitian form \( f \) or \((\sigma)\)-quadratic form \( q \), as in (2.5)(I)-(III). Assume that the points of \( S \) generate \( W \) and that \( S \) contains lines.

Let \( v_1 \) and \( v_2 \) be two isotropic vectors such that \( f(v_1, v_2) = 1 \). Then \( x = \langle v_1 \rangle \) and \( y = \langle v_2 \rangle \) are two non-collinear points of \( S \). The 2-dimensional space \( H := \langle v_1, v_2 \rangle \) is non-degenerate and \( W = H \perp H^\perp \).

#### 2.7 The hyperbolic line \( \{x, y\}^\perp \) equals \( \{v \in \langle v_1, v_2 \rangle \cap S_f \} \) or \( \{v \in \langle v_1, v_2 \rangle + \mathrm{Rad}(W_0, f) \} \) \( \cap S_q \).

**Proof.** Suppose \( z \in \{x, y\}^\perp \). As, by (2.2), \( S \) is generated by \( x^\perp \) and \( y \), (and similarly \( y^\perp \) and \( x \)) we see that \( \{x, y\}^\perp = H^\perp \). That implies that \( \{x, y\}^\perp \) is contained in \( H + \mathrm{Rad}(W_0, f) \).

Assume that \( S = S_f \) for some \((\sigma, -1)\)-hermitian form \( f \) as in (I). Then the isotropic 1-spaces inside \( H \) are \( \langle v_2 \rangle \) and the spaces \( \langle v_1 \lambda + v_2 \rangle \) where \( \lambda \in \mathbb{K}_{\sigma,-1} = \mathbb{K}^{\sigma,-1} \).

In particular, as \( 1 \in \mathbb{K}_{\sigma,-1} \neq \{0\} \), the hyperbolic lines are thick. Denote by \( H \) the set of hyperbolic lines of \( S \). Then \( \Pi = (\mathcal{P}, H) \) is a partial linear space.

Now suppose \( v_3 \in V \setminus H \) is an isotropic vector with \( f(v_3, v_2) = 1 \) and \( f(v_1, v_3) = 0 \). Then \( v_2 + v_3 \) is also an isotropic vector. Now consider the non-degenerate 2-spaces \( H_1 = \langle v_1, v_2 + v_3 \rangle \) and \( H_2 = \langle v_1 \lambda + v_2, v_3 \mu + v_2 \rangle \), where \( \lambda \neq 0 \) and \( \mu \neq 0, 1 \) are distinct elements in \( \mathbb{K}_{\sigma,-1} \). The intersection \( H_1 \cap H_2 \) is the 1-space \( \langle v_1 \lambda (1 - \mu^{-1}) + v_2 + v_3 \rangle \).

This is a point of \( S \) if and only if \( \lambda \mu^{-1} \in \mathbb{K}_{\sigma,-1} \). It means that inside \( \Pi \) the three points \( \langle v_1 \rangle, \langle v_2 \rangle \) and \( \langle v_3 \rangle \) are in a subspace isomorphic to a dual affine plane, then for all \( \lambda \) and \( \mu \neq 0 \) in \( \mathbb{K}_{\sigma,-1} \) we have \( \lambda \mu^{-1} \in \mathbb{K}_{\sigma,-1} \). But that implies that \( \mathbb{K}_{\sigma,-1} \) is a (commutative) field.

Conversely, if \( \mathbb{K}_{\sigma,-1} \) is a commutative field, then all points of the form \( \langle v_1 \lambda + v_2 \mu + v_3 \mu \rangle \) with \( \lambda, \mu, \rho \in \mathbb{K}_{\sigma,-1} \) different from \( \langle v_1 - v_3 \rangle \) form a subspace of \( \Pi \) isomorphic to a dual affine plane.

Now assume that \( S = S_q \) for some non-degenerate quadratic or pseudo-quadratic form \( q \) with associated \((\sigma,-1)\)-hermitian form \( f \). Suppose \( \langle v_1 \rangle \) and \( \langle v_2 \rangle \) are two isotropic points with \( f(v_1, v_2) = 1 \). Then the points on the hyperbolic line through these two points are, besides \( \langle v_2 \rangle \), the unique isotropic points in each of the subspaces \( \langle v_1 \lambda + v_2 + \mathrm{Rad}(W, f) \rangle \), where \( \lambda \in \Phi_q \). So, hyperbolic lines of \( S \) are thick if and only if the set \( \Phi_q \) contains at least 2 elements. Suppose \( \Phi_q \) contains at least 2 elements. Let \( \mathcal{H} \) be the set of all hyperbolic lines and set \( \Pi := (\mathcal{P}, \mathcal{H}) \). As above we can prove that any triple of points \( p, q, r \) with \( p \sim q \sim r \) and \( p \) are in a subspace of \( \Pi \) isomorphic to a dual affine plane if and only if for all \( \lambda, \mu \in \Phi_q \) we have \( \lambda \mu^{-1} \in \Phi_q \). Since \( 1 \in \Phi_q \), we find \( \Phi_q \) to be a commutative field.
From the above we now easily deduce the following.

2.8 Proposition. Let $\mathcal{S}$ be a non-degenerate classical polar space $S_f$ or $S_q$ defined by $(\sigma, \epsilon)$-hermitian form $f$ as in (2.5)(I) or (pseudo-)quadratic form $q$ with associated $(\sigma, \epsilon)$-hermitian form $f$ as in (2.5)(II) or (III), respectively.

Suppose $\mathcal{S}$ contains lines and let $\Pi$ be the corresponding geometry of hyperbolic lines. Then $\Pi$ is a connected and co-connected thick partial linear space satisfying the conditions (b)-(d) of Theorem 1.1 if and only if

(a) $f$ is as in (2.5)(I) and $K_{\sigma, \epsilon}$ is a commutative field;

(b) $q$ is as in (2.5)(II) or (III) and $\Phi_q$ is a commutative field.

We finish this section with a description of Fischer spaces, some of which appear in the conclusion of Theorem 1.3.

2.9 Fischer spaces. A Fischer space is a partial geometry $\Pi = (P, L)$ of order 2 (i.e., all lines contain $2 + 1 = 3$ points) in which any pair of intersecting lines is contained in a subspace isomorphic to a dual affine plane or an affine plane. Fischer spaces are closely related to groups generated by 3-transpositions. Indeed, if $D$ is a normal set of 3-transpositions in a group $G$, that is a normal set of involutions with the property that $de$ is of order at most 3 for any two elements $d, e \in D$, then the space with $D$ as point set and as lines the triple of elements from $D$ in the subgroups of the form $\langle d, e \rangle$ with $d, e \in D$ with $de$ of order 3 is a Fischer space. On the other and if $\Pi$ is a Fischer space, then to each point $p \in P$ we can attach an involutory automorphism $\tau_p$ of $\Pi$ which fixes $p$ and all points not collinear to $p$ and switches all points $q, r$ with $\{p, q, r\}$ being a line. The set $\{\tau_p \mid p \in P\}$ is a normal set of 3-transpositions in the group $\text{Aut}(\Pi)$.

By exploiting this relationship all connected Fischer spaces containing two non-collinear points have been classified by Cuypers and Hall [11, 10]. We briefly describe the so-called irreducible Fischer spaces (see [11, 10]).

(a) Symmetric Fischer spaces. Let $\Omega$ be a set and $D$ the class of transpositions in $G = \text{Sym}(\Omega)$ the symmetric group on $\Omega$. Then $D$ is a normal set of 3-transpositions in $G$. The corresponding Fischer space is called a symmetric Fischer space.

(b) Symplectic and unitary Fischer spaces. These are the geometries of hyperbolic lines of a non-degenerate symplectic polar space over the field $\mathbb{F}_2$ or a unitary polar space over $\mathbb{F}_4$.

(c) Orthogonal Fischer spaces over $\mathbb{F}_2$. Let $q$ be a non-degenerate quadratic form on a vector space $V$ over $\mathbb{F}_2$ with trivial radical. Then the space whose point set is the set of all non-isotropic 1-spaces and whose lines are the elliptic lines is an orthogonal Fischer space over $\mathbb{F}_2$. 

8
Orthogonal Fischer spaces over $\mathbb{F}_3$. Let $q$ be a non-degenerate quadratic form on a vector space $V$ over $\mathbb{F}_3$. Take the space with as points the non-isotropic 1-spaces and as lines the sets of three points contained in a 2-space tangent to the quadric. Then any connected component of this geometry is called an orthogonal Fischer space over $\mathbb{F}_3$.

Sporadic Fischer spaces. These are the Fischer spaces corresponding to the classes of 3-transpositions in a sporadic Fischer group $\text{Fi}_{22}$, $\text{Fi}_{23}$, $\text{Fi}_{24}$, $\Omega(8, 2) : \text{Sym}_3$ or $\Omega(8, 3) : \text{Sym}_3$.

3 A Reduction Theorem

In this section we start with a proof of Theorem 1.1. Actually we first consider a more general situation as in the following.

3.1 Setting. Let $\Pi = (P, L)$ be a connected partial linear space satisfying the following conditions.

(a) All lines in $L$ contain at least 4 points;

(b) every triple of points $x, y, z \in P$ with $x \sim y \sim z$ and $x \perp z$ is contained in a subspace isomorphic to the dual of an affine plane;

(c) if $x$ and $y$ are points with $x^\perp \subseteq y^\perp$ then $x^\perp = y^\perp$;

(d) there exist non-collinear points;

(e) if $X$ is a subspace of $\Pi$ isomorphic to a dual affine plane and $p$ a point collinear with two non-collinear points $q$ and $r$ of $X$, then $p$ is collinear to all points of $q^\perp \cap X$.

It is the purpose of this section to show that we can reduce the above setting to the setting of Theorem 1.1.

Let $\Pi$ be as in (3.1).

3.2 If $x$ is a point and $l$ a line of $\Pi$, then $x$ is collinear with no point on $l$, with all but one point on $l$ or with all points of $l$.

Proof. Suppose $x$ is collinear to some but not all points of the line $l$. Then we can find a point $y \in l \cap x^\sim$ and a point $z \in l \cap x^\perp$. The points $x, y, z$ form a triangle and hence are are contained in a subspace $X$ of $\Pi$ isomorphic to a dual affine plane. Clearly $l \subseteq X$. Within $X$ we see that the point $x$ is collinear to all but one of the points of $l$. □

The above shows that $\Pi$ is a so-called $\Delta$-space.
3.3 Let \( x \in P \) be a point then \( x^\perp \) and \( x^\perp \setminus \{ x \} \) are subspaces of \( \Pi \).

Proof. Let \( l \) be a line in \( L \) meeting \( x^\perp \) or \( x^\perp \setminus \{ x \} \) in at least two points. Then (3.2) implies that \( l \) is contained in \( x^\perp \setminus \{ x \} \). Thus \( x^\perp \) and \( x^\perp \setminus \{ x \} \) are subspaces of \( \Pi \). \( \square \)

3.4 Every connected subspace of \( \Pi \) has diameter at most 2.

Proof. Suppose \( v, x, y, z \) is a path of length 3 in the collinearity graph of \( \Pi \). By (3.2) the points \( v \) and \( z \) are collinear to all but one of the points on the line \( l = xy \) on \( x \) and \( y \). Since \( l \) contains at least 3 points, there is a point \( u \in l \) collinear to both \( v \) and \( z \). This clearly implies that the diameter of any connected component is at most 2. \( \square \)

In a dual affine plane each point is in a unique maximal coclique of the collinearity graph of the plane. Such a coclique is called a transversal coclique of the plane. Any set \( T \) of non-collinear points in \( \Pi \) which is the transversal coclique of some subspace of \( \Pi \) isomorphic to a dual affine plane will be called a transversal coclique of \( \Pi \).

3.5 Let \( \pi \) be a dual affine plane of \( \Pi \) and \( x \) a point not in \( \pi \). Then \( x^\perp \cap \pi \) is empty, a point, a line, \( \pi \) itself or a transversal coclique of \( \pi \).

Proof. As the intersection of two subspace is again a subspace, (3.3) implies that \( x^\perp \cap \pi \) is a subspace of \( \pi \). But then the lemma follows easily by condition (d) of (1.1). \( \square \)

3.6 Let \( \pi \) be a dual affine plane and \( l \) a line meeting \( \pi \) in a point \( x \). If for some \( y \in l \setminus \{ x \} \) we have \( y^\perp \cap \pi \) being a transversal coclique of \( \pi \), then for all \( y \in l \setminus \{ x \} \) the intersection \( y^\perp \cap \pi \) is a transversal coclique of \( \pi \).

If for some \( y \in l \setminus \{ x \} \) we have \( y^\perp \cap \pi \) being a line of \( \pi \), then for all \( y \in l \setminus \{ x \} \) the intersection \( y^\perp \cap \pi \) is a line of \( \pi \).

Proof. Suppose \( y \) and \( z \) are two points of \( l \setminus \{ x \} \) with \( y^\perp \cap \pi \) being a transversal coclique \( T_y \) of \( \pi \). Let \( m \) be a line of \( \pi \) on \( x \). Then \( \langle m, l \rangle \) is a dual affine plane. So \( m \) contains a point different from \( x \) which is non-collinear to \( z \). As the above is true for every line \( m \) in \( \pi \) on \( x \), we find that \( z^\perp \cap \pi \) contains at least 2 points. But then \( z^\perp \cap \pi \) is either a transversal coclique or a line. Suppose we are in the latter case and \( n = z^\perp \cap \pi \) is a line. Then this line \( n \) meets \( T_y \) in some point, \( u \) say. Inside the dual affine plane \( \pi \) we see that all points of \( T_y \), and hence in particular \( u \), are collinear to \( x \). However, \( u^\perp \) contains \( y \) and \( z \) and by (3.2) all points of \( l \) including \( x \). This contradiction proves the first statement.

The second follows immediately from (3.5) and the above. \( \square \)

3.7 Let \( \pi_1 \) and \( \pi_2 \) be two planes intersecting in two non-collinear points \( x \) and \( y \). Suppose \( z_1 \) is a point in \( \pi_1 \setminus (\pi_1 \cap \pi_2) \) with \( z_1^\perp \cap \pi_2 \) being a line. Then for all points \( z_2 \) in \( \pi_2 \setminus (\pi_1 \cap \pi_2) \) we have that \( z_2^\perp \cap \pi_1 \) contains a line, and \( \pi_1 \cap \pi_2 \) consists of all but one of the points of the transversal coclique on \( \pi_1 \) containing \( x \) and \( y \), where \( i = 1, 2 \).
**Proof.** First of all notice that \( z_1 \) is collinear to both \( x \) and \( y \), for otherwise \( π_2 ⊆ \pi_2^⊥ \). Suppose \( l \) is the line in \( π_2 \) contained in \( \pi_2^⊥ \). This line meets the transversal coclique of \( π_2 \) on \( x \) and \( y \) in a point \( z \) say. Hence \( π_1 ⊆ \pi_2^⊥ \).

Let \( z_2 \) be a point in \( π_2 \) collinear with \( x \) and \( y \). As for any point \( z' \) on \( xz_2 \) different from \( x \) we have that \( z_2 \cap π_2 \) is also a line, see (3.6), we may without loss of generality assume that \( z_2 \) is collinear to \( z_1 \).

The lines \( xz_2 \) and \( yz_2 \) meet \( l \) and hence the planes generated by \( z_1 \), \( z_2 \) and \( x \), respectively, \( y \) are dual affine. Let \( u \) be the unique point on \( z_1z_2 \) in \( y^⊥ \). Denote by \( x' \) respectively \( y' \) the point on \( xz_1 \) respectively \( yz_1 \) being non-collinear with \( z_2 \). Then the line \( x'u \) meets \( xz_2 \) in a point \( x'' \) say. Let \( y'' \) be the intersection point of the lines \( x''z \) and \( yz_2 \). This point \( y'' \) is collinear with \( u \) and the line \( y''u \) meets the line \( yz_1 \) in a point \( v \). If \( v \sim x' \), then the line \( x'v \) meets the line \( x''z \), which is impossible as \( x', v \in z^⊥ \). Thus \( x' \perp v \), and both points are on the transversal coclique of \( (x', x'', z) \) on \( z \). As \( z_2 \sim z \) condition (3.1)(e) implies that \( v \sim z_2 \). Thus \( v \neq y' \) and hence \( x' \sim y' \) showing that \( z_2^⊥ \cap π_1 \) is a line.

Now suppose \( m \) is a line through \( x'' \) meeting \( z_2y \) in a point \( w \) different from \( y'' \). If \( w = y \) then \( m \) meets \( π_1 \). If \( w \neq y \), then let \( w' \) be the point of intersection of \( z_1y \) and \( wu \). Then \( w' \neq y \) and thus \( w' \sim x' \). Furthermore, as \( m \) meets \( z_2z \) which is contained in \( x'^⊥ \), we find that \( (x', x'', w) \) is a dual affine plane. Thus \( m \) meets \( x''w' \) and therefore also \( π_1 \). This shows that \( π_1 \cap π_2 \) consists of all the points of the transversal coclique of \( π_2 \) on \( x \) and \( y \) except for the point \( z \). Hence, for any point \( z_2 \) of \( π_2 \) that is not in \( π_1 \) we have that \( z_2^⊥ \cap π_1 \) contains a line.

Now by reversing the role of \( π_1 \) and \( π_2 \) the lemma follows immediately. \( \Box \)

### 3.8 The connected components of \((P, \perp)\) have diameter at most 2.

**Proof.** Let \( a \perp b \perp c \perp d \) be a path of length 3 in \((P, \perp)\). Then \((a, c, d)\) is a dual affine plane. Since \( b \) is not collinear with the points on the line \( ac \) we can apply the above lemma (3.7). Fix a point \( e \) on the transversal coclique on \( c \) in \((a, c, d)\) different from \( c \) and \( d \). Then \((b, d, e)\) is a dual affine plane meeting \((a, c, d)\) in all points of the transversal coclique in \((a, c, d)\) different from \( c \). Again using (3.7) we see that there is a point \( f \) on the transversal coclique on \( d \) in \((b, d, e)\) which is not in \((a, d, e)\). Then \( f \) and \( a \) are not collinear. Thus we have found a path of length 2 from \( a \) to \( d \). This shows that the diameter of the connected components of \((P, \perp)\) is at most 2. \( \Box \)

### 3.9 Corollary Connected components of \((P, \perp)\) of diameter 2 are connected subspaces of \( \Pi \).

**Proof.** If \( x \) and \( y \) are two collinear points in a connected component of \((P, \perp)\), then by the above (3.8) we can find a point \( z \) non-collinear to both \( x \) and \( y \). But it follows from (3.2) that all points on the line \( xy \) are in \( z^⊥ \). Hence \( xy \) is contained in the connected component of \((P, \perp)\) containing \( x \), \( y \) and \( z \). This shows that this component is a subspace of \( \Pi \).
It remains to prove connectedness in $\Pi$ of a connected component $C$ of $(P, \perp)$. Thus let $x$ and $y$ be two non-collinear points in $C$, and $z$ a point collinear to both of them. We may assume that $z$ is in a connected component of $(P, \perp)$ different from $C$. Let $u$ be a point in the dual affine plane generated by $x$, $y$ and $z$, different from $x$ and $y$, but on the transversal coclique on $x$ and $y$ inside this plane. If $x^\perp \neq u^\perp$, then by condition (3.1)(c) there exists a point $v \in u^\perp$ collinear to $x$. The point $v$ is in $C$ and, by condition (3.1)(d) also collinear to $y$. So, Similarly $y^\perp \neq u^\perp$ implies that there is a point in $C$ coclique to both $x$ and $y$.

If $x^\perp = y^\perp = u^\perp$, then, as $C$ has diameter 2, there is point $v \in C$ collinear to all three points $x$, $y$ and $u$. In any case we have found a point in $C$ coclique to both $x$ and $y$. \hfill \Box

**3.10** Suppose $(P, \perp)$ contains a connected component of diameter 1. Then all components of $(P, \perp)$ have diameter 1.

*Proof.* Suppose $C_1, C_2$ are connected components of $(P, \perp)$ with $C_1$ having diameter 1 and $C_2$ having diameter 2. Suppose $x, y, z$ is a path of length 2 in $C_2$ and $u$ is a point of $C_1$. Fix a point $p$ on $xu$ different from $x$ and $u$ and a point $q$ on $uz$ different from $z$ and $u$. The subspaces of $\Pi$ spanned by $u, y, x$ and by $u, y, z$ are both dual affine planes. Inside these planes we find a points $r \in py$ and $s \in qz$ not collinear with $u$. Clearly $r, s \in C_1$. So $r \perp s$. The subspace $\langle y, s, u \rangle$ of $\Pi$ is also a dual affine plane. The intersection of $r^\perp$ with $\langle y, s, u \rangle$ equals the transversal coclique of $\langle y, s, u \rangle$ containing $u$ and $s$. Applying (3.6) to the line $yr$ and plane $\langle y, q, u \rangle$ implies that $p^\perp \cap \langle y, q, u \rangle$ is also a transversal coclique. Applying (3.6) again but now to the line $up$ and $\langle y, q, u \rangle$ shows that also $x^\perp$ meets $\langle u, y, q \rangle$ in a transversal coclique. However, $y$ is in $x^\perp$, this transversal coclique is the unique transversal coclique containing $y$ and $z$. This contradicts that $x$ and $z$ are collinear. \hfill \Box

On the point set $P$ of $\Pi$ we can define the relation $\equiv$ by

$$x \equiv y \Leftrightarrow x^\perp = y^\perp.$$ 

This relation is clearly an equivalence relation on $P$. The $\equiv$-equivalence class of a point $x$ is denoted by $[x]$. For each line $l \in L$ define $[l]$ to be the set $\{x \mid x \in L\}$ and let $L/\equiv$ be $\{[l] \mid l \in L\}$.

**3.11** Suppose $x, y \in P$. Then $x \perp y$ if and only if $x' \perp y'$ for all $x' \in [x]$ and $y' \in [y]$.

*Proof.* Suppose $x \perp y$, $x' \in [x]$ and $y' \in [y]$. Then $y \in x^\perp = x'^\perp$. So $x' \in y^\perp$ which equals $y'^\perp$. \hfill \Box

The above Lemma 3.11 justifies the following definitions. If $x, y \in P$, then we write $[x] \perp [y]$ if and only if $x \perp y$. Moreover, $[x]^\perp$ denotes the set of all classes $[y]$ with $x \perp y$.

12
3.12 Reduction Theorem. \( \Pi/ \equiv := (P/ \equiv, L/ \equiv) \) is a connected partial linear space satisfying:

(a) all lines contain at least 4 points;

(b) every triangle is contained in a subspace isomorphic to the dual of an affine plane;

(c) if \( X \) and \( Y \) are points with \( X^\perp \subseteq Y^\perp \) then \( X = Y \);

(d) if a point is not collinear with two points of a transversal coclique, then it is not collinear to any point of that transversal coclique.

(e) \( (P/ \equiv, L/ \equiv) \) is linear if and only if \( (P, \perp) \) has only components of diameter 1.

Proof. If \( p \) and \( q \) are collinear points, then \( [p] \neq [q] \), so all elements of \( L/ \equiv \) contain at least 4 points of \( P/ \equiv \). This proves (a) and connectedness of \( \Pi/ \equiv \).

Let \( X \) and \( Y \) be points of \( P/ \equiv \). Then \( X \perp Y \) if and only if \( x \perp y \) for some points \( x \in X \) and \( y \in Y \), if and only if \( x \perp y \) for all points \( x \in X \) and \( y \in Y \). Thus if \( X^\perp \subseteq Y^\perp \), then for each \( x \in X \) and \( y \in Y \) we have \( x^\perp \subseteq y^\perp \), hence \( x \equiv y \) and \( X = Y \). This proves (c).

Now suppose \( X \) and \( Y \) are two points in \( P/ \equiv \) on a line \([l], l \in L\). Then, for any two points \( x \in X \) and \( y \in Y \) we have \( x \sim y \). Fix \( x \in X \) and \( y \in Y \) and let \( z \) be a point on the line through \( x \) and \( y \). We will show that \([z] \in [l]\). First assume that \( x \in l \neq xy \). Then 〈\( l, x, y \)〉 is a dual affine plane, and \( l \) contains a unique point \( z' \in z^\perp \).

Suppose \( u \perp z' \). If \( u \perp x \), then \( u \perp a \) for all \( a \in 〈l, y〉 \). Thus \( u \perp z \). If \( u \sim x \), then there is a point on \( ux \) non-collinear to the unique point in \([y] \cap l\) and therefore also to \( y \) and all points in the transversal coclique of 〈\( l, y \)〉 on \( y \). But then (3.6) implies that \( u^\perp \cap 〈l, y〉 \) is also a transversal coclique, in particular the transversal coclique on \( z' \) which contains \( z \). Thus \( u \perp z \). This shows that \( z^\perp \subseteq x^\perp \) and hence \( z' \in [z] \).

Now suppose \( xy \) does not meet \( l \). Then let \( x' \) be the point on \( l \) which is in \([x] \). Consider the line \( x'y \). By the above there is a point \( z' \in x'y \cap [z] \) and a point \( z'' \in l \cap [z'] \). But then \( z'' \in [z] \cap l \). Hence, in any case \([l] \) contains \([z] \).

This proves that \( (P/ \equiv, L/ \equiv) \) is a partial linear space. Furthermore, it shows that if \( (X, Y, Z) \) is a triangle in \( (P/ \equiv, L/ \equiv) \), then there is a triangle \( (x, y, z) \) in \( \Pi \) with \( (X, Y, Z) = 〈[x], [y], [z]〉 \). No two distinct points in 〈\( x, y, z \)〉 are \( \equiv \)-equivalent. Indeed, suppose \( u \perp v \) for two distinct points in 〈\( x, y, z \)〉 with \( u \equiv v \). Then \( u \equiv v' \) for every point \( u' \) on the transversal of 〈\( x, y, z \)〉 containing \( u \) and \( v \). So, without loss of generality, we can assume \( u \) to be on the line through \( y \) and \( x \) and \( v \) to be on the line through \( y \) and \( z \). Moreover, since \([x] \neq [z] \), we find that \( u \neq x \) and \( v \neq z \). Let \( w \in x^\perp \setminus z^\perp \). Then \( w \not\in u^\perp \) for otherwise it is in \( x^\perp \cap u^\perp \cap v^\perp \), which is contained in \( z^\perp \). So, 〈\( w, y, x \)〉 is a dual affine plane containing a point \( w' \neq u \) which is in \( u^\perp \) and thus also in \( v^\perp \).

So \( w'^\perp \) meets 〈\( x, y, z \)〉 in a transversal and, by (3.6), so does \( w^\perp \), which contradicts
that $w \not\in z^\perp$. Hence $\{u \mid u \in \langle x, y, z \rangle\}$ is a subspace of $(P/ \equiv, L/ \equiv)$ isomorphic to $\langle x, y, z \rangle$ and hence to the dual of an affine plane. Furthermore, (d) follows now easily by the above.

Finally, $(P/ \equiv, L/ \equiv)$ is linear if and only if $x^\perp = y^\perp$ for all $x, y \in P$ with $x \perp y$. But that is equivalent with $(P, \perp)$ having only components of diameter 1. Now (3.10) implies (e). $\square$

4 The geometry of hyperbolic lines

Now we focus on the proof of Theorem 1.1. So, assume that $\Pi = (P, L)$ is a partial linear space satisfying the hypothesis of that theorem.

4.1 Let $x$ be a point in $P$. The subspace $x^\perp \setminus \{x\}$ of $\Pi$ is connected.

Proof. Suppose $y$ and $z$ are two non-collinear points in $x^\perp \setminus \{x\}$. Since $\Pi$ is connected and of diameter at most 2, there is a point $u$ collinear with $y$ and $z$. If $u \perp x$ then we are done, thus suppose that $u$ and $x$ are collinear. Then $\langle x, u, y \rangle$, $\langle x, u, z \rangle$ and $\langle y, u, z \rangle$ are dual affine planes. By condition (1.1)(c) and our assumption that all $\equiv$-classes are of size 1, there is a point $v$ collinear with $y$ but not with $x$. If $v$ and $z$ are collinear, we are done. Assume $z \perp v$. Thus by (3.5) and condition (1.1)(d) $z$ is the only point on the transversal coclique $T$ on $z$ in the dual affine plane $\langle y, u, z \rangle$ not collinear with $v$.

Again by condition (1.1)(c) there is a point $w$ in $x^\perp$ collinear with $z$. By the same reasoning as above we can assume that $y$ is the unique point on $T$ not collinear with $v$. Since $T$ contains at least 3 points, there is a point on $T$ collinear with $u$ and $v$, and by condition (1.1)(d) not to $x$. But that means that we have found a path from $y$ to $z$ in the collinearity graph of $\Pi$ that is completely inside $x^\perp \setminus \{x\}$.

The following lemma will be of great importance in constructing a polar space on the point set $P$.

4.2 Let $x$ and $y$ be two non-collinear points and suppose $z$ is collinear to both $x$ and $y$. Then there is a transversal coclique $T$ on $x$ and $y$ that contains a unique point in $z^\perp$.

Proof. Let $x$ and $y$ be two non-collinear points and $z$ a point collinear to both of them. Then by (3.8) there is a point $u$ in $x^\perp \cap z^\perp$.

First suppose that $u$ and $y$ are collinear. The $u^\perp \cap \langle x, y, z \rangle$ is the line $xz$. Fix a point $w$ in the transversal coclique on $x$ in $\langle x, y, z \rangle$ different from $x$ and $y$. By (3.6) there is a unique line on $u$ in the plane $\langle u, w, y \rangle$ that is contained in $z^\perp$. Suppose $v$ is the unique point on this line in $y^\perp$. Fix a point $u'$ on $uy$, then $u'^\perp \cap \langle x, y, z \rangle$ is a line through $x$ meeting $wz$. In particular, the plane generated by $w$, $z$ and $u'$ is dual affine and contains a point $p \in w^\perp \cap zu'$ different from $u'$. Clearly $p$ is collinear with

14
\(x,\) and hence \(y.\) So \(w^\perp\) meets the dual affine plane generated by \(y,\) \(z\) and \(w\) in the line \(yp.\) Fix the point \(t\) of this line that is not collinear with \(z.\) Then \(t\) is collinear with \(x\) and \(y\) but not with \(z,\) and \(w.\) Thus the dual affine plane \((x, t, y)\) meets \((x, y, z)\) in all points of the transversal coclique on \(x\) except for the point \(w\) and contains by (3.7) the transversal coclique we are looking for. (Notice \(t^\perp\) contains the line \(wz!\)

Now assume that \(u \perp y.\) By the previous lemma there is a point \(v\) in \(x^\perp\) collinear with both \(u\) and \(y.\) If \(v \perp z\) we are in the above situation with \(v\) instead of \(u.\) Thus we can assume that \(v\) and \(z\) are collinear. By the arguments in the preceding paragraph there is a transversal coclique on \(x\) and \(v\) containing a point \(u'\) in \(z^\perp.\) Since \(y\) and \(v\) are collinear but \(y\) and \(x\) are not, we have that \(y\) and \(u'\) are collinear. So we can apply the above with \(u'\) instead of \(u.\)

\[\tag{\textbf{4.3}}\text{Let } x \text{ be a point and } y \text{ and } z \text{ two points in } x^\perp \text{ different from } x. \text{ If } x^\perp \cap y^\perp \subseteq x^\perp \cap z^\perp, \text{ then } x^\perp \cap y^\perp = x^\perp \cap z^\perp.\]

\textit{Proof.} Suppose \(x^\perp \cap y^\perp\) is a proper subset of \(x^\perp \cap z^\perp.\) Then \(y \perp z.\) By (4.1) and (3.4) there is a point \(u \in x^\perp\) collinear to both \(y\) and \(z.\) By the above lemma there is a transversal coclique on \(y\) and \(z\) containing a unique point, \(w\) say, in \(u^\perp.\) Clearly \(w \in x^\perp.\) If \(w = x\) then \(x,\) \(y\) and \(z\) are in a transversal coclique and by condition (d) of Theorem 1.1 we also have \(x^\perp \cap z^\perp \subseteq y^\perp.\) Thus we may assume that \(w \neq x.\) But then \(x^\perp \cap y^\perp = x^\perp \cap y^\perp \cap z^\perp\) is properly contained in \(x^\perp \cap w^\perp.\) Now let \(v\) be a point in \(x^\perp \cap z^\perp\) collinear with \(y.\) Again by the above lemma, there is a transversal coclique on \(z\) and \(v\) containing a unique point \(t\) in \(u^\perp.\) This point \(t\) is in \(x^\perp \cap z^\perp,\) but collinear with \(y.\) Applying (4.2) one more time, we find a transversal coclique on \(u\) and \(t\) containing a point \(s\) in \(y^\perp.\) Then \(s\) is in \(x^\perp,\) but \(s\) is collinear with \(z,\) since \(z \perp t\) but \(z \sim u.\) A contradiction, which proves the lemma.

Now we are in a position to define a set \(\mathcal{L}\) on \(P\) such that \((P, \mathcal{L})\) is a polar space.

Let \(x\) and \(y\) be two distinct points of \(\Pi.\) Then define

\[(xy)^{\perp \perp} = \{z \in P \mid z \perp u \text{ for all } u \in x^\perp \cap y^\perp\}.

Now let \(\mathcal{L}\) be the set of all subsets \((xy)^{\perp \perp}\) of \(P,\) with \(x\) and \(y\) two non-collinear points in \(P.\) The elements of \(\mathcal{L}\) will be called \textit{singular lines}. If \(x \sim y,\) then \((xy)^{\perp \perp}\) is called the \textit{hyperbolic line} through \(x\) and \(y.\)

\[\tag{\textbf{4.4}}\text{Let } x \text{ and } y \text{ be two non-collinear points. Then } \text{for all distinct points } s \text{ and } t \text{ in } (xy)^{\perp \perp} \text{ we have } s \perp t \text{ and } (st)^{\perp \perp} = (xy)^{\perp \perp}.\]

\textit{Proof.} Suppose \(s \in (xy)^{\perp \perp}\) different from \(x.\) Then \(s \perp x, y.\) By (4.3) we have \(x^\perp \cap y^\perp = x^\perp \cap s^\perp.\) Hence \((xy)^{\perp \perp} = (xs)^{\perp \perp}.\) Applying this argument one more time we see \((xs)^{\perp \perp} = (st)^{\perp \perp}.\)

\[\tag{\textbf{4.5}}\text{Proposition. } (P, \mathcal{L}) \text{ is a non-degenerate polar space of rank at least 2.}\]
**Proof.** By the previous lemma and the definition of singular lines we see that for any singular line \( l \) and point \( x \) we have \( x^\perp \cap l \) is empty, consists of a single point or equals \( l \). In view of the Buekenhout-Shult axiom and condition (c) of Theorem 1.1, we only have to show that this intersection is never empty. So fix two points \( y \) and \( z \) in \( l \) collinear with \( x \). Then \( l = (yz)^\perp \perp \) and by (4.2) there is a transversal coclique on \( y \) and \( z \) containing a point in \( x^\perp \). By condition (d) of Theorem 1.1 this transversal coclique is contained in \( l \). Thus \( x^\perp \) meets \( l \) non-trivially.

Since \( \Pi \) contains non-collinear points, we find \((P, \mathcal{L})\) to contain lines. This implies that \((P, \mathcal{L})\) has rank at least 2. \( \square \)

Proposition 4.5 finishes the proof of Theorem 1.1.

## 5 Hyperbolic lines in classical polar spaces of finite order

In this section we prove Theorem 1.2. So, let \( \Pi = (P, \mathcal{L}) \) be a partial linear space as in the hypothesis of (1.2).

**5.1** Let \( \pi \) be a linear plane of \( \Pi \). Any three non-collinear points in \( \pi \) generate \( \pi \).

**Proof.** Let \( x, y \) and \( z \) be three non-collinear points in \( \pi \). Then the lines \( xy \) and \( xz \) are distinct and contained in the planes \( \pi \) and \( \langle x, y, z \rangle \). Since \( \Pi \) is planar, these planes are the same. \( \square \)

**5.2** Let \( T \) be a transversal coclique and \( x \) a point in \( \Pi \). If \( x^\perp \cap T \) contains more than one element, then it contains \( T \).

**Proof.** Let \( \pi \) be a dual affine plane containing \( T \) and suppose \( y \) and \( z \) are two points of \( T \cap x^\perp \). If \( T \) is not contained in \( x^\perp \), then there is a line \( l \) in \( \pi \) completely in \( x^\perp \).

So \( x \) and \( l \) generate a linear plane. Fix two points \( u \) and \( v \) on \( l \) but not in \( T \). Let \( w \) be the intersection point of \( yu \) and \( zv \). Let \( w' \) be the point on \( l \) not collinear with \( w \).

Since \( x \) is not collinear with \( y \), there is a point \( y' \) on \( xu \) not collinear to \( w \). As \( x \perp z \), there is also a point \( z' \) on \( xv \) not collinear to \( w \). By condition (b) of Theorem 1.2, the lines \( w'y' \) and \( w'z' \) are distinct. Hence \( \langle x, l \rangle = \langle w', y', z' \rangle \subseteq w^\perp \), which is clearly a contradiction. Thus \( T \subseteq x^\perp \). \( \square \)

With the help of Theorem 1.1 and Section 3 we obtain the following.

**5.3 Proposition** \((P, \perp)\) is a disjoint union of non-degenerate polar graphs of rank at least 2.

**Proof.** Consider a connected component \( C \) of \((P, \perp)\). By (3.9), this component \( C \) is a connected subspace of \( \Pi \). As a subspace of \( \Pi \) it satisfies the conditions (a)-(d) of Theorem 1.1. So, by Theorem 1.1 we have that the component \( C \) of \((P, \perp)\) is indeed a non-degenerate polar graph. \( \square \)
Assume that \((P, \perp)\) is connected. Let \(L\) be the set of singular lines. We first consider the case where \((P, L)\) is a generalized quadrangle. Since every point in \(P\) is on at least \(q + 1\) singular lines and every singular line contains at least \(q\) points, all singular lines have the same number of points, and every point is on the same number (possibly infinite) of singular lines. Suppose the generalized quadrangle \((P, L)\) has order \((s, t)\), where \(s, t \in \mathbb{N} \cup \{\infty\}\).

**5.4** \(s\) and \(t\) are finite.

**Proof.** First we consider \(t\). Suppose there are more than \(q + 1\) singular lines on a point \(x\). Then \(x^\perp \setminus \{x\}\) contains a linear plane \(\pi\) say. Let \(y\) be a point on a singular line through \(x\), then \(y^\perp \cap \pi\) contains a point \(z\) say and \(y \perp x, z\). Hence \(y\) is contained in the singular line through \(x\) and \(z\). This implies that there are at most \(|\pi|\) singular lines through \(x\) and \(t\) is finite.

Now consider \(s\). Suppose there are more than \(q + 1\) points on a singular line \(l\). Fix distinct points \(x, y\) and \(z\) on \(l\) and let \(m\) be an ordinary line through \(y\) contained in \(x^\perp\). Let \(n\) be a singular line on \(z\) different from \(l\). Then \(n\) does not meet \(m\) and contains a point \(x'\) that is collinear to all points on \(m\). Let \(\pi\) be the linear plane generated by \(x'\) and \(m\). For every point \(p\) of \(\pi \setminus \{y\}\) there is a unique point in \(p^\perp \cap l\). Thus either \(l\) contains at most \(|\pi|\) points and hence \(s\) is finite, or it contains a point \(y'\) different from \(y\) that is collinear with all points of \(\pi\) except for \(y\). Suppose we are in the latter situation. Then let \(k\) be a singular line on \(y'\) different from \(l\). Then again, for every point \(p \in \pi\), there is a unique point in \(k \cap p^\perp\). By assumption \((1.2)(e)\), all points of \(k\) are collinear (in \(\Pi\)) to some point of \(\pi\). So, \(|\pi| \geq |k|\) and \(s\) is finite.

Next we show that the elements of \(L\) are the hyperbolic lines of the generalized quadrangle \((P, L)\).

**5.5** Let \(x\) and \(y\) be two collinear points and \(X\) a set contained in \((xy)^{\perp \perp}\). Then \(\langle X \rangle\) is a linear subspace contained in \((xy)^{\perp \perp}\).

**Proof.** This is a direct consequence of the Delta space property of \(\Pi\), and the fact that in a generalized quadrangle hyperbolic and singular lines meet in at most one point.

**5.6** The elements of \(L\) are the hyperbolic lines of \((P, L)\).

**Proof.** Let \(x\) and \(y\) be two collinear points, then by the above lemma \(xy\) is contained in the hyperbolic line \((xy)^{\perp \perp}\). Now suppose there is a point \(z\) in \((xy)^{\perp \perp}\) which is not in \(xy\). Then the plane \(\pi = \langle x, y, z \rangle\) is contained in the hyperbolic line on \(x\) and \(y\). Let \(p\) and \(p'\) be distinct points in \(x^\perp \cap y^\perp\). If \(l\) is a singular line on \(p\) (or \(p'\)), then for any point \(q \neq p\) on \(l\) but not in \(\pi\), we find, by condition \((d)\) of Theorem 1.2, a
point \( r \in q^\perp \cap \pi \subseteq P^\perp \). So, \( r \in l \). So, all singular lines on \( p \) (and also on \( p' \)) meet \( \pi \), which therefore has cardinality \( t+1 \). By 1.4.2 of [20] we have that \( s \geq t \). Thus on the singular line \((zp')^\perp \perp \) we can find a point \( z' \) not in \( \pi \) but collinear with all points on \( xy \). Consider the plane \( \pi' = \langle x, y, z' \rangle \). This is contained in \( P^\perp \) and contains therefore \( t+1 \) points. Now every point of \( \pi' \) is non collinear with a unique point on \((pz)^\perp \perp \) and every point of \((pz)^\perp \perp \) is non collinear to some point in \( \pi' \). Since \( xy \) is contained in \( P^\perp \) this implies that \((pz)^\perp \perp \) contains at most \( |\pi'| - q \) points. Hence \( s < t \), a contradiction. 

\[ \square \]

5.7 Proposition. Suppose \((P, \perp)\) is connected. If \((P, L)\) has rank 2, then it is a classical symplectic or unitary generalized quadrangle defined over the field \( \mathbb{F}_q \) or \( \mathbb{F}_{q^2} \), respectively.

Proof. By the above \((P, L)\) is a finite generalized quadrangle in which two intersecting hyperbolic lines generate a linear subspace of \( \Pi \) or a dual affine plane. Now Theorem 5.6.8 of [20] proves the proposition. \( \square \)

Now suppose \((P, L)\) is a polar space of rank at least 3 and possibly infinite.

5.8 Proposition. Suppose \((P, \perp)\) is connected. Then \((P, L)\) is a classical symplectic or unitary polar space defined over the field \( \mathbb{F}_q \) or \( \mathbb{F}_{q^2} \), respectively.

Proof. If the rank \( r \) of \((P, L)\) is finite, then by considering the subspaces \( x^\perp \cap y^\perp \) for two points \( x \) and \( y \) with \( x \sim y \), we obtain a polar space of rank \( r-1 \), which by the Delta space property of \( \Pi \) is also a subspace of \( \Pi \). Thus by induction all singular lines contain either \( q+1 \) points, or \( q^2 + 1 \) points. Now it follows easily from the classification of polar spaces of rank at least 3, see [19, 26], that \( \Pi \) is one of the spaces of the conclusion of Theorem 1.2.

If the rank of the polar space is infinite, then it is embeddable in a projective space such that hyperbolic lines are contained in projective lines, see [12, 19, 26]. Assume we have such an embedding of \((P, L)\). By intersecting the polar space with an appropriate finite dimensional projective subspace we can get a non-degenerate polar subspace of finite rank which is also a subspace of \( \Pi \). So again we find that singular lines are of size \( q \) or \( q^2 \) and \( L \) is the set of hyperbolic lines in \((P, L)\) and it follows from the classification of polar spaces of rank at least 3, (see [19, 26]), that \( \Pi \) is isomorphic to one of the spaces in the conclusion of Theorem 1.2. \( \square \)

To finish the proof of Theorem 1.2 we still have to prove that the graph \((P, \perp)\) is connected. But this is done in the next lemma.

5.9 The graph \((P, \perp)\) is connected.

Proof. Suppose \( C \) is a connected component of \((P, \perp)\) and \( x \) a point not in \( C \). By the above results, \( C \) is the polar graph of a classical symplectic or unitary polar space. In particular singular lines of \( C \) contain \( q+1 \) or \( q^2 + 1 \) points.
Fix a singular line $l$ of $C$ and a line $m$ of $\Pi$ on $x$ meeting $l$ in a point. If $n$ is a second line on $x$ meeting $l$, then $\langle m, n \rangle_{\Pi}$ is a dual affine plane meeting $l$ in a transversal. As each dual affine plane is generated by any two of its lines, we find that there are $(|l| - 1)/(q - 1)$ dual affine planes on $m$ meeting $l$ in a transversal. But, that contradicts $q > 2$ and we can conclude that $(P, \perp)$ is connected.

\[\square\]

6 Fischer Spaces

In this section we prove Theorem 1.3. Suppose that $\Pi = (P, L)$ is a partial linear space satisfying the hypothesis of Theorem 1.3. Then if all lines of $L$ contain more than three points, Theorem 1.2 applies and we find $\Pi$ to be a space as in case (a) of the conclusion of Theorem 1.3. So, now assume that all lines of $L$ have exactly 3 points. In that case $\Pi$ is a Fischer space and we can apply the results of Cuypers and Hall [11]. The conditions in (1.3)(c) imply that $\Pi$ is both $\theta$- and $\tau$-reduced in the terminology of [11]. Now we easily deduce from the results in [11] that $\Pi$ is one of the spaces in the conclusion of Theorem [16].

7 Applications

In this final section we briefly discuss three applications of our results. The first is concerned with groups generated by so-called $K$-transvections, $K$ being a field, the second with Lie algebras generated by special elements called extremal, and the final one on local recognition of graphs.

7.1 $K$-Transvection groups. Suppose $K$ is a field. Let $G$ be a group generated by a conjugacy class of Abelian subgroups $\Sigma$ such that for any two elements $A, B \in \Sigma$ we have one of the following:

(a) $\langle A, B \rangle$ is isomorphic to $(P)\text{SL}(2, K)$ and $A$ and $B$ are full unipotent subgroups of $\langle A, B \rangle$;

(b) $[A, B] = 1$.

Then $\Sigma$ is called a class of $K$-transvection (sub)groups of $G$. Groups generated by $K$-transvection groups have been studied by Aschbacher [1], Aschbacher and Hall [2], Fischer [16] and others, and in particular by Timmesfeld [24].

Our main result can be viewed as a geometric version of some of the results Timmesfeld. This connection has been explored in [15].

Let $G$ be a group generated by a conjugacy class of $K$-transvection groups $\Sigma$. We construct a geometry $\Pi(\Sigma)$ with point set $\Sigma$. Lines are defined as follows. Then for each non-commuting pair $A, B \in \Sigma$ the line on $A$ and $B$ is the set of elements $C$ from $\Sigma$ with $C \leq \langle A, B \rangle$. 

19
Let $\Sigma$ be a class of $K$-transvection subgroups of the group $G$. If $K$ contains at least 4 elements, then in [15] we are able to show that $\Pi(\Sigma)$ satisfies the following conditions as given in 3.1, which implies that we can invoke our main results, Theorem 3.12 and Theorem 1.1. This leads to the following result (for details, see [15]).

**7.2 Theorem.** Let $K$ be a field and $G$ be a group generated by a class $\Sigma$ of $K$-transvection subgroups. Suppose that $k$ contains at least 4 elements and that $G$ has no nontrivial nilpotent normal subgroup. If $\Sigma$ contains two commuting elements $A$ and $B$ with $C_\Sigma(A) \neq C_\Sigma(B)$, then $(\Sigma, \perp)$ is the polar graph of a non-degenerate polar space of rank at least 2 and of Moufang type.

The above result together with the work of Steinbach [23] and Timmesfeld [24] provides a full classification of the groups $G$ as in the hypothesis of the above theorem.

When we allow the field $K$ to have order 2 or 3, the situation is different. In that case we obtain a geometry in which 1.1 (d) might be violated. The additional geometries appearing in these cases are the so-called generalized Fischer spaces, see [7, 15]. Combining the results of this paper, [15] and various results on 3-transposition groups leads to a classification of all groups generated by a class $\Sigma$ of $K$-transvection groups, $K$ an arbitrary field, having no nontrivial nilpotent normal subgroups. For details the reader is referred to [15].

**7.3 Extremal elements in Lie algebras.** Another situation in which one can apply our main result Theorem 1.1 is the following. Let $L$ be a Lie algebra over the field $K$. An element $x \in L$ is called extremal if $[x, [x, L]] \subseteq Kx$. Extremal elements in Lie algebras are studied by Cohen et al. [5] and Cohen and Ivanyos [6]. In [5] it has been shown that for any two distinct extremal elements $x, y \in L$ we have one of the following:

(a) $[x, y] = 0$;
(b) $[x, y]$ is an extremal element;
(c) $\langle x, y \rangle$ is isomorphic to $sl(2, K)$.

If case (b) does not occur, then Cohen and Ivanyos [6] show that the geometry with as points the 1-dimensional subspace $Kx$ of $L$ with and as lines the sets of points contained in an $sl(2, K)$-subalgebra generated by two extremal elements is again a geometry satisfying the conditions of the Setting 3.1. Theorem 3.12 and Theorem 1.1 can now be invoked to obtain an important step in the classification of Lie algebras generated by extremal elements.

**7.4 Local recognition of graphs.** Our final application is concerned with local recognition of graphs. Let $\Gamma$ be a graph and $x$ a vertex of $\Gamma$. By $\Gamma_x$ we denote the subgraph of $\Gamma$ induced on the set of neighbors of $x$.  

20
Consider the graph $\Gamma = \Gamma(S)$ with as vertices the set the hyperbolic lines of a non-degenerate classical polar space $S$, two vertices being adjacent if they are perpendicular to each other. Then for each vertex $x$ of $\Gamma$ the induced subgraph $\Gamma_x$ is isomorphic to $\Gamma(T)$, where $T$ is the polar subspace $x^\perp$ of $S$. If $S$ is a non-degenerate symplectic polar space of rank $n$, then $T$ is a symplectic space of rank $n - 1$. It has been proved by Gramlich [18] that in case $n > 5$ the graph $\Gamma$ is the unique connected graph such that for each vertex $x$ the graph $\Gamma_x$ is isomorphic to $\Gamma(T)$. In the proof of that result, Gramlich uses the main result of [8] to identify $\Gamma(S)$. The Theorems 1.1 and 1.2 provide tools to extend Gramlich’s methods to obtain similar local recognition results on the graphs of hyperbolic lines of arbitrary polar spaces.

References


