Port-Hamiltonian modelling of fluid dynamics models with variable cross-section ⋆

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Abstract: Many single- and multi-phase fluid dynamical systems are governed by non-linear evolutionary equations. A key aspect of these systems is that the fluid typically flows across spatially and temporally varying cross-sections. We, first, show that not any choice of state-variables may be apt for obtaining a port-Hamiltonian realization under spatially varying cross-section. We propose a modified choice of the state-variables and then represent fluid dynamical systems in port-Hamiltonian representations. We define these port-Hamiltonian representations under spatial variation in the cross-section with respect to a new proposed state-dependent and extended Stokes- Dirac structure. Finally, we account for temporal variations in the cross-section and obtain a suitable structure that respects key properties, such as, for instance, the property of dissipation inequality.

Keywords: multi-phase, non-linear, evolutionary equations, varying cross-sections, port-Hamiltonian, Stokes-Dirac structure, dissipation inequality.

1. INTRODUCTION

We are interested in developing a modelling, simulation, optimization, and control framework for an automated Managed Pressure Drilling (MPD) system, the set-up of which has been introduced in Naderi Lordejani et al. (2020). In MPD, the drill string and the bottomhole assembly (BHA) are part of a system through, and around, which the flow of single-phase and multi-phase fluids takes place. These flow paths have different geometrical specifications. Consequently, the flow area in the annular section of the well varies along the spatial location in the well. In addition, the flow area changes dynamically due to the axial movements of the integrated drill string and the BHA system. This dynamical change depends on the position of the drill string and the BHA inside the well. Hence, the dynamic model must take into account cross-sectional area variations, which affect the downhole pressure. The variation in cross-section alters the pressure transmission between the top and down-hole parts of the well, because part of the pressure wave is reflected and part of it is transmitted at the point where the cross-section changes. Oscillatory pressures profiles may be induced more frequently compared to the case where there are no cross-sectional changes along the well. The convergence to a steady-state situation may become slower with the inclusion of cross-sectional change. Moreover, the geometrical cross-section across which the fluid flows can vary over time during some drilling operations. For instance, during tripping, the drill string moves at a certain speed, and this results in temporally varying flow cross-section across different parts of the annulus. This motivates the need to develop a framework for (single or multi-phase) fluid dynamical systems admitting flows across spatially and temporally varying cross-sections. This aspect is also relevant and encountered in many other practical applications. For instance, fluid (single- or multi-phase) flows across components with different cross-sections in blood flow through a stenosis as shown in Sankar (2010), and many more.

Port-Hamiltonian (PH) framework has recently emerged as a powerful strategy for robust, and modular, first principles, energy-based modelling, simulation, optimization, and control for multiphysics problems (e.g., finite- and

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infinite-dimensional dynamical systems that are characterized by differential, algebraic or mixture of differential and algebraic equations); see van der Schaft (2020); van der Schaft and Maschke (2020, 2018); Jacob and Zwart (2012); Duindam et al. (2009). PH systems are the backbone for developing passivity- and energy-preserving representations of (interconnected) mathematical models governing physical processes. A PH framework has also helped to integrate finite- and infinite-dimensional components and preserve key system-theoretic properties, such as compositionality (Pasumarthy and van der Schaft (2007)). PH representations and its corresponding structure-preserving discretization and model order reduction have been gaining a lot of momentum recently. Some relevant works include van der Schaft (2020); Kotyczka et al. (2018); Altmann and Schulze (2017); Chaturantabut et al. (2016); Zhou et al. (2015); Trang VU et al. (2012); Martins et al. (2010); Maschke and van der Schaft (2005); Macchelli et al. (2004); van der Schaft and Maschke (2002), and Maschke and van der Schaft (1992). In de Wilde (2015), a PH formulation for single-phase models for flows across constant cross-sections is already given with several different choices for the equation of state. Moreover, in Bansal et al. (2021), a PH formulation has been presented for two-phase models with fluids flowing across constant cross-sections. However, to the best of our knowledge, no works have considered a PH representation of single- and two-phase flow models across spatially and temporally varying cross-sections. In view of the advantages of the PH framework and the current state-of-the-art, we seek to develop a PH-based modelling framework for (distributed-parameter) fluid dynamical models admitting flows across variable cross-sections.

Infinite-dimensional PH systems can be described through a geometric structure known as Stokes-Dirac structure; see e.g., Le Gorrec et al. (2005) and Duindam et al. (2009). This geometric structure helps to gain insight in describing the consistent boundary port-variables. Such a structure has been associated to canonical skew-symmetric differential operators in Le Gorrec et al. (2005). Furthermore, in the same paper, the notion of Stokes-Dirac structures has been extended to skew-symmetric differential operators of any order. Existing works have focused on the state-independent operators and have also considered an extended structure to account for dissipative effects (which may include differential terms), while mostly dealing with quadratic Hamiltonian functionals. However, single-phase or multi-phase flow models possess non-quadratic Hamiltonian functionals. Moreover, in general, most of the research in the field of PH systems has not dealt with the spatial and temporal variations in the parameters of the mathematical model, such as, the cross-sectional area. It is of great interest to investigate whether these aspects require mathematical modifications to the existing theory of PH systems, which is quite rich for linear problems (see Jacob and Zwart (2012)) and promises a lot of interesting research in the scope of non-linear problems with non-quadratic Hamiltonian functionals.

The structure of this paper is as follows. The models governing single- and two-phase flow across a variable cross-section are introduced in Section 2. We, then, consider only spatially varying geometry and present a dissipative Hamiltonian representation, and propose an extended, state-dependent, Stokes-Dirac structure in Section 3 for both mathematical models of interest. Section 4 discusses the corresponding PH structure under both spatial and temporal variations in the cross-sectional area. We finally end the paper with conclusions and potential future works.

2. MODEL INTRODUCTION

A single-phase flow is mathematically modeled by isothermal Euler equations as in LeVeque (2002):

$$\begin{cases} \partial_t(A\rho) + \partial_x(A\rho v) = 0, \\ \partial_t(A\rho v) + \partial_x(A\rho v^2 + AP) = AS + P\partial_x A, \\ \rho = \rho_0 + \dfrac{P}{c_\ell^2}, \\ S = -\rho g \sin\theta - \dfrac{32\mu v}{d^2}, \end{cases} \tag{1}$$

where $t \in \mathbf{R}_{\geq 0}$ and $x \in [a, b]$ are respectively the time and the spatial domain. Here, variables ρ, v, P, A, g, μ, d and θ respectively, refer to density, velocity, pressure, cross-section area, gravitational constant, fluid viscosity, the diameter of the pipe, and, the (constant) pipe inclination.

A two-phase flow across a geometry with variable cross-section can be modelled by the Drift Flux Model as in Aarsnes et al. (2014), which consists of a combined set of differential equations and algebraic closure laws. The differential equations read as follows:

$$\begin{cases} \partial_t(Am_\ell) + \partial_x(Am_\ell v) = 0, \\ \partial_t(Am_g) + \partial_x(Am_g v) = 0, \\ \partial_t(A(m_\ell v + m_g v)) + \partial_x(A(m_\ell + m_g)v^2) + A\partial_x P = \tilde{S}. \end{cases} \tag{2}$$

Here the abbreviations $m_\ell := \alpha_\ell \rho_\ell$ and $m_g := \alpha_g \rho_g$ have been used. The model is completed via the following algebraic closure laws:

$$\begin{cases} \alpha_g + \alpha_\ell = 1, \\ \rho_g = \dfrac{P}{c_g^2}, \\ \rho_l = \rho_0 + \dfrac{P}{c_\ell^2}, \\ \tilde{S} = -A\Big(g(m_g + m_\ell)\sin\theta - \dfrac{32\mu_m v}{d^2}\Big). \end{cases} \tag{3}$$

The variables α_ℓ and α_g respectively denote liquid and gas void fraction. Variables ρ_ℓ and ρ_g refer to the density of the liquid and the gaseous phase, respectively. v is the velocity of the phases (no slip assumed), μ_m is the mixture viscosity, and, c_g and c_ℓ, respectively, are the speed of sound in the gaseous and the liquid phase.

Remark 2.1. Using elimination of variables, the system (1) can be rewritten in terms of two partial differential equations in two unknowns. Similarly, the set of equations (2) and (3) can be expressed in terms of three partial differential equations in three unknowns. We omit this model reformulation in this work and instead refer to Bansal et al. (2021) for further insights on similar models.

Remark 2.2. We only consider smooth spatial area variations in this work. Non-smooth (discontinuous) area variations will be considered in future works.

3. PH MODELING - SPATIAL AREA VARIATIONS

We focus on accounting for only spatial cross-section variations and developing a corresponding PH model representation(s) in this section. We first introduce dissipative Hamiltonian representations i.e., without boundary effects/under the assumption of zero boundary conditions for the mathematical models under consideration. The resulting formal skew-adjoint operator(s) and the resistive matrix are used as a tool to define a candidate geometrical structure, which is later shown to be a non-canonical/extended Stokes-Dirac structure. This geometric object yields a way to describe the boundary port-variables ultimately leading to the port-Hamiltonian representation of the models of interest. These PH representations inherit properties from the Stokes-Dirac structures.

3.1 Dissipative Hamiltonian Representation

Considering the total energy of the system, the Hamiltonian functional, consisting of kinetic, internal and potential energy, given by:

$$\mathcal{H}_s = \int_\Omega A(\rho\frac{v^2}{2} + \rho c_l^2 \ln\rho + c_\ell^2\rho_0 + \rho g x \sin\theta)\mathrm{d}x, \qquad (4)$$

where $\Omega = [a, b]$ refers to the spatial domain.

Remark 3.1. The above functional is similar to the functional used in de Wilde (2015). However, here $\mathcal{H}_s$ is distinct as it accounts for the effects of area (A). Moreover, the equation of state (an algebraic relation relating density and pressure) is also different.

We, first, choose a state coordinate vector comprised of non-conservative variables i.e., ρ and v, and aim to develop a port-Hamiltonian framework for Isothermal Euler equations (governed by the set of equations (1)) across a variable cross-section. This case is used as a test-bed to emphasize that not any choice of state-variables may be apt to obtain a structure with the required properties.

The isothermal Euler equations in (1) can be re-written as follows:

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} \partial_t\rho \\ \partial_t v \end{pmatrix} = \Bigg[\underbrace{\begin{pmatrix} 0 & -\partial_x(\cdot) \\ -\partial_x(\cdot) + \frac{1}{A}\partial_x A & 0 \end{pmatrix}}_{M} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{32\mu}{\rho^2 d^2} \end{pmatrix}\Bigg]\begin{pmatrix} \frac{\delta\mathcal{H}_s}{\delta\rho} \\ \frac{\delta\mathcal{H}_s}{\delta v} \end{pmatrix}. \qquad (5)$$

We omit the derivation as the above formulation can be obtained in a straightforward manner.

We decompose the operator M, introduced in (5), as follows:

$$M := \begin{pmatrix} 0 & -\partial_x(\cdot) \\ -\partial_x(\cdot) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{A}\partial_x A & 0 \end{pmatrix}. \qquad (6)$$

It is trivial to see that the first term in the right-hand side of (6) is formally skew-adjoint. However, the second term in (6) is not formally skew-adjoint under a spatial variation in the cross-sectional area. As a result, the operator M is not formally skew-adjoint and, hence, the representation in (5) is not in a dissipative Hamiltonian form. It is, however, worth stressing that the system written in terms of non-conservative state variables can be formulated in a dissipative Hamiltonian representation with special care; see Definition 4.2.3 in Bansal (2020), which is inspired from the port-Hamiltonian descriptor realization introduced in Mehrmann and Morandin (2019). Alternatively, the use of the standard $\mathcal{L}^2$ inner product could be a hurdle in obtaining the Hamiltonian representation under the choice of the primitive variables. To this end, similar to Matignon and Helie (2013), the choice of weighted $\mathcal{L}^2$ inner product, where the cross-section represents the weight, can be adopted in the pursuit of obtaining dissipative Hamiltonian realizations for the model(s) of interest.

Remark 3.2. It is clear that the second term in the right-hand side of (6) would be the zero matrix (which is trivially formally skew-adjoint) under constant cross-section. Hence, the operator M would be formally skew-adjoint in that case.

The above observations illustrate that the non-conservative state variables may not always have the desired properties attributed to general (port-) Hamiltonian representations. However, the conservative state variables (generally) yield relevant structural properties. We now define the state vector in terms of conservative variables. In addition, we extend the reduced version of (1) (obtained upon elimination of variable P) by an extra equation $\partial_t A = 0$, which means that only spatial variations of A are allowed. Finally, by invoking these proposed modifications, we demonstrate the dissipative Hamiltonian representation for the single-phase flow model while accounting for (smooth) spatial cross-sectional area variations.

We re-write the Hamiltonian functional in terms of the chosen set of state-variables $q = [q_1,\ q_2,\ q_3]^T := [A,\ A\rho,\ A\rho v]^T$. This yields

$$\mathcal{H}_s = \int_\Omega \frac{q_3^2}{2q_2} + q_2 c_\ell^2 \ln(\frac{q_2}{q_1}) + q_1 c_\ell^2 \rho_0 + q_2 g x \sin\theta \mathrm{d}x. \qquad (7)$$

We now present the dissipative Hamiltonian representation [1] for the single-phase model.

Theorem 1. Considering the governing equations (1), the associated dissipative Hamiltonian representation is given by

$$\partial_t q = (\mathcal{J}_s(q) - \mathcal{R}_s(q))\delta_q\mathcal{H}_s(q), \qquad (8)$$

with the Hamiltonian functional (7), where

$$\mathcal{J}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_x(q_2\cdot) \\ 0 & -q_2\partial_x(\cdot) & -q_3\partial_x(\cdot) - \partial_x(q_3\cdot) \end{pmatrix}, \qquad (9)$$

is a formally skew-adjoint operator with respect to the $\mathcal{L}^2$ inner product, and,

$$\mathcal{R}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_1\frac{32\mu}{d^2} \end{pmatrix}, \qquad (10)$$

is symmetric and positive semi-definite matrix.

Proof. We evaluate the variational derivatives with respect to the states. These are

$$\frac{\delta\mathcal{H}_s}{\delta q_1} = -\frac{q_2}{q_1}c_\ell^2 + \rho_0 c_\ell^2, \qquad (11)$$

[1] *The dissipative Hamiltonian representation refers to the model representation abiding by the non-increasing behavior of the Hamiltonian functional along the solutions of the model.*

$$\frac{\delta \mathcal{H}_s}{\delta q_2} = -\frac{q_3^2}{2q_2^2} + c_\ell^2 \ln(\frac{q_2}{q_1}) + c_\ell^2 + gx\sin\theta, \quad (12)$$

$$\frac{\delta \mathcal{H}_s}{\delta q_3} = \frac{q_3}{q_2}. \quad (13)$$

Using these variational derivatives, the claim that (8) is equivalent to a reformulated version of (1) (with additional $\partial_t A = 0$) follows in a manner similar to the derivation discussed in-depth in Theorem 2. Hence, we omit the derivation here.

The positive semi-definiteness and symmetric nature of $\mathcal{R}_s$ follows immediately from the positivity of q_1, μ and d and the structure of the matrix. The formal skew-adjointness of $\mathcal{J}_s$ essentially follows from integration by parts and neglecting the boundary conditions. The operator $\mathcal{J}_s$ has terms similar to the skew-adjoint operator in Bansal et al. (2021). For the sake of brevity, we omit the proof and instead refer to Bansal et al. (2021) for a similar derivation.

Using the properties of $\mathcal{J}_s$ and $\mathcal{R}_s$, the following dissipation inequality holds:

$$\begin{aligned}\frac{d\mathcal{H}_s}{dt} &= \int_\Omega (\delta_q \mathcal{H}_s(q))^T \partial_t q \ \mathrm{d}x \\ &= \int_\Omega (\delta_q \mathcal{H}_s(q))^T (\mathcal{J}_s(q) - \mathcal{R}_s(q))\delta_q \mathcal{H}_s(q) \ \mathrm{d}x \quad (14) \\ &= \int_\Omega (\delta_q \mathcal{H}_s(q))^T (-\mathcal{R}_s(q))\delta_q \mathcal{H}_s(q) \ \mathrm{d}x \leq 0.\end{aligned}$$

This completes the proof.

We now consider a two-phase Drift Flux Model without slip i.e., (2) and (3), and show the corresponding dissipative Hamiltonian representation under the choice of conservative state-variables. Following the choice of candidate Hamiltonian functional in Bansal et al. (2021), we now choose the Hamiltonian functional in the following manner:

$$\mathcal{H}_t = \int_\Omega A(m_g \frac{v^2}{2} + m_\ell \frac{v^2}{2} + m_g c_g^2 \ln \rho_g + m_\ell c_\ell^2 \ln \rho_\ell + (1-\alpha_g)\beta + (m_g + m_\ell) gx \sin\theta)\mathrm{d}x,$$

where $\beta = \rho_0 c_\ell^2$. The above functional can be expressed in terms of the following choice of state-variables $\tilde{q} = [\tilde{q}_1, \ \tilde{q}_2, \ \tilde{q}_3, \ \tilde{q}_4]^T := [A, \ Am_g, \ Am_\ell, \ A(m_g + m_\ell)v]^T$ as follows:

$$\mathcal{H}_t = \int_\Omega \Big(\tilde{q}_1(\frac{\tilde{q}_2}{2\tilde{q}_1}v^2 + \frac{\tilde{q}_3}{2\tilde{q}_1}v^2) + \tilde{q}_2 c_g^2 \ln(\frac{P}{c_g^2}) + \tilde{q}_3 c_\ell^2 \ln(\frac{P+\beta}{c_\ell^2}) + \tilde{q}_1(1-\alpha_g)\beta + (\tilde{q}_2 + \tilde{q}_3)gx\sin\theta\Big)\mathrm{d}x, \quad (15)$$

where v can be expressed in terms of the chosen state-variables by a relation $v = \frac{\tilde{q}_4}{\tilde{q}_2 + \tilde{q}_3}$. Moreover, we use the relations in Aarsnes et al. (2014) to obtain the gas void fraction α_g from the mass variables, which is given by:

$$\alpha_g = -\frac{\tilde{q}_2}{\tilde{q}_1}\frac{c_g^2}{2\beta} - \frac{\tilde{q}_3}{\tilde{q}_1}\frac{c_\ell^2}{2\beta} + \frac{1}{2} + \sqrt{\Delta}, \quad (16)$$

where

$$\Delta = \Big(\Big(\frac{\tilde{q}_2}{\tilde{q}_1}\frac{c_g^2}{2\beta} + \frac{\tilde{q}_3}{\tilde{q}_1}\frac{c_\ell^2}{2\beta} - \frac{1}{2}\Big)^2 + \frac{\tilde{q}_2}{\tilde{q}_1}\frac{c_g^2}{\beta}\Big). \quad (17)$$

The pressure P can be computed in the following way:

$$P = \frac{\tilde{q}_2}{\tilde{q}_1}c_g^2 + \frac{\tilde{q}_3}{\tilde{q}_1}c_\ell^2 - \beta(1-\alpha_g). \quad (18)$$

Next, we discuss the dissipative Hamiltonian representation for the two-phase model. We consider a model reformulation of the governing equations (2) along with the closure equations (3), and, express these as a system composed of three equations in three unknowns (state-variables). Moreover, as before, we consider an additional equation $\partial_t A = 0$. We refer to the resulting model as Σ in the sequel.

Theorem 2. The dissipative Hamiltonian representation of the reformed model Σ in the scope of two-phase flow models takes the following form:

$$\partial_t \tilde{q} = (\mathcal{J}_t(\tilde{q}) - \mathcal{R}_t(\tilde{q}))\delta_{\tilde{q}}\mathcal{H}_t(\tilde{q}), \quad (19)$$

with the Hamiltonian functional (15)*, and where*

$$\mathcal{J}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_x(\tilde{q}_2 \cdot) \\ 0 & 0 & 0 & -\partial_x(\tilde{q}_3 \cdot) \\ 0 & -\tilde{q}_2\partial_x(\cdot) & -\tilde{q}_3\partial_x(\cdot) & -\partial_x(\tilde{q}_4 \cdot) - \tilde{q}_4\partial_x(\cdot) \end{pmatrix}, \quad (20)$$

and,

$$\mathcal{R}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{q}_1\frac{32\mu_m}{d^2} \end{pmatrix}. \quad (21)$$

Proof. We first compute the variational derivatives. The variational derivatives [2] are:

$$\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_2} = -\frac{\tilde{q}_4^2}{2(\tilde{q}_2 + \tilde{q}_3)^2} + c_g^2 \ln(\frac{P}{c_g^2}) + c_g^2 + gx\sin\theta, \quad (22)$$

$$\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_3} = -\frac{\tilde{q}_4^2}{2(\tilde{q}_2 + \tilde{q}_3)^2} + c_\ell^2 \ln(\frac{P+\beta}{c_\ell^2}) + c_\ell^2 + gx\sin\theta, \quad (23)$$

$$\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4} = \frac{\tilde{q}_4}{\tilde{q}_2 + \tilde{q}_3} = v. \quad (24)$$

We now prove the claim equation by equation. The first line holds trivially as we assume that the cross-sectional area only varies spatially. The second line reads

$$\partial_t(Am_g) = -\partial_x(\tilde{q}_2 \frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) = -\partial_x(Am_g v). \quad (25)$$

Similarly, the third line results in

$$\partial_t(Am_\ell) = -\partial_x(\tilde{q}_3 \frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) = -\partial_x(Am_\ell v). \quad (26)$$

Finally, the fourth line yields

$$\partial_t(\tilde{q}_4) = -\tilde{q}_2\partial_x(\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_2}) - \tilde{q}_3\partial_x(\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_3}) - \partial_x(\tilde{q}_4\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) - \tilde{q}_4\partial_x(\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}) - \tilde{q}_1\frac{32\mu}{d^2}\frac{\delta \mathcal{H}_t}{\delta \tilde{q}_4}.$$

Substituting the variational derivatives, we have

$$\begin{aligned}\partial_t(\tilde{q}_4) = &-Am_g\partial_x(-\frac{v^2}{2} + c_g^2\ln(\frac{P}{c_g^2}) + c_g^2) \\ &- Am_\ell\partial_x(-\frac{v^2}{2} + c_\ell^2\ln(\frac{P+\beta}{c_\ell^2}) + c_\ell^2) \\ &- \partial_x(A(m_g + m_\ell)v^2) - A(m_g + m_\ell)v\partial_x v \\ &- A(m_g + m_\ell)g\sin\theta - A\frac{32\mu_m v}{d^2}. \quad (27)\end{aligned}$$

[2] *The variational derivative with respect to q_1 can also be computed. However, we omit its computation as the corresponding elements in the operator $\mathcal{J}_t$ and the matrix $\mathcal{R}_t$ are zero.*

This simplifies to:

$$\partial_t(A(m_g+m_\ell)v) = -\partial_x(A(m_g+m_\ell)v^2) - A\partial_x P - A(m_g+m_\ell)g\sin\theta - A\frac{32\mu_m v}{d^2}, \quad (28)$$

where we have used the identity

$$-Am_g c_g^2\partial_x(\ln\frac{P}{c_g^2}) - Am_\ell c_\ell^2\partial_x(\ln\frac{P+\beta}{c_\ell^2}) =: -A\partial_x P.$$

This completes the proof.

Remark 3.3. We have only used constant pipe-inclination θ in this work. However, it is straightforward to account for spatially varying pipe inclinations; see Bansal et al. (2021).

The formal skew-adjointness of $\mathcal{J}_t$ with respect to the $\mathcal{L}^2$ inner product and the symmetric positive semi-definiteness of $\mathcal{R}_t$ can directly be recognized in (20), (21) by following the line of reasoning as outlined in earlier proofs.

3.2 Stokes-Dirac Structures

The properties of the Stokes-Dirac structure can be exploited in the development of energy-based boundary control laws for distributed port-Hamiltonian systems. We do not recall the formal definition of infinite-dimensional Stokes-Dirac structure and instead refer to Duindam et al. (2009); Le Gorrec et al. (2005), and Bansal et al. (2021).

Next, we propose two variants of extended Stokes-Dirac structures. Firstly, the PH representation for the two-phase model will be defined with respect to the structure in Proposition 3. Secondly, the Stokes-Dirac structure in Proposition 4 will be used to define PH representation for the single-phase model.

We first show the Stokes-Dirac structure representation that will be useful in the scope of the Drift Flux Model without slip. Hereto, we introduce the following notations

$$\mathbf{f}_t = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_R & f_a^B & f_b^B \end{bmatrix}^T,$$

$$\mathbf{e}_t = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_R & e_a^B & e_b^B \end{bmatrix}^T,$$

$$\mathbf{f}_{tr} = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_R]^T,$$

and,

$$\mathbf{e}_{tr} = [e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_R]^T,$$

and define the space of flow variables in the following manner:

$$\mathcal{F}_t = \mathcal{L}^2(\Omega)^5 \times \mathcal{L}^2(\partial\Omega)^2, \quad (29)$$

where $\mathcal{L}^2(\Omega)$ is the space of square-integrable functions and

$$\mathcal{L}^2(\Omega)^p = \mathcal{L}^2(\Omega)\times\mathcal{L}^2(\Omega)\times ...\times\mathcal{L}^2(\Omega) \quad (p-times). \quad (30)$$

The space of effort variables can be analogously defined as follows:

$$\mathcal{E}_t = \mathcal{L}^2(\Omega)^5 \times \mathcal{L}^2(\partial\Omega)^2. \quad (31)$$

Functions in $H^1(\Omega)$ and $H_0^1(\Omega)$ are also considered in the sequel. $H^1(\Omega)$ denotes the Sobolev space of functions that also possess a weak derivative. $H_0^1(\Omega)$ denotes the functions in $H^1(\Omega)$ that have zero boundary values.

The non-degenerated bilinear product on $\mathcal{F}_t\times\mathcal{E}_t$ is defined in the following way:

$$< \mathbf{f}_t \mid \mathbf{e}_t >= \int_\Omega (f_1e_1+f_2e_2+f_3e_3+f_4e_4+ f_Re_R)\mathrm{d}x + f_b^Be_b^B + f_a^Be_a^B. \quad (32)$$

Proposition 3. Let $\mathcal{Z}_t = \mathcal{L}^2(\Omega)^5$. Consider the bond space, a trivial bundle over $\mathcal{Z}_t$: $\mathcal{B}_t = \mathcal{Z}_t\times(\mathcal{F}_t\times\mathcal{E}_t)$, where $\mathcal{F}_t$ and $\mathcal{E}_t$ are as given in (29) and (31). We assume that $\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{q}_4 \in H^1(\Omega)$ and that $\tilde{q}_2+\tilde{q}_3>0$ on Ω. Then, for any $\tilde{q}\in\mathcal{Z}_t$, the linear subset $\mathcal{D}_t\subset\mathcal{F}_t\times\mathcal{E}_t$ given by:

$$\mathcal{D}_t = \Big\{ (\mathbf{f}_t,\mathbf{e}_t)\in\mathcal{F}_t\times\mathcal{E}_t \mid \begin{pmatrix}\tilde{q}_2e_2+\tilde{q}_3e_3\\ \tilde{q}_2e_4\\ e_4\end{pmatrix}\in H^1(\Omega)^3,$$

$$\mathbf{f}_{tr} = \mathcal{J}_{ext}\mathbf{e}_{tr},$$

$$\begin{pmatrix}f_a^B\\ e_a^B\end{pmatrix} = \begin{pmatrix}-\tilde{q}_2 & -\tilde{q}_3 & -\tilde{q}_4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}e_2\\ e_3\\ e_4\end{pmatrix}|_a,$$

$$\begin{pmatrix}f_b^B\\ e_b^B\end{pmatrix} = \begin{pmatrix}\tilde{q}_2 & \tilde{q}_3 & \tilde{q}_4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}e_2\\ e_3\\ e_4\end{pmatrix}|_b \Big\}, \quad (33)$$

with

$$\mathcal{J}_{ext} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\partial_x(\tilde{q}_2\cdot) & 0\\ 0 & 0 & 0 & -\partial_x(\tilde{q}_3\cdot) & 0\\ 0 & -D(\tilde{q}_2\cdot)\&D(\tilde{q}_3\cdot) & & -\partial_x(\tilde{q}_4\cdot)-\tilde{q}_4\partial_x & -I\\ 0 & 0 & 0 & I & 0\end{pmatrix}, \quad (34)$$

is a pointwise Stokes-Dirac structure with respect to the symmetric pairing given by:

$$\ll \begin{bmatrix}\mathbf{f}_t\\ \mathbf{e}_t\end{bmatrix}, \begin{bmatrix}\tilde{\mathbf{f}}_t\\ \tilde{\mathbf{e}}_t\end{bmatrix} \gg = < \mathbf{f}_t \mid \tilde{\mathbf{e}}_t > + < \tilde{\mathbf{f}}_t \mid \mathbf{e}_t >, \quad \begin{bmatrix}\mathbf{f}_t\\ \mathbf{e}_t\end{bmatrix}, \begin{bmatrix}\tilde{\mathbf{f}}_t\\ \tilde{\mathbf{e}}_t\end{bmatrix} \in \mathcal{F}_t\times\mathcal{E}_t, \quad (35)$$

where the pairing $< \cdot \mid \cdot >$ is given in (32). Furthermore, the notation $(\cdot)\mid_a$ (similarly for $(\cdot)\mid_b$) refers to the function value evaluated at the boundary $x=a$ (similarly for $x=b$). Moreover, $D(\tilde{q}_2\cdot)\&D(\tilde{q}_3\cdot)$ is the operator with domain all $e_2,e_3\in\mathcal{L}^2(\Omega)$ such that $\tilde{q}_2e_2+\tilde{q}_3e_3\in H^1(\Omega)$ and the action of this operator is

$$D(\tilde{q}_2e_2)\&D(\tilde{q}_3e_3) = \partial_x(\tilde{q}_2e_2+\tilde{q}_3e_3) - e_2\partial_x\tilde{q}_2 - e_3\partial_x\tilde{q}_3. \quad (36)$$

The above action is an extension of the normal action of the operator, which for all $e_2,e_3\in H^1$ will take the following form:

$$D(\tilde{q}_2e_2)\&D(\tilde{q}_3e_3) = \tilde{q}_2\partial_xe_2+\tilde{q}_3\partial_xe_3.$$

Proof. The proof consists of two parts. The first part comprises of the proof $\mathcal{D}_t\subset\mathcal{D}_t^\perp$. And, the second part comprises of the proof $\mathcal{D}_t^\perp\subset\mathcal{D}_t$. For the first part of the proof, we begin with considering two pairs of flow and effort variables belonging to the Dirac structure i.e., $(\mathbf{f}_t,\mathbf{e}_t)\in\mathcal{D}_t$ and $(\tilde{\mathbf{f}}_t,\tilde{\mathbf{e}}_t)\in\mathcal{D}_t$. Using the earlier introduced notations, the pairing (35) gives:

$$\int_\Omega (f_1\tilde{e}_1+f_2\tilde{e}_2+f_3\tilde{e}_3+f_4\tilde{e}_4+f_R\tilde{e}_R)\mathrm{d}x + \int_\Omega (\tilde{f}_1e_1+\tilde{f}_2e_2+\tilde{f}_3e_3+\tilde{f}_4e_4+\tilde{f}_Re_R)\mathrm{d}x + f_a^B\tilde{e}_a^B + f_b^B\tilde{e}_b^B + \tilde{f}_a^Be_a^B + \tilde{f}_b^Be_b^B. \quad (37)$$

Using (33), (34) and (36) in (37), we obtain:

$$\int_\Omega \Big((-\partial_x \tilde{q}_2 e_4)\tilde{e}_2 + (-\partial_x \tilde{q}_3 e_4)\tilde{e}_3 + \Big(-\partial_x(\tilde{q}_2 e_2 + \tilde{q}_3 e_3) + e_2\partial_x \tilde{q}_2 + e_3 \partial_x \tilde{q}_3 \Big)\tilde{e}_4 - \partial_x(\tilde{q}_4 e_4)\tilde{e}_4 - \tilde{q}_4(\partial_x e_4)\tilde{e}_4 - e_R\tilde{e}_4 + e_4\tilde{e}_R \Big)\mathrm{d}x + \int_\Omega \Big((-\partial_x \tilde{q}_2 \tilde{e}_4) e_2 + (-\partial_x \tilde{q}_3 \tilde{e}_4) e_3 + \Big(-\partial_x(\tilde{q}_2 \tilde{e}_2 + \tilde{q}_3 \tilde{e}_3) + \tilde{e}_2\partial_x \tilde{q}_2 + \tilde{e}_3\partial_x \tilde{q}_3 \Big) e_4 - \partial_x(\tilde{q}_4 \tilde{e}_4) e_4 - \tilde{q}_4(\partial_x \tilde{e}_4) e_4 - \tilde{e}_R e_4 + \tilde{e}_4 e_R \Big)\mathrm{d}x + f_a^B \tilde{e}_a^B + f_b^B \tilde{e}_b^B + \tilde{f}_a^B e_a^B + \tilde{f}_b^B e_b^B. \quad (38)$$

Performing integration by parts on few terms in the above equation, (38) can be easily shown to be zero and hence, $\mathcal{D}_t \subset \mathcal{D}_t^\perp$. This concludes the first part of the proof, which carries the symbolism of power-conserving structure.

We now prove the converse part: $\mathcal{D}_t^\perp \subset \mathcal{D}_t$. The proof consists of several small but repeated steps. Hence, we summarize the key steps that are followed in each step while proving the converse part. We take $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{D}_t^\perp$ i.e., $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{F}_t \times \mathcal{E}_t$ such that $\ll \begin{bmatrix}\mathbf{f}_t\\ \mathbf{e}_t\end{bmatrix}, \begin{bmatrix}\tilde{\mathbf{f}}_t\\ \tilde{\mathbf{e}}_t\end{bmatrix} \gg = 0$ for all $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$. Furthermore, we make a certain choice on the effort variables (which can be freely chosen as per the definition of the Stokes-Dirac structure) in each step. We also exploit the fundamental lemma of calculus of variations to obtain several identities. Each step (and the associated choices) is described below:

Step 1: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_2, e_3, e_4, e_R = 0$ and $e_1(a) = e_1(b) = 0$. Following the procedure leads to:

$$\int_\Omega \tilde{f}_1 e_1 \mathrm{d}x = 0 \quad \forall e_1 \in \mathcal{L}^2(\Omega). \quad (39)$$

Thus $\tilde{f}_1 = 0$.

Step 2: We now consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_3, e_4, e_R = 0$ and $e_2(a) = e_2(b) = 0$. Plugging the flow-effort relations (33) in (37) under the aforementioned considerations gives:

$$\int_\Omega \Big((-\tilde{q}_2 \partial_x e_2)\tilde{e}_4 + \tilde{f}_2 e_2 \Big)\mathrm{d}x + b.c. = 0 \quad \forall e_2 \in H^1(\Omega). \quad (40)$$

The fundamental lemma of calculus of variations gives

$$\tilde{q}_2 \tilde{e}_4 \in H^1(\Omega) \quad \text{and} \quad \tilde{f}_2 = -\partial_x(\tilde{q}_2 \tilde{e}_4). \quad (41)$$

Considering $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with e_1, e_2, e_4 and $e_R = 0$ along with $e_3(a) = e_3(b) = 0$ gives by a similar argument that

$$\tilde{q}_3 \tilde{e}_4 \in H^1(\Omega) \quad \text{and} \quad \tilde{f}_3 = -\partial_x(\tilde{q}_3 \tilde{e}_4). \quad (42)$$

Now using $\tilde{q}_2 \tilde{e}_4 \in H^1(\Omega)$ and $\tilde{q}_3 \tilde{e}_4 \in H^1(\Omega)$, we have that $(\tilde{q}_2 + \tilde{q}_3)\tilde{e}_4 \in H^1(\Omega)$. Furthermore, using $\tilde{q}_2, \tilde{q}_3 \in H^1(\Omega)$ along with $\tilde{q}_2 + \tilde{q}_3 > 0$ on Ω, we have that $\tilde{e}_4 \in H^1(\Omega)$.

Step 3: Now choosing $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_R = 0$ and $e_4 \in H_0^1(\Omega)$ gives

$$\int_\Omega \Big(-\partial_x(\tilde{q}_2 e_4)\tilde{e}_2 - \partial_x(\tilde{q}_3 e_4)\tilde{e}_3 - \partial_x(\tilde{q}_4 e_4)\tilde{e}_4 - (\tilde{q}_4 \partial_x e_4)\tilde{e}_4 + e_4 \tilde{e}_R + \tilde{f}_4 e_4 \Big)\mathrm{d}x = 0. \quad (43)$$

We rewrite (43) as

$$\int_\Omega \Big(-e_4 \tilde{e}_2 \partial_x \tilde{q}_2 - e_4 \tilde{e}_3 \partial_x \tilde{q}_3 - (\tilde{q}_2 \tilde{e}_2 + \tilde{q}_3 \tilde{e}_3)\partial_x e_4 - \partial_x(\tilde{q}_4 e_4)\tilde{e}_4 - (\tilde{q}_4 \partial_x e_4)\tilde{e}_4 + e_4 \tilde{e}_R + \tilde{f}_4 e_4 \Big)\mathrm{d}x = 0 \quad \forall e_4 \in H_0^1(\Omega). \quad (44)$$

As a result of the fundamental lemma of calculus of variations, we obtain the following identity:

$$\tilde{f}_4 = -\partial_x(\tilde{q}_2 \tilde{e}_2 + \tilde{q}_3 \tilde{e}_3) + \tilde{e}_2 \partial_x \tilde{q}_2 + \tilde{e}_3 \partial_x \tilde{q}_3 - \partial_x(\tilde{q}_4 \tilde{e}_4) - \tilde{q}_4 \partial_x \tilde{e}_4 - \tilde{e}_R. \quad (45)$$

Step 4: Let us consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_4 = 0$. The identity that follows under these considerations is:

$$-e_R \tilde{e}_4 + \tilde{f}_R e_R = 0 \implies \tilde{f}_R = \tilde{e}_4. \quad (46)$$

Step 5: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_3, e_4, e_R = 0$ and $e_2(a) = 0$ and $e_2(b) \neq 0$. Following the procedure similar to earlier steps yields:

$$-\tilde{q}_2 e_2 \tilde{e}_4 \mid_b + \tilde{e}_b^B(\tilde{q}_2 e_2) \mid_b = 0. \quad (47)$$

The identity that follows is:

$$\tilde{e}_b^B = \tilde{e}_4 \mid_b . \quad (48)$$

Step 6: We now let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_3, e_4, e_R = 0$ and $e_2(b) = 0$ and $e_2(a) \neq 0$. We follow the procedure similar to Step 5 and obtain the following identity:

$$\tilde{e}_a^B = \tilde{e}_4 \mid_a . \quad (49)$$

Step 7: Consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_R = 0$ and $e_4(a) = 0$ and $e_4(b) \neq 0$. Following the outlined procedure and using $\tilde{e}_b^B$ from (48), we have:

$$\tilde{f}_b^B e_4 \mid_b + \tilde{e}_4 \mid_b (\tilde{q}_4 e_4) \mid_b - \tilde{q}_2 e_4 \tilde{e}_2 \mid_b - \tilde{q}_3 e_4 \tilde{e}_3 \mid_b - \tilde{q}_4 e_4 \tilde{e}_4 \mid_b - \tilde{q}_4 e_4 \tilde{e}_4 \mid_b = 0. \quad (50)$$

This results in the following identity:

$$\tilde{f}_b^B = \Big(\tilde{q}_2 \tilde{e}_2 + \tilde{q}_3 \tilde{e}_3\Big) \mid_b + \Big(\tilde{q}_4 \tilde{e}_4\Big) \mid_b . \quad (51)$$

Step 8: We now consider $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_1, e_2, e_3, e_R = 0$ and $e_4(b) = 0$ and $e_4(a) \neq 0$. Under these considerations, we follow the procedure similar to Step 7 and also use $\tilde{e}_a^B$ from (49) to obtain the following identity:

$$\tilde{f}_a^B = -\Big(q_2 \tilde{e}_2 + q_3 \tilde{e}_3\Big) \mid_a - \Big(q_4 \tilde{e}_4\Big) \mid_a . \quad (52)$$

Thus, we have shown that $\mathcal{D}_t^\perp \subset \mathcal{D}_t$ and, hence, $\mathcal{D}_t$ is a Stokes-Dirac structure. This completes the proof.

The obtained boundary flow and effort variables can be interpreted physically. Ignoring the sign associated to the boundary port variables; the effort variables, e_a^B and e_b^B, can, respectively, be interpreted as common flow velocity at the left and right end of the spatial domain. The flow variable at the left boundary, f_a^B, and the corresponding variable at the right boundary, f_b^B, have physical dimensions of energy per unit mass per unit cross-sectional area. The boundary port-variables can normally be split into inputs, outputs, and homogeneous boundary conditions, see, e.g., Jacob and Zwart (2012). Mathematically, we can express the inputs u, outputs y, and homogeneous boundary conditions as follows:

$$\begin{bmatrix} u \\ y \\ 0 \end{bmatrix} = W \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}, \quad (53)$$

where W is a mapping matrix, and f_∂ and e_∂ denote the vector constituting boundary port flow and effort variables, respectively. Here, the first two lines indicate the inputs and outputs, and the third line indicates the homogeneous boundary condition(s). Splitting the boundary port variables as shown above aids in relating the boundary ports of a PH formulation to the boundary conditions of a system governed by partial differential equations.

We now discuss the Stokes-Dirac structure representation that will be useful in the scope of the single-phase model.

We introduce $\mathbf{f}_s = \begin{bmatrix} f_1 & f_2 & f_3 & f_R & f_a^B & f_b^B \end{bmatrix}^T$ and $\mathbf{e}_s = \begin{bmatrix} e_1 & e_2 & e_3 & e_R & e_a^B & e_b^B \end{bmatrix}^T$. Using these notations, we define the space of flow variables as follows:

$$\mathcal{F}_s = \mathcal{L}^2(\Omega)^4 \times \mathcal{L}^2(\partial\Omega)^2. \tag{54}$$

Similarly, the space of effort variables is defined as follows:

$$\mathcal{E}_s = \mathcal{L}^2(\Omega)^4 \times \mathcal{L}^2(\partial\Omega)^2. \tag{55}$$

The non-degenerated bilinear product on $\mathcal{F}_s \times \mathcal{E}_s$ is defined as:

$$< \mathbf{f}_s \mid \mathbf{e}_s >= \int_\Omega (f_1 e_1 + f_2 e_2 + f_3 e_3 + f_R e_R) \mathrm{d}x + f_b^B e_b^B + f_a^B e_a^B. \tag{56}$$

Proposition 4. Let $\mathcal{Z}_s = \mathcal{L}^2(\Omega)^4$. Consider the bond space, a trivial bundle over $\mathcal{Z}_s$: $\mathcal{B}_s = \mathcal{Z}_s \times (\mathcal{F}_s \times \mathcal{E}_s)$, where $\mathcal{F}_s$ and $\mathcal{E}_s$ are as given in (54) and (55). Additionally, we consider $q_1, q_2, q_3 \in H^1(\Omega)$ and q_2 (or $A\rho$) is invertible. Then, for any $q \in \mathcal{Z}_s$, the linear subset $\mathcal{D}_s \subset \mathcal{F}_s \times \mathcal{E}_s$ defined as:

$$\mathcal{D}_s = \Bigg\{ (\mathbf{f}_s, \mathbf{e}_s) \in \mathcal{F}_s \times \mathcal{E}_s \mid \begin{pmatrix} q_2 e_2 \\ e_3 \end{pmatrix} \in H^1(\Omega)^2,$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_x(q_2 \cdot) & 0 \\ 0 & -D(q_2 \cdot) & -\partial_x(q_3 \cdot) - q_3 \partial_x & -I \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_R \end{pmatrix},$$

$$\begin{pmatrix} f_{a,s}^B \\ e_{a,s}^B \end{pmatrix} = \begin{pmatrix} -q_2 & -q_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} |_a,$$

$$\begin{pmatrix} f_{b,s}^B \\ e_{b,s}^B \end{pmatrix} = \begin{pmatrix} q_2 & q_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} |_b \Bigg\}, \tag{57}$$

is a pointwise Stokes-Dirac structure with respect to the symmetric pairing given by:

$$\ll \begin{bmatrix} \mathbf{f}_s \\ \mathbf{e}_s \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{f}}_s \\ \tilde{\mathbf{e}}_s \end{bmatrix} \gg = < \mathbf{f}_s \mid \tilde{\mathbf{e}}_s > + < \tilde{\mathbf{f}}_s \mid \mathbf{e}_s >,$$

$$\begin{bmatrix} \mathbf{f}_s \\ \mathbf{e}_s \end{bmatrix}, \begin{bmatrix} \tilde{\mathbf{f}}_s \\ \tilde{\mathbf{e}}_s \end{bmatrix} \in \mathcal{F}_s \times \mathcal{E}_s, \tag{58}$$

where the pairing $< \cdot \mid \cdot >$ is given in (56). Moreover, $D(q_2 \cdot)$ is the operator with domain all $e_2 \in \mathcal{L}^2(\Omega)$ such that $q_2 e_2 \in H^1(\Omega)$ and the extended action of the operator is

$$D(q_2 e_2) = \partial_x(q_2 e_2) - e_2 \partial_x q_2.$$

Remark 3.4. We do not prove the Proposition 4. The proof of the corresponding (extended) Stokes-Dirac structure can be easily demonstrated by following the similar lines of reasoning as in the proof of the Proposition 3.

The boundary port variables can be interpreted physically in the scope of a single-phase flow model as well. Ignoring the sign associated to the boundary port variables; the effort variables, $e_{a,s}^B$ and $e_{b,s}^B$, can, respectively, be interpreted as the flow velocity (of the phase under consideration) at the left and right end of the spatial domain. The flow variables, $f_{a,s}^B$ and $f_{b,s}^B$, can, respectively, be interpreted as the quantities having physical dimensions of energy per unit mass per unit cross-sectional area at the left and right end of the spatial domain.

Remark 3.5. We have associated a particular choice of boundary-port variables with a Stokes-Dirac structure. In principle, it would be ideal to derive an admissible set of boundary conditions in a parametrized way similar to Le Gorrec et al. (2005), where a parametrization was derived for a canonical skew-symmetric differential operator. However, the structures derived in our current paper are non-canonical and eventually have a non-invertible matrix (Q as per the notation in Le Gorrec et al. (2005)), that hinders the elegant parametrization for the class of systems under discussion. An elegant parametrization of boundary port-variables will be considered in future works for the class of non-linear PH systems with non-quadratic Hamiltonian functionals.

4. MODELING TEMPORAL VARIATIONS

We now briefly consider temporal variations in the geometrical cross-section i.e., $\partial_t A \neq 0$. We consider that the evolution of the area is described as:

$$\partial_t A = r_1(t, z), \tag{59}$$

where r_1 is a function, which say is known a-priori or can be determined via some control law.

Allowing for temporal variations in area can be viewed as the structure (with state-variable z), which contains additional terms (relative to the structure with only spatial variations) that can be perceived as state and time-dependent control inputs. See the following theorem.

Theorem 5. Consider the system Σ_t [3] governed by the combination of (2), (3) and (59). Then, it can be formulated in the dissipative Hamiltonian representation of the following form:

$$\partial_t z = (\mathcal{J}(z) - \mathcal{R}(z))\delta_z \mathcal{H}(z) - r(t, z). \tag{60}$$

Remark 4.1. In the scope of two-phase models, $z = \tilde{q}$, $\mathcal{J} = \mathcal{J}_t$, $\mathcal{R} = \mathcal{R}_t$ and $\mathcal{H} = \mathcal{H}_t$. Equivalently, the structure holds in the scope of single-phase models with corresponding state-variables, interconnection (formal skew-adjoint) operator, resistive matrix and the Hamiltonian functional.

The above structure can be viewed as a special case of the representation in Mehrmann and Morandin (2019). If we ignore the boundary ports in the pHDAE definition of Mehrmann and Morandin (2019) and use slightly different notations for the sake of consistency in this paper, then we obtain:

$$\mathcal{E}\dot{z} = (\mathcal{J}(z) - \mathcal{R}(z))s - r(t, z), \tag{61}$$

where $r(t, z)$ is of the form: $[r_1(t, z) \quad 0 \quad 0 \quad 0]^T$. The reasoning behind the choice of this form is apparent from the comment in the footnote.

We consider the mapping $\mathcal{E} = I$, $\partial_z \mathcal{H} = s$ and $\partial_t \mathcal{H} = s^T r$ and follow Mehrmann and Morandin (2019) to obtain the dissipation inequality.

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= (\partial_z \mathcal{H})^T \dot{z} + \partial_t \mathcal{H} \\ &= s^T \Big((\mathcal{J} - \mathcal{R})s - r \Big) + s^T r \\ &= -s^T \mathcal{R} s \leq 0. \end{aligned} \tag{62}$$

The structural representation as in (61) has already been shown to be a Dirac structure in Mehrmann and Morandin (2019). Hence, we refer the reader to Mehrmann and Morandin (2019) for further details.

[3] *The first equation of the composed system Σ_t is (59). The rest of the equations in the composed system are the mass and the momentum conservation laws.*

Remark 4.2. *Structure* (60) *or* (61) *has been presented in rather general sense. It is worth mentioning that a desirable structure is realizable for the models governing the single-phase and two-phase fluid flow across variable geometrical cross-section by using specific choice of state-variables and the associated interconnection operator and the dissipation matrix.*

5. CONCLUSION

The main results of this paper are the dissipative Hamiltonian realizations and definition of (extended) state-dependent Stokes-Dirac structure consequently leading to port-Hamiltonian representations for both single-phase and two-phase models governing fluid flow across spatially and temporally varying cross-section. Future works will deal with developing structure preserving numerical schemes for the obtained representations.

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