Port-Hamiltonian Formulation of Two-phase Flow Models[⋆]

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ARTICLE INFO

Keywords:
Two-Fluid Model
Drift Flux Model
non-quadratic Hamiltonian
skew-adjoint
Stokes-Dirac structures
Port-Hamiltonian

ABSTRACT

Two-phase flows are frequently modelled and simulated using the Two-Fluid Model (TFM) and the Drift Flux Model (DFM). This paper proposes Stokes-Dirac structures with respect to which port-Hamiltonian representations for such two-phase flow models can be obtained. We introduce a non-quadratic candidate Hamiltonian function and present dissipative Hamiltonian representations for both models. We then use the structure of the corresponding formally skew-adjoint operator to derive a Stokes-Dirac structure in the scope of the two variants of multi-phase flow models. Moreover, we present a numerical counter example to demonstrate that only a special form of the DFM (without slip between the phases) can be cast in a port-Hamiltonian representation and that the DFM with the Zuber-Findlay slip conditions is not an energy consistent model for two-phase flow.

1. Introduction

In this paper, we develop a port-Hamiltonian (pH) formulation for modelling multi-phase flow dynamics in pipes. Multi-phase flows are important in a large range of industrial applications, such as within the oil and gas industry, chemical and process industry (including heat-pumping systems) as well as the safety analysis of nuclear power plants [1, 2, 3]. Within the oil and gas industry, such models are used for virtual drilling scenario testing [1, 2]. The multi-phase aspect is particularly relevant in these applications in case of gas influx occurring from a reservoir.

A pH model formulation is known to provide a modular framework for multi-physics and interconnected systems [4]. The pH structure allows for non-zero energy flow through the boundary and guarantees power preservation [5]. Moreover, structure-preserving methods for discretization and the model order reduction of infinite-dimensional pH systems can preserve certain original system-theoretic properties such as stability and passivity [6, 7]. Additionally, the pH framework supports the development of control strategies [8].

In the literature, the infinite-dimensional pH structure has been exploited in several domains of science and engineering. For instance, some well-known fluid dynamical systems such as the shallow water equations [7], reactive Navier Stokes equations [9], and reaction diffusion processes [10] have already been formulated in the pH formalism. Such a representation is also prevalent in the fields of structural dynamics [8] and fluid-structure interaction [11].

Multi-phase flows are mathematically governed by conservation laws. Several conservation laws have previously been converted to pH representations [12, 13]. Some work on Hamiltonian modeling for multi-phase hydrodynamics has been done in [14]. However, (dissipative) Hamiltonian representations do not exist for the Two-Fluid Model (TFM) and the Drift Flux Model (DFM) [15]. Moreover, until now, to the best of our knowledge, pH modeling for fluid dynamics only encompasses single-phase models [16].

Matrix/operator theory for *linear* distributed parameter port-Hamiltonian systems on one-dimensional domains is owed to some pioneering works [17, 18]. The central theme of the current paper is to extend and propose modifications to the existing theory for *non-linear* distributed parameter systems. We exploit the existing theory in the scope of linear systems and arrive at new results from an operator theoretic viewpoint, including further generalizations in the scope of non-linear distributed parameter port-Hamiltonian systems.

The main contributions of this paper are as follows: *(i)* (dissipative) Hamiltonian representations of the TFM and the DFM, and *(ii)* proposition of state-dependent Stokes Dirac structures for both the TFM and the DFM along with the proof of the corresponding representation obtained in the scope of the TFM.

The paper is organized as follows. In Section 2, we introduce the two mathematical models governing 1-D multi-phase flow dynamics and mention under which conditions these are equivalent. The (dissipative) Hamiltonian representations of these models are presented in Section 3. Then, the corresponding geometric properties are discussed and proved in Section 4. This section also includes a non-unique

[⋆]The first author has been funded by the Shell NWO/ FOM PhD Programme in Computational Sciences for Energy Research. The second author is supported by the DFG Collaborative Research Center 1029, project A02. The third author has carried out this research in the HYDRA project, which has received funding from European Union's Horizon 2020 research and innovation program under grant agreement No 675731.

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parametrization of the boundary port-variables. Afterwards, Section 5 deals with the reasons behind formulating the DFM without slip between the two phases in a pH representation instead of a general DFM with the Zuber-Findlay slip conditions. Finally, Section 6 closes with conclusions.

Notations: We first introduce few notations that are used in the sequel. $\mathcal{L}^2(\Omega)$ is the space of square-integrable functions over the spatial domain Ω, and

$$\mathcal{L}^2(\Omega)^p = \mathcal{L}^2(\Omega)\times\mathcal{L}^2(\Omega)\times...\times\mathcal{L}^2(\Omega) \quad \text{(p-times)}. \tag{1}$$

$H^1(\Omega)$ denotes the Sobolev space of functions that also possess a weak derivative. Furthermore, $H_0^1(\Omega)$ denotes the functions in $H^1(\Omega)$ that have zero boundary values. $H^1(\Omega)^p$ is defined in a manner anologous to $\mathcal{L}^2(\Omega)^p$. And, $\mathbb{R}$ denotes the space of real numbers.

2. Multi-phase flow models

In this section, we present two sets of nonlinear conservation laws, namely, the TFM and the DFM.

2.1. Two-Fluid Model (TFM)

The TFM is a set of Partial Differential Equations (PDEs) and algebraic closure relations. The PDEs expressing mass and momentum conservation for each phase are as follows:

$$\partial_t\left(\alpha_g\rho_g\right)+\partial_x\left(\alpha_g\rho_g v_g\right)=0, \tag{2a}$$
$$\partial_t\left(\alpha_\ell\rho_\ell\right)+\partial_x\left(\alpha_\ell\rho_\ell v_\ell\right)=0, \tag{2b}$$
$$\partial_t\left(\alpha_g\rho_g v_g\right)+\partial_x\left(\alpha_g\rho_g v_g^2\right)=-\partial_x\left(\alpha_g p\right)+M_g, \tag{2c}$$
$$\partial_t\left(\alpha_\ell\rho_\ell v_\ell\right)+\partial_x\left(\alpha_\ell\rho_\ell v_\ell^2\right)=-\partial_x\left(\alpha_\ell p\right)+M_\ell, \tag{2d}$$

where $t\in\mathbb{R}_{\geq 0}$ and $x\in[a,b]$ are, respectively, the temporal and spatial variables (a and b refer to the location of the left and the right boundary of the one-dimensional spatial domain). The model contains seven unknown variables, namely, liquid and gas void fraction, α_ℓ and α_g, liquid and gas phase velocity, v_ℓ and v_g, liquid and gas phase density, ρ_ℓ and ρ_g, and the common pressure p.

To complete the model, we use one set of the most widely applied closure laws as in [15]:

$$\alpha_g+\alpha_\ell=1, \tag{3a}$$
$$M_g+M_\ell=0, \tag{3b}$$
$$M_g=p\partial_x\alpha_g+M_{ig}, \tag{3c}$$
$$M_{ig}=b_g^M(v_\ell-v_g),\quad \text{with } b_g^M\geq 0, \tag{3d}$$
$$\rho_g=\frac{p}{c_g^2}, \tag{3e}$$
$$\rho_\ell=\rho_{\ell 0}+\frac{p-p_{\ell 0}}{c_\ell^2}, \tag{3f}$$

where (3a) expresses that any pipe segment is occupied by the combination of gas and liquid. The terms M_g and M_ℓ with the constant b_g^M in (3b)–(3d) account for the force interaction between the phases. Finally, (3e)–(3f) define the equation of state of each phase with the reference density and pressure as $\rho_{\ell 0}$ and $p_{\ell 0}$, and c_g and c_ℓ are the constant speeds of sound in the gas and liquid phase, respectively.

Remark 2.1. *We do not consider gravitational and frictional effects in the above TFM description for the sake of simplicity. However, in principle, the TFM can be formulated with the additional terms accounting for these effects [15].*

The TFM, governed by the set of equations (2) and (3), can be written in terms of only four physical variables. We introduce the following shorthand notations: $m_g := \alpha_g\rho_g$ and $m_\ell := \alpha_\ell\rho_\ell$.

Assumption 1. *The gas void fraction, the liquid void fraction, the liquid and the gaseous phase densities along with $\beta=\rho_{\ell 0}c_\ell^2-p_{\ell 0}$ are positive.*

Lemma 2.2. *By considering m_g, m_ℓ, v_g and v_ℓ as state variables, the system of equations (2) and (3) can be re-written in the following form:*

$$\partial_t m_g+\partial_x\left(m_g v_g\right)=0, \tag{4a}$$
$$\partial_t m_\ell+\partial_x\left(m_\ell v_\ell\right)=0, \tag{4b}$$
$$\partial_t v_g+\partial_x\left(\frac{v_g^2}{2}\right)=-c_g^2\partial_x\left(\ln p\right)+\frac{b_g^M}{m_g}v_r, \tag{4c}$$
$$\partial_t v_\ell+\partial_x\left(\frac{v_\ell^2}{2}\right)=-c_\ell^2\partial_x\left(\ln\left(p+\beta\right)\right)-\frac{b_g^M}{m_\ell}v_r, \tag{4d}$$

where $v_r=\left(v_\ell-v_g\right)$, and

$$p\left(m_g,m_\ell,\alpha_g\right)=m_g c_g^2+m_\ell c_\ell^2-\beta\left(1-\alpha_g\right), \tag{5}$$

$$\alpha_g\left(m_g,m_\ell\right)=-m_g\frac{c_g^2}{2\beta}-m_\ell\frac{c_\ell^2}{2\beta}+\frac{1}{2}+\sqrt{\left(m_g\frac{c_g^2}{2\beta}+m_\ell\frac{c_\ell^2}{2\beta}-\frac{1}{2}\right)^2+m_g\frac{c_g^2}{\beta}}. \tag{6}$$

We refer the reader to [2] for the detailed proof of the expression for $\alpha_g\left(m_g,m_\ell\right)$. In summary, the set of equations (4) is equivalent to (2) and (3).

2.2. Drift Flux Model (DFM)

The DFM can be obtained from the TFM via a slip relation of the form

$$v_g-v_\ell=\Phi\left(m_g,m_\ell,v_g\right), \tag{7}$$

where m_g and m_ℓ are as introduced above. Since the slip relation (7) determines the coupling between the velocities of the two phases, only one momentum equation is required contrary to the two momentum equations in the TFM (2). Several models of the form (7) exist depending on the choice of the function Φ [15]. In the simplest case, without slip, $\Phi := 0$. Another case is the Zuber-Findlay relation [15]:

$$\Phi := \frac{(K-1)v_g+S}{K\alpha_\ell}\rightarrow v_g=K(\alpha_g v_g+\alpha_\ell v_\ell)+S, \tag{8}$$

where K and S are flow-regime dependent parameters, which are assumed to be constant in this study.

Using the abbreviations $I_g := m_g v_g$ and $I_\ell := m_\ell v_\ell$, the governing equations for the DFM are:

$$\partial_t m_g + \partial_x I_g = 0, \tag{9a}$$

$$\partial_t m_\ell + \partial_x I_\ell = 0, \tag{9b}$$

$$\partial_t\big(I_g + I_\ell\big) + \partial_x\big(I_g v_g + I_\ell v_\ell\big) = -\partial_x p + Q_g + Q_v \tag{9c}$$

completed with closure equations (3a), (3e), (3f), (7) and gravitational effects Q_g and frictional effects Q_v defined by [19]:

$$Q_g = -g\left(m_g + m_\ell\right)\sin\theta, \tag{10a}$$

$$Q_v = -\frac{32\mu_m(\alpha_g v_g + \alpha_\ell v_\ell)}{d^2}, \tag{10b}$$

with gravitational constant g, space-dependent pipe inclination $\theta(x)$, mixture viscosity $\mu_m > 0$, and pipe diameter d.

Remark 2.3. *Similar to Lemma 2.2, the governing equations (9) associated with $v := v_g = v_\ell$ (DFM without slip), the closure equations (3a), (3e), (3f) and (10), upon elimination of auxiliary variables, can be rewritten as a system of PDEs with as many unknowns as equations. We omit the discussion for the sake of brevity.*

The TFM can be adapted to behave exactly like the DFM if the term M_{ig} in (3d) is replaced with the term stated in the following theorem. For the proof, we refer to [15].

Theorem 2.4. *Under zero gravitational and frictional effects, the DFM (9) together with (3a) and (7) is equivalent to the TFM (2) with (3a)–(3c), and*

$$M_{ig} = -\alpha_g\alpha_\ell\frac{\rho_g - \zeta\rho_\ell}{m_g + \zeta m_\ell}\partial_x p - \frac{m_g m_\ell}{m_g + \zeta m_\ell}\Big(v_\ell\partial_x v_\ell - \zeta v_g\partial_x v_g + \mu_g\partial_x(m_g v_g) + \mu_\ell\partial_x(m_\ell v_\ell)\Big), \tag{11}$$

with $\mu_g := \frac{\partial\Phi}{\partial m_g}$, $\mu_\ell := \frac{\partial\Phi}{\partial m_\ell}$, $\zeta := 1 - \frac{\partial\Phi}{\partial v_g}$.

Remark 2.5. *The equivalence of the DFM and the TFM can also be shown in the presence of gravitational and frictional effects; see [15], for further details.*

The model equivalence, stated above, will play a crucial role in drawing a conclusion about the behavior of the Hamiltonian along the solutions of the DFM by using the theoretical analysis conducted for the TFM (see Section 5).

3. Dissipative Hamiltonian Formulations

Port-Hamiltonian (pH) systems have several useful properties for system analysis and control. Basic properties of pH systems include passivity and compositionality. The pH model formulation is appealing as it helps to characterize the energy exchange across the boundaries and thus accounts for the interaction between the system and the environment. Such a framework generalizes the classical Hamiltonian framework by the definition of boundary ports. We restrict ourselves to pH systems (with state-variable z) of the form

$$\begin{aligned}\partial_t z &= \Big(\mathcal{J}(z) - \mathcal{R}(z)\Big)\delta_z\mathcal{H}(z),\\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} &= \mathcal{M}\begin{pmatrix} (\delta_z\mathcal{H}(z))(b) \\ (\delta_z\mathcal{H}(z))(a) \end{pmatrix},\end{aligned} \tag{12}$$

where $\mathcal{H}$ is the Hamiltonian functional, $\delta_z\mathcal{H}(z)$ its variational derivative, and $\mathcal{M}$ is a state-dependent bijective mapping. Furthermore, for every z, $\mathcal{J}(z)$ is formally skew-adjoint with respect to the L^2 inner product, i.e., for e_1, e_2 sufficiently smooth and zero at the boundary there holds

$$\int_\Omega e_1^T(\mathcal{J}(z))e_2\mathrm{d}x + \int_\Omega e_2^T(\mathcal{J}(z))e_1\mathrm{d}x = 0, \tag{13}$$

where Ω refers to the spatial domain, and $\mathcal{R}$ is formally self-adjoint with respect to the L^2 inner product and positive semi-definite. Finally, f_∂, e_∂ are the boundary ports.

The dissipation inequality, which expresses that energy cannot be generated within the system, is a property which directly follows from the definition of a pH system. In particular, ignoring the boundary conditions,

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \int_\Omega (\delta_z\mathcal{H}(z))^T\partial_t z\,\mathrm{d}x\\ &= \int_\Omega (\delta_z\mathcal{H}(z))^T(\mathcal{J}(z) - \mathcal{R}(z))\delta_z\mathcal{H}(z)\,\mathrm{d}x\\ &= \int_\Omega (\delta_z\mathcal{H}(z))^T(-\mathcal{R}(z))\delta_z\mathcal{H}(z)\,\mathrm{d}x \le 0.\end{aligned} \tag{14}$$

Thus, $\mathcal{R}$ is the dissipative component of the system. In the presence of boundary conditions, the behavior of the Hamiltonian along the solutions of the mathematical model is governed by the following balance equation:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_\Omega (\delta_z\mathcal{H}(z))^T(-\mathcal{R}(z))\delta_z\mathcal{H}(z)\,\mathrm{d}x + \text{b.t.}, \tag{15}$$

where b.t. denotes the boundary terms. Normally f_∂, e_∂ are chosen such that the boundary terms equal $\langle f_\partial, e_\partial\rangle$ w.r.t. some inner product. In our case, this will be the standard inner product on Euclidean space. Associated to the operators $\mathcal{J}$ and $\mathcal{R}$, we can identify an underlying geometric object called Stokes-Dirac structure. This is crucial as the pH systems can be defined with respect to these infinite-dimensional Stokes-Dirac structures [20]. Often, this structure is only associated to $\mathcal{J}$. This geometric object yields a manner to describe the boundary port variables, i.e., f_∂ and e_∂, see (12).

We first introduce (dissipative) Hamiltonian representations, i.e., without boundary effects for the mathematical models under consideration. The resulting formally skew-adjoint operators and formally self-adjoint operators are used as a tool to derive a non-canonical Stokes-Dirac structure, and hence the boundary port variables.

In the models discussed in Section 2, the Hamiltonian is dependent on the kinetic, gravitational potential and internal energy. To derive the internal energy of the system, consider the following remark.

Remark 3.1. *The internal energy $u_i, i \in \{\ell, g\}$, can be interpreted as the energy causing the expansion of the i-th compressed phase or compression of the i-th expanded phase. In order to derive this energy component, the Gibbs relation [21] under barotropic and isentropic flow considerations for an infinitesimal part of the phase is used, i.e.,*

$$\rho_i^2 \mathrm{d}u_i = p\mathrm{d}\rho_i, \qquad i \in \{\ell, g\}.$$

Using (3e)–(3f) and integrating the above equation leads to

$$u_\ell = -\frac{p_{\ell 0}}{\rho_\ell} + c_\ell^2 \ln \rho_\ell + \frac{\rho_{\ell 0} c_\ell^2}{\rho_\ell} + K_1, \tag{16a}$$

$$u_g = c_g^2 \ln \rho_g + K_2, \tag{16b}$$

where K_1 and K_2 are the integration constants.

Considering the total energy of the system (neglecting the gravitational potential energy), we define a candidate for the Hamiltonian as follows:

$$\mathcal{H} := \int_\Omega \left(\alpha_g \rho_g \frac{v_g^2}{2} + \alpha_\ell \rho_\ell \frac{v_\ell^2}{2} + \alpha_g \rho_g u_g + \alpha_\ell \rho_\ell u_\ell\right) \mathrm{d}x, \tag{17}$$

where $\Omega = [a, b]$ refers to the spatial domain.

Inserting (16) into (17), the Hamiltonian for a flow across a (unit) constant cross-section takes the following form:

$$\mathcal{H} := \int_\Omega \Big(\alpha_g \rho_g \frac{v_g^2}{2} + \alpha_\ell \rho_\ell \frac{v_\ell^2}{2} + \alpha_g \rho_g \left(c_g^2 \ln \rho_g + K_2\right) + \alpha_\ell \rho_\ell \left(c_\ell^2 \ln \rho_\ell + K_1\right) + \alpha_\ell (c_\ell^2 \rho_{\ell 0} - p_{\ell 0})\Big) \mathrm{d}x. \tag{18}$$

It should be noted that when $\rho_i \to 0$, $\rho_i \ln \rho_i \to 0$. The term $\rho_i \ln \rho_i$ is bounded from below, i.e., $\rho_i \ln \rho_i \geq -1/e$. So, the Hamiltonian (18) is bounded from below. Due to the high bulk modulus of the liquid phase, we usually have $\rho_{\ell 0} c_\ell^2 \gg p_{\ell 0}$ [19]; therefore, the positivity of the Hamiltonian (18) can be ensured by appropriately choosing K_1 and K_2 or even adding some constants to the Hamiltonian. For simplicity, we set $K_1 := 0$ and $K_2 := 0$ henceforth.

Remark 3.2. *The discussion in the above paragraph is reasonable from a physical perspective. However, numerically, solutions of the TFM and DFM may not be guaranteed to have non-negative density and non-negative void fractions.*

3.1. Dissipative Hamiltonian Formulation for the Two-Fluid Model

We now present the dissipative Hamiltonian framework for the TFM.

Theorem 3.3. *The governing equations (2) together with the closure equations (3) can be written in the following dissipative Hamiltonian form:*

$$\partial_t q = \left(\mathcal{J}_T(q) - \mathcal{R}_T\right) \delta_q \mathcal{H}(q) \tag{19}$$

with $q = [q_1,\ q_2,\ q_3,\ q_4]^T := [m_g,\ m_\ell,\ I_g,\ I_\ell]^T$, the Hamiltonian functional (18), and where

$$\mathcal{J}_T(q) = -\begin{bmatrix} 0 & 0 & \partial_x(q_1 \cdot) & 0 \\ 0 & 0 & 0 & \partial_x(q_2 \cdot) \\ q_1 \partial_x(\cdot) & 0 & \partial_x(q_3 \cdot) + q_3 \partial_x(\cdot) & 0 \\ 0 & q_2 \partial_x(\cdot) & 0 & \partial_x(q_4 \cdot) + q_4 \partial_x(\cdot) \end{bmatrix}$$

is a formally skew-adjoint operator with respect to the $\mathcal{L}^2$ inner product, and

$$\mathcal{R}_T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_g^M & -b_g^M \\ 0 & 0 & -b_g^M & b_g^M \end{bmatrix}$$

is a symmetric and positive semi-definite matrix.

Proof: Similar to (4), the TFM with respect to the state variables q can be straightforwardly formulated. We omit the model reformulation here for the sake of brevity.

The Hamiltonian (18) in terms of q_1, q_2, q_3 and q_4 is rewritten as follows:

$$\mathcal{H}(q_1, q_2, q_3, q_4) := \int_\Omega \frac{q_3^2}{2q_1} + \frac{q_4^2}{2q_2} + q_1 c_g^2 \ln\left(\frac{p}{c_g^2}\right) + q_2 c_\ell^2 \ln\left(\frac{p+\beta}{c_\ell^2}\right) + \left(1 - \alpha_g\right) \beta \, \mathrm{d}x, \tag{20}$$

where p and α_g can be replaced by the relations (5) and (6), respectively.

The variational derivatives are:

$$\frac{\delta \mathcal{H}}{\delta q_1} = -\frac{1}{2}\frac{q_3^2}{q_1^2} + c_g^2 \ln\left(\frac{p}{c_g^2}\right) + c_g^2, \quad \frac{\delta \mathcal{H}}{\delta q_3} = \frac{q_3}{q_1},$$

$$\frac{\delta \mathcal{H}}{\delta q_2} = -\frac{1}{2}\frac{q_4^2}{q_2^2} + c_\ell^2 \ln\left(\frac{p+\beta}{c_\ell^2}\right) + c_\ell^2, \quad \frac{\delta \mathcal{H}}{\delta q_4} = \frac{q_4}{q_2}.$$

For the sake of brevity, we omit detailed calculations here. Instead, we argue that the TFM exhibits similarities in structure with the model presented in [22], where the Hamiltonian structure was discussed for single-phase dynamics. The TFM with $b_g^M = 0$ can be viewed as two separately existing phases. The contributions due to the non-zero b_g^M enter into the dissipation matrix $\mathcal{R}_T$. The proof of the symmetric and positive semi-definite nature of $\mathcal{R}_T$ is straightforward.

The operator $\mathcal{J}_T$ is formally skew-adjoint (with respect to the $\mathcal{L}^2$ inner product). To prove formal skew-adjointness of $\mathcal{J}_T$, we check whether $\langle \mathbf{e^1}, \mathcal{J}_T \mathbf{e^2} \rangle_{\mathcal{L}^2(\Omega)} + \langle \mathcal{J}_T \mathbf{e^1}, \mathbf{e^2} \rangle_{\mathcal{L}^2(\Omega)} = 0$ for smooth $\mathbf{e^1}, \mathbf{e^2}$ which are zero at the boundary, where we define $\mathbf{e^i} = (e_1^i, e_2^i, e_3^i, e_4^i)^T$. Here, the variable e_j^i refers to the j-th element of $\mathbf{e^i}$. $\mathcal{J}_T$ is formally skew-adjoint with respect to the $\mathcal{L}^2$ inner product as

$$-\langle \mathbf{e^1}, \mathcal{J}_T \mathbf{e^2} \rangle_{\mathcal{L}^2(\Omega)} - \langle \mathcal{J}_T \mathbf{e^1}, \mathbf{e^2} \rangle_{\mathcal{L}^2(\Omega)} =$$

$$\int_\Omega \; e_1^1\partial_x(q_1e_3^2) + q_1e_3^2\partial_x e_1^1 + \\ e_1^2\partial_x(q_1e_3^1) + q_1e_3^1\partial_x e_1^2 + \\ e_2^1\partial_x(q_2e_4^2) + q_2e_4^2\partial_x e_2^1 + \\ e_2^2\partial_x(q_2e_4^1) + q_2e_4^1\partial_x e_2^2 + \\ e_3^1\left[\partial_x(q_3e_3^2) + q_3\partial_x e_3^2\right] + \\ e_3^2\left[\partial_x(q_3e_3^1) + q_3\partial_x e_3^1\right] + \\ e_4^1\left[\partial_x(q_4e_4^2) + q_4\partial_x e_4^2\right] + \\ e_4^2\left[\partial_x(q_4e_4^1) + q_4\partial_x e_4^1\right] dx = \tag{21}$$

$$\left(\begin{bmatrix} e_1^1 & e_2^1 & e_3^1 & e_4^1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \\ q_1 & 0 & 2q_3 & 0 \\ 0 & q_2 & 0 & 2q_4 \end{bmatrix}}_{Q} \begin{bmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \\ e_4^2 \end{bmatrix} \right)\Big|_a^b,$$

which vanishes under our assumptions on the boundary conditions. □

3.2. Dissipative Hamiltonian Formulation for the Drift Flux Model

So far, we focused on the dissipative Hamiltonian representation for the TFM. We will now deal with the DFM under gravitational and frictional effects, and present a corresponding dissipative Hamiltonian formulation. For the DFM, we focus only on a case in which there is no slip between the phases, i.e., $v := v_\mathrm{g} = v_\ell$ (the reason for adopting this no-slip assumption is provided in Section 5). Since gravitation is considered, the gravitational potential energy needs to be added to the Hamiltonian. The Hamiltonian now takes the following form:

$$\mathcal{H}_D(m_\mathrm{g}, m_\ell, v) = \int_\Omega m_\mathrm{g}\frac{v^2}{2} + m_\ell\frac{v^2}{2} + \\ m_\ell c_\ell^2 \ln\left(\frac{p+\beta}{c_\ell^2}\right) + m_\mathrm{g} c_g^2 \ln\left(\frac{p}{c_g^2}\right) + \\ \alpha_\ell\beta + \left(m_\mathrm{g}+m_\ell\right)\left(\int_a^x g\sin(\theta(\xi))\mathrm{d}\xi\right)\mathrm{d}x. \tag{22}$$

Using the above candidate Hamiltonian function $\mathcal{H}_D$, a dissipative Hamiltonian representation of a special case of the DFM is shown below.

Theorem 3.4. *The governing equations (9) together with $v := v_\mathrm{g} = v_\ell$ (case of no slip), the closure equations (3a), (3e), (3f) and (10) can be written in dissipative Hamiltonian form as follows:*

$$\partial_t z_D = \left(\mathcal{J}_D(z_D) - \mathcal{R}_D(z_D)\right)\delta_{z_D}\mathcal{H}_D(z_D) \tag{23}$$

with $z_D := [m_\mathrm{g},\ m_\ell,\ v]^T$, the Hamiltonian functional (22), where

$$\mathcal{J}_D(z_D) = -\begin{bmatrix} 0 & 0 & \partial_x\left(\frac{m_\mathrm{g}}{m_\mathrm{g}+m_\ell}\cdot\right) \\ 0 & 0 & \partial_x\left(\frac{m_\ell}{m_\mathrm{g}+m_\ell}\cdot\right) \\ \frac{m_\mathrm{g}}{m_\mathrm{g}+m_\ell}\partial_x(\cdot) & \frac{m_\ell}{m_\mathrm{g}+m_\ell}\partial_x(\cdot) & 0 \end{bmatrix}$$

is a formally skew-adjoint operator with respect to the $\mathcal{L}^2$ inner product, and

$$\mathcal{R}_D(z_D) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{32\mu_m}{d^2\left(m_\mathrm{g}+m_\ell\right)^2} \end{bmatrix}$$

is a symmetric and positive semi-definite matrix.

Proof: First note that, using (9a) and (9b), the left-hand side of equation (9c) can be rewritten as

$$\left(m_\mathrm{g}+m_\ell\right)\partial_t v + v\partial_t\left(m_\mathrm{g}+m_\ell\right) + \partial_x\left(\left(m_\mathrm{g}+m_\ell\right)v^2\right) \\ = \left(m_\mathrm{g}+m_\ell\right)\left(\partial_t v + \partial_x\left(\frac{v^2}{2}\right)\right).$$

Thus, instead of (9c) we can also consider

$$\partial_t v + \partial_x\left(\frac{v^2}{2}\right) = \frac{1}{m_\mathrm{g}+m_\ell}\left(-\partial_x p + Q_g + Q_v\right). \tag{24}$$

The variational derivatives of $\mathcal{H}_D$ are given by:

$$\frac{\delta\mathcal{H}_D}{\delta m_\mathrm{g}} = c_g^2\ln\left(\frac{p}{c_g^2}\right) + \frac{v^2}{2} + c_g^2 + \int_a^x g\sin(\theta(\xi))\mathrm{d}\xi,$$

$$\frac{\delta\mathcal{H}_D}{\delta m_\ell} = c_\ell^2\ln\left(\frac{p+\beta}{c_\ell^2}\right) + \frac{v^2}{2} + c_\ell^2 + \int_a^x g\sin(\theta(\xi))\mathrm{d}\xi,$$

$$\frac{\delta\mathcal{H}_D}{\delta v} = \left(m_\mathrm{g}+m_\ell\right)v.$$

Next, we prove the claim equation by equation. The first line of (23) reads

$$\partial_t m_\mathrm{g} = -\partial_x\left(\frac{m_\mathrm{g}}{m_\mathrm{g}+m_\ell}\left(m_\mathrm{g}+m_\ell\right)v\right) = -\partial_x\left(m_\mathrm{g}v\right). \tag{25}$$

Similarly, the second line is

$$\partial_t m_\ell = -\partial_x\left(\frac{m_\ell}{m_\mathrm{g}+m_\ell}\left(m_\mathrm{g}+m_\ell\right)v\right) = -\partial_x\left(m_\ell v\right). \tag{26}$$

Let us introduce a short-hand notation $G = \int_a^x g\sin(\theta(\xi))\mathrm{d}\xi$.

Then, the third line yields

$$\begin{aligned}\partial_t v = &-\frac{m_g}{m_g + m_\ell}\partial_x \left(c_g^2 \ln\left(\frac{p}{c_g^2}\right) + \frac{v^2}{2} + c_g^2 + G \right) \\ &- \frac{m_\ell}{m_g + m_\ell}\partial_x \left(c_\ell^2 \ln\left(\frac{p+\beta}{c_\ell^2}\right) + \frac{v^2}{2} + c_\ell^2 + G \right) \\ &- \frac{32\mu_m}{d^2 \left(m_g + m_\ell\right)^2} \left(m_g + m_\ell\right) v \\ &= -\partial_x \left(\frac{v^2}{2}\right) - \frac{1}{\left(m_g + m_\ell\right)}(\partial_x p + Q_g + Q_v).\end{aligned} \tag{27}$$

The claim follows by observing that (25), (26), and (27) are identical to (9a), (9b), and (24), respectively.

The symmetric and positive semi-definite nature of $\mathcal{R}_D$ follows immediately from the positivity of μ_m. The formal skew-adjointness of $\mathcal{J}_D$ essentially follows from integration by parts and neglecting the boundary conditions. The operator $\mathcal{J}_D$ contains terms similar to the skew-adjoint operator $\mathcal{J}_T$, the formal skew-adjointness of which was discussed extensively in the proof of Theorem 3.3. For the sake of brevity, we refer the reader to follow similar lines of reasoning to show the formal skew-adjointness of $\mathcal{J}_D$. □

4. Geometrical properties of the system: Stokes-Dirac structures

We now define a geometric structure, a generalization of symplectic and Poisson structures, called a Stokes-Dirac structure.

Definition 4.1. *[17, 20] Consider $\mathcal{F}$ and $\mathcal{E}$ as real Hilbert spaces which are isometrically isomorphic. The subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a Stokes-Dirac structure if $\mathcal{D} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ denotes the orthogonal complement which is defined as*

$$\mathcal{D}^\perp := \{(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}) \in \mathcal{F} \times \mathcal{E} \mid \ll (\tilde{\mathbf{f}}, \tilde{\mathbf{e}}), (\mathbf{f}, \mathbf{e}) \gg = 0 \quad \forall (\mathbf{f}, \mathbf{e}) \in \mathcal{D}\}. \tag{28}$$

Here, $\ll (\tilde{\mathbf{f}}, \tilde{\mathbf{e}}), (\mathbf{f}, \mathbf{e}) \gg$ is defined as follows:

$$\ll (\tilde{\mathbf{f}}, \tilde{\mathbf{e}}), (\mathbf{f}, \mathbf{e}) \gg := \langle \tilde{\mathbf{f}} \mid \mathbf{e} \rangle + \langle \mathbf{f} \mid \tilde{\mathbf{e}} \rangle, \tag{29}$$

where the notation $\langle \mathbf{f} \mid \mathbf{e} \rangle$ indicates a non-degenerate bilinear form defined on the bond space $\mathcal{B} = \mathcal{F} \times \mathcal{E}$.

This structure relates the composing elements of a system in a power-conserving manner [7]. Such geometric structures often have a compositionality property [7, 23, 24].

For (f, e) element of a Stokes-Dirac structure, it is easy to see that $\langle f \mid e \rangle = 0$, and thus there is a close relation to (formally) skew-adjoint operators, see also (13). However, if $f = \mathcal{J} e$ for all $(f, e) \in D$, and J is formally skew-adjoint, then $\mathcal{D} \subset \mathcal{D}^\perp$. To make such a $\mathcal{D}$ into a Stokes-Dirac structure, it is required that $\mathcal{D} = \mathcal{D}^\perp$ holds. The formally skew-adjoint part of a pH system will form the foundation of the associated Stokes-Dirac structure, as we will show as well.

Non-linearity encoded within the Hamiltonian along with a linear Stokes-Dirac structure constitutes a favorable representation of PDEs. Such a structure facilitates the analysis of non-linear systems as the linearity of the Stokes-Dirac structure can be exploited to assess system behavior. Stokes-Dirac structures can also be used to formulate boundary control systems [17].

In the existing results [17, 20, 25], the skew-adjoint operator yields a symmetric bilinear form on the space of the boundary variables. An important tool used in that framework is the trace operator, which, in earlier works [17, 20, 25], requires that the effort variables $\mathbf{e}$ belong to the function class $H^1(\Omega)$. Given the state-dependent nature of skew-adjoint operators in (19) and (23) (unlike in [17]), a combination of the states and the effort variables have to belong to the function class $H^1(\Omega)$ or suitable conditions have to be imposed on the state variables in order to have effort variables belonging to the function class $H^1(\Omega)$ (see Theorems 4.4 and 4.6). Boundary port-variables have been parametrized in [17] using the trace operators. However, such an elegant parametrization is limited to the case of a non-singular matrix Q (synonymous to (21)) arising in linear problems with state-independent operators. To the best of our knowledge, the work [25] is the only work in the scope of parametrization of boundary port-variables for a singular matrix Q, thereby enlarging the class of systems that can be dealt. Villegas in [25] demonstrated the approach to define the non-degenerate bilinear form under singular Q and consequently modified the definition of the boundary port-variables. However, [25] was limited to the setting of *state-independent* Stokes-Dirac structures. In this work, we extend the definition of boundary port-variables to eventually obtain *state-dependent* Stokes-Dirac structures with boundary ports for non-linear problems with non-quadratic Hamiltonian functional. It should be mentioned that the authors in [5] have also considered state-dependent Stokes-Dirac structures for problems (for instance, ideal isentropic fluid) with non-quadratic Hamiltonian functional by using a differential geometric viewpoint. We, contrarily, use the matrix or operator-theoretic viewpoint in the consideration of such geometric structures in the scope of the compressible two-phase flow models.

Remark 4.2. *Boundary port-variables, in our setting, will remain unchanged in the presence of dissipation. This is only true since our resistive operator ($\mathcal{R}$) does not include any differential operator. In general, the boundary ports could also include contributions from the resistive part. In this work, we only consider Stokes-Dirac structures without accounting for resistive ports (for the above mentioned reason) and finally arrive at a definition of the boundary port-variables, which is practical for pH representations.*

We recall the following fundamental lemma of calculus of variations.

Lemma 4.3. *If the pair $(h, m) \in \mathcal{L}^2(\Omega)^2$ satisfies*

$$\int_a^b [h(x)\partial_x f(x) + m(x) f(x)]\mathrm{d}x = 0, \tag{30}$$

for all $f \in H_0^1(\Omega)$, *then*

$$h \in H^1(\Omega), \quad \text{and} \quad \partial_x h = m(x). \tag{31}$$

Lemma 4.3 will be extensively used in order to prove that a certain structure is a Stokes-Dirac structure.

Using the above mathematical preliminaries, we first propose a Stokes-Dirac structure for the TFM and present a corresponding proof, and then we propose it for the DFM without slip.

4.1. Stokes-Dirac structure representation for the Two-Fluid Model

We, first, introduce the notations

$$\mathbf{f}_t = \begin{bmatrix} f_{m_g} & f_{m_\ell} & f_{I_g} & f_{I_\ell} & f_{a,t}^B & f_{b,t}^B \end{bmatrix}^T, \tag{32a}$$

$$\mathbf{e}_t = \begin{bmatrix} e_{m_g} & e_{m_\ell} & e_{I_g} & e_{I_\ell} & e_{a,t}^B & e_{b,t}^B \end{bmatrix}^T, \tag{32b}$$

$$\mathbf{f}_{tr} = \begin{bmatrix} f_{m_g} & f_{m_\ell} & f_{I_g} & f_{I_\ell} \end{bmatrix}^T, \tag{32c}$$

$$\mathbf{e}_{tr} = \begin{bmatrix} e_{m_g} & e_{m_\ell} & e_{I_g} & e_{I_\ell} \end{bmatrix}^T \tag{32d}$$

with $\mathbf{f}_t \in \mathcal{F}_t, \mathbf{e}_t \in \mathcal{E}_t$ where $\mathcal{F}_t = \mathcal{E}_t = \mathcal{L}^2(\Omega)^4 \times \mathbb{R}^2 \times \mathbb{R}^2$. On $\mathcal{F}_t \times \mathcal{E}_t$ the following non-degenerate bilinear form is defined:

$$\langle \mathbf{f}_t \mid \mathbf{e}_t \rangle = \int_\Omega (f_{m_g} e_{m_g} + f_{m_\ell} e_{m_\ell} + f_{I_g} e_{I_g} + f_{I_\ell} e_{I_\ell}) \mathrm{d}x + (f_{b,t}^B)^T e_{b,t}^B + (f_{a,t}^B)^T e_{a,t}^B. \tag{33}$$

Using these notations, the Stokes-Dirac structure corresponding to the dissipative Hamiltonian representation of the TFM can be expressed as follows.

Theorem 4.4. *Consider $\mathcal{F}_t$ and $\mathcal{E}_t$ as introduced above. Moreover, assume that $m_g, m_\ell, I_g, I_\ell =: q_1, q_2, q_3, q_4 \in H^1(\Omega)$. We also assume that $q_1, q_2 > 0$ on Ω. Then, the linear subset $\mathcal{D}_t \subset \mathcal{F}_t \times \mathcal{E}_t$ defined as follows:*

$$\mathcal{D}_t = \Big\{ (\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{F}_t \times \mathcal{E}_t \mid \mathbf{e}_{tr} \in H^1(\Omega)^4, \mathbf{f}_{tr} = \mathcal{J}_t(q)\mathbf{e}_{tr},$$
$$\begin{pmatrix} f_{b,t}^B \\ e_{b,t}^B \end{pmatrix} = \begin{pmatrix} f_{b1,t}^B \\ f_{b2,t}^B \\ e_{b1,t}^B \\ e_{b2,t}^B \end{pmatrix} = \left(\begin{pmatrix} q_1 & 0 & q_3 & 0 \\ 0 & q_2 & 0 & q_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{m_g} \\ e_{m_\ell} \\ e_{I_g} \\ e_{I_\ell} \end{pmatrix} \right)(b), \tag{34}$$
$$\begin{pmatrix} f_{a,t}^B \\ e_{a,t}^B \end{pmatrix} = \begin{pmatrix} f_{a1,t}^B \\ f_{a2,t}^B \\ e_{a1,t}^B \\ e_{a2,t}^B \end{pmatrix} = \left(\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ q_1 & 0 & q_3 & 0 \\ 0 & q_2 & 0 & q_4 \end{pmatrix} \begin{pmatrix} e_{m_g} \\ e_{m_\ell} \\ e_{I_g} \\ e_{I_\ell} \end{pmatrix} \right)(a) \Big\},$$

where

$$\mathcal{J}_t(q) = -\begin{bmatrix} 0 & 0 & \partial_x(m_g \cdot) & 0 \\ 0 & 0 & 0 & \partial_x(m_\ell \cdot) \\ m_g \partial_x(\cdot) & 0 & \partial_x(I_g \cdot) + I_g \partial_x(\cdot) & 0 \\ 0 & m_\ell \partial_x(\cdot) & 0 & \partial_x(I_\ell \cdot) + I_\ell \partial_x(\cdot) \end{bmatrix} \tag{35}$$

is a Stokes-Dirac structure with respect to the symmetric pairing given by

$$\ll (\mathbf{f}_t, \mathbf{e}_t), (\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \gg = \langle \mathbf{f}_t \mid \tilde{\mathbf{e}}_t \rangle + \langle \tilde{\mathbf{f}}_t \mid \mathbf{e}_t \rangle, \quad (\mathbf{f}_t, \mathbf{e}_t), (\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{F}_t \times \mathcal{E}_t, \tag{36}$$

where the pairing $\langle \cdot \mid \cdot \rangle$ is given in (33).

Proof: The proof is divided into two parts. We first prove that $\mathcal{D}_t \subset \mathcal{D}_t^\perp$.

We consider two pairs of flow and effort variables belonging to the Stokes-Dirac structure, i.e., $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ and $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{D}_t$. Using the earlier introduced notations, we obtain:

$$\begin{aligned} &\ll (\mathbf{f}_t, \mathbf{e}_t), (\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \gg = \\ &\int_\Omega (f_{m_g} \tilde{e}_{m_g} + f_{m_\ell} \tilde{e}_{m_\ell} + f_{I_g} \tilde{e}_{I_g} + f_{I_\ell} \tilde{e}_{I_\ell}) \mathrm{d}x + \\ &\int_\Omega (\tilde{f}_{m_g} e_{m_g} + \tilde{f}_{m_\ell} e_{m_\ell} + \tilde{f}_{I_g} e_{I_g} + \tilde{f}_{I_\ell} e_{I_\ell}) \mathrm{d}x + \\ &(f_{a,t}^B)^T \tilde{e}_{a,t}^B + (f_{b,t}^B)^T \tilde{e}_{b,t}^B + (\tilde{f}_{a,t}^B)^T e_{a,t}^B + (\tilde{f}_{b,t}^B)^T e_{b,t}^B. \end{aligned} \tag{37}$$

Substituting the mappings between the flow and the effort variables, the total sum within the integrals of (37) becomes

$$\begin{aligned} &\Big[-\partial_x(q_1 e_{I_g}) \tilde{e}_{m_g} - \partial_x(q_2 e_{I_\ell}) \tilde{e}_{m_\ell} + \\ &\quad \Big(-q_1 \partial_x e_{m_g} - \partial_x(q_3 e_{I_g}) - q_3 \partial_x e_{I_g} \Big) \tilde{e}_{I_g} + \\ &\quad \Big(-q_2 \partial_x e_{m_\ell} - \partial_x(q_4 e_{I_\ell}) - q_4 \partial_x e_{I_\ell} \Big) \tilde{e}_{I_\ell} \Big] \\ &+ \Big[-\partial_x(q_1 \tilde{e}_{I_g}) e_{m_g} - \partial_x(q_2 \tilde{e}_{I_\ell}) e_{m_\ell} + \\ &\quad \Big(-q_1 \partial_x \tilde{e}_{m_g} - \partial_x(q_3 \tilde{e}_{I_g}) - q_3 \partial_x \tilde{e}_{I_g} \Big) e_{I_g} + \\ &\quad \Big(-q_2 \partial_x \tilde{e}_{m_\ell} - \partial_x(q_4 \tilde{e}_{I_\ell}) - q_4 \partial_x \tilde{e}_{I_\ell} \Big) e_{I_\ell} \Big) \Big] \\ &= -\partial_x(q_1 \tilde{e}_{m_g} e_{I_g}) - \partial_x(q_1 e_{m_g} \tilde{e}_{I_g}) \\ &\quad - \partial_x(q_2 e_{I_\ell} \tilde{e}_{m_\ell}) - \partial_x(q_2 \tilde{e}_{I_\ell} e_{m_\ell}) \\ &\quad - \partial_x(q_3 e_{I_g} \tilde{e}_{I_g}) - \partial_x(q_3 e_{I_g} \tilde{e}_{I_g}) \\ &\quad - \partial_x(q_4 e_{I_\ell} \tilde{e}_{I_\ell}) - \partial_x(q_4 e_{I_\ell} \tilde{e}_{I_\ell}). \end{aligned}$$

Performing integration on the above expression, it equals minus the last expressions in (37) and hence, $\mathcal{D}_t \subset \mathcal{D}_t^\perp$. This concludes the first part of the proof.

We now prove the converse part, i.e., $\mathcal{D}_t^\perp \subset \mathcal{D}_t$. For this, we follow the steps similar to Proposition 4.1 in [20]. The proof consists of several repeated steps, which are summarized below. We take $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{D}_t^\perp$ i.e., $(\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \in \mathcal{F}_t \times \mathcal{E}_t$ such that $\ll (\mathbf{f}_t, \mathbf{e}_t), (\tilde{\mathbf{f}}_t, \tilde{\mathbf{e}}_t) \gg = 0 \quad \forall (\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$. To this end, we use the freedom in the choice of the effort variables and exploit Lemma 4.3.

Step 1: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_\ell}, e_{I_g}, e_{I_\ell} = 0$ and $e_{m_g}(a) = e_{m_g}(b) = 0$. Using (37), we find that

$$\int_\Omega -(q_1 \partial_x e_{m_g}) \tilde{e}_{I_g} + \tilde{f}_{m_g} e_{m_g} \mathrm{d}x = 0 \quad \forall e_{m_g} \in H_0^1(\Omega). \tag{38}$$

Lemma 4.3 gives

$$q_1\tilde{e}_{I_g} \in H^1(\Omega) \quad \text{and} \quad \tilde{f}_{m_g} = -\partial_x(q_1\tilde{e}_{I_g}). \tag{39}$$

Using $q_1 \in H^1(\Omega)$ along with $q_1 > 0$ on Ω, we obtain that $\tilde{e}_{I_g} \in H^1(\Omega)$.
Step 2: Considering $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_g}, e_{I_g}, e_{I_\ell} = 0$ and $e_{m_\ell} \in H_0^1(\Omega)$, we have by (37) that

$$\int_\Omega -(q_2\partial_x e_{m_\ell})\tilde{e}_{I_\ell} + \tilde{f}_{m_\ell} e_{m_\ell} \mathrm{d}x = 0 \quad \forall e_{m_\ell} \in H_0^1(\Omega). \tag{40}$$

Now using Lemma 4.3 leads to

$$q_2\tilde{e}_{I_\ell} \in H^1(\Omega) \quad \text{and} \quad \tilde{f}_{m_\ell} = -\partial_x(q_2\tilde{e}_{I_\ell}). \tag{41}$$

As before, using $q_2 \in H^1(\Omega)$ along with $q_2 > 0$ on Ω, we have that $\tilde{e}_{I_\ell} \in H^1(\Omega)$.
Step 3: For $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_g}, e_{m_\ell}, e_{I_\ell} = 0$ and $e_{I_g} \in H_0^1(\Omega)$, we obtain:

$$\int_\Omega -\partial_x(q_1 e_{I_g})\tilde{e}_{m_g} - \partial_x(q_3 e_{I_g})\tilde{e}_{I_g} - (q_3\partial_x e_{I_g})\tilde{e}_{I_g} + \tilde{f}_{I_g} e_{I_g} \mathrm{d}x = 0 \quad \forall e_{I_g} \in H_0^1(\Omega).$$

We rewrite the above equation as follows:

$$\int_\Omega -(\partial_x q_1)(e_{I_g}\tilde{e}_{m_g}) - (\partial_x q_3)(e_{I_g}\tilde{e}_{I_g}) - (\partial_x e_{I_g})\cdot \left(q_1\tilde{e}_{m_g} + 2q_3\tilde{e}_{I_g}\right) + \tilde{f}_{I_g} e_{I_g} \mathrm{d}x = 0 \quad \forall e_{I_g} \in H_0^1(\Omega).$$

As a result of Lemma 4.3, we have that $q_1\tilde{e}_{m_g} + 2q_3\tilde{e}_{I_g} \in H^1(\Omega)$. Moreover, we obtain the following identity:

$$\tilde{f}_{I_g} = -\partial_x(q_1\tilde{e}_{m_g} + 2q_3\tilde{e}_{I_g}) + \tilde{e}_{m_g}\partial_x q_1 + \tilde{e}_{I_g}\partial_x q_3. \tag{42}$$

Using $q_1, q_3, \tilde{e}_{I_g} \in H^1(\Omega)$ and that $q_1 > 0$, it can easily be deduced that $\tilde{e}_{m_g} \in H^1(\Omega)$, and so (42) can be written as

$$\tilde{f}_{I_g} = -q_1\partial_x\tilde{e}_{m_g} - \partial_x(q_3\tilde{e}_{I_g}) - q_3\partial_x\tilde{e}_{I_g}. \tag{43}$$

Step 4: Considering $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_g}, e_{m_\ell}, e_{I_g} = 0$ and $e_{I_\ell} \in H_0^1(\Omega)$, we obtain:

$$\int_\Omega -\partial_x(q_2 e_{I_\ell})\tilde{e}_{m_\ell} - \partial_x(q_4 e_{I_\ell})\tilde{e}_{I_\ell} - (q_4\partial_x e_{I_\ell})\tilde{e}_{I_\ell} + \tilde{f}_{I_\ell} e_{I_\ell} \mathrm{d}x = 0 \quad \forall e_{I_\ell} \in H_0^1(\Omega).$$

Re-writing the above equation as in the previous step and using Lemma 4.3, we have that $q_2\tilde{e}_{m_\ell} + 2q_4\tilde{e}_{I_\ell} \in H^1(\Omega)$ and also obtain:

$$\tilde{f}_{I_\ell} = -\partial_x(q_2\tilde{e}_{m_\ell} + 2q_4\tilde{e}_{I_\ell}) + \tilde{e}_{m_\ell}\partial_x q_2 + \tilde{e}_{I_\ell}\partial_x q_4. \tag{44}$$

Using $q_2, q_4, \tilde{e}_{I_\ell} \in H^1(\Omega)$ and that $q_2 > 0$, it can easily be deduced that $\tilde{e}_{m_\ell} \in H^1(\Omega)$ and so

$$\tilde{f}_{I_\ell} = -q_2\partial_x\tilde{e}_{m_\ell} - \partial_x(q_4\tilde{e}_{I_\ell}) - q_4\partial_x\tilde{e}_{I_\ell}. \tag{45}$$

Step 5: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_\ell} = e_{I_g} = e_{I_\ell} = 0$ and $e_{m_g}(a) = 0, e_{m_g}(b) \neq 0$. Using the procedure outlined above, we obtain the following identity: $\tilde{e}^B_{b1,t} = \tilde{e}_{I_g} \mid_b$.
Step 6: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_g} = e_{I_g} = e_{I_\ell} = 0$ and $e_{m_\ell}(a) = 0, e_{m_\ell}(b) \neq 0$. We now observe that $\tilde{e}^B_{b2,t} = \tilde{e}_{I_\ell} \mid_b$ holds.
Step 7: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_g} = e_{m_\ell} = e_{I_\ell} = 0$ and $e_{I_g}(a) = 0, e_{I_g}(b) \neq 0$. Using the outlined procedure, we now obtain:

$$-(q_1\tilde{e}_{m_g} e_{I_g}) \mid_b - (q_3\tilde{e}_{I_g} e_{I_g}) \mid_b + \tilde{f}^B_{b1,t} e_{I_g} \mid_b = 0. \tag{46}$$

Finally, we obtain the following identity:

$$\tilde{f}^B_{b1,t} = \left(q_1\tilde{e}_{m_g} + q_3\tilde{e}_{I_g}\right) \mid_b . \tag{47}$$

Step 8: Let $(\mathbf{f}_t, \mathbf{e}_t) \in \mathcal{D}_t$ with $e_{m_g} = e_{m_\ell} = e_{I_g} = 0$ and $e_{I_\ell}(a) = 0, e_{I_\ell}(b) \neq 0$. Using the outlined procedure, we now obtain the following identity:

$$\tilde{f}^B_{b2,t} = \left(q_2\tilde{e}_{m_\ell} + q_4\tilde{e}_{I_\ell}\right) \mid_b . \tag{48}$$

The boundary port-variables $f^B_{a1,t}, f^B_{a2,t}, e^B_{a1,t}$ and $e^B_{a2,t}$ can be obtained in a manner similar to the one outlined for computing the boundary port-variables at the right boundary of the spatial domain Ω.

Thus, in summary we have shown $\mathcal{D}_t^\perp \subset \mathcal{D}_t$ and, hence, $\mathcal{D}_t$ is a Stokes-Dirac structure. □

Remark 4.5. *The formally skew-adjoint operator $\mathcal{J}_T(q)$ in Theorem 3.3 is equal to the skew-adjoint operator $\mathcal{J}_t(q)$ associated to the Stokes-Dirac structure representation in Theorem 4.4. These operators are found to be equal only because of the assumptions on the state variables q; see Theorem 4.4 for details. In general, the formally skew-adjoint operator and the skew-adjoint operator associated to the Stokes-Dirac structure representation need not be the same. For instance, see Theorem 4.6.*

We now discuss the representation of the Stokes-Dirac structure corresponding to the skew-adjoint operator $\mathcal{J}_D$ in the scope of the Drift Flux Model without slip.

4.2. Stokes-Dirac structure representation for the Drift Flux Model

We introduce the notations

$$\mathbf{f}_d = \begin{bmatrix} f_{m_g,d} & f_{m_\ell,d} & f_{v,d} & f^B_{a,d} & f^B_{b,d} \end{bmatrix}^T, \tag{49a}$$

$$\mathbf{e}_d = \begin{bmatrix} e_{m_g,d} & e_{m_\ell,d} & e_{v,d} & e^B_{a,d} & e^B_{b,d} \end{bmatrix}^T, \tag{49b}$$

$$\mathbf{f}_{dr} = \begin{bmatrix} f_{m_g,d} & f_{m_\ell,d} & f_{v,d} \end{bmatrix}^T, \tag{49c}$$

$$\mathbf{e}_{dr} = \begin{bmatrix} e_{m_g,d} & e_{m_\ell,d} & e_{v,d} \end{bmatrix}^T. \tag{49d}$$

A Stokes-Dirac structure for the dissipative Hamiltonian representation of the DFM can be expressed as follows.

Theorem 4.6. *Consider $\mathcal{F}_d = \mathcal{E}_d = \mathcal{L}^2(\Omega)^3 \times \mathbb{R}^2$. We assume that $A_g := \frac{m_g}{m_g+m_\ell}$, $A_\ell := \frac{m_\ell}{m_g+m_\ell} \in H^1(\Omega)$. We also consider that the non-degenerate bilinear form on $\mathcal{F}_d \times \mathcal{E}_d$ is defined in the following way:*

$$\langle \mathbf{f}_d \mid \mathbf{e}_d \rangle = \int_\Omega (f_{m_g,d} e_{m_g,d} + f_{m_\ell,d} e_{m_\ell,d} + f_{v,d} e_{v,d}) \mathrm{d}x + f^B_{b,d} e^B_{b,d} + f^B_{a,d} e^B_{a,d}. \quad (50)$$

Then, the linear subset $\mathcal{D}_d \subset \mathcal{F}_d \times \mathcal{E}_d$ given by

$$\mathcal{D}_d = \Bigg\{ (\mathbf{f}_d, \mathbf{e}_d) \in \mathcal{F}_d \times \mathcal{E}_d, \begin{pmatrix} A_g e_{m_g,d} + A_\ell e_{m_\ell,d} \\ e_{v,d} \end{pmatrix} \in H^1(\Omega)^2, \mathbf{f}_{dr} = \mathcal{J}_d(z_D)\mathbf{e}_{dr}, \begin{pmatrix} f^B_{a,d} \\ e^B_{a,d} \end{pmatrix} = \left(\begin{pmatrix} -A_g & -A_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{m_g,d} \\ e_{m_\ell,d} \\ e_{v,d} \end{pmatrix} \right)(a), \begin{pmatrix} f^B_{b,d} \\ e^B_{b,d} \end{pmatrix} = \left(\begin{pmatrix} A_g & A_\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{m_g,d} \\ e_{m_\ell,d} \\ e_{v,d} \end{pmatrix} \right)(b) \Bigg\}, \quad (51)$$

where

$$\mathcal{J}_d(z_D) = \begin{pmatrix} 0 & 0 & -\partial_x(A_g \cdot) \\ 0 & 0 & -\partial_x(A_\ell \cdot) \\ -D(A_g\cdot)\&D(A_\ell\cdot) & & 0 \end{pmatrix}$$

is a Stokes-Dirac structure with respect to the symmetric pairing given by, see (50):

$$\ll (\mathbf{f}_d, \mathbf{e}_d), (\tilde{\mathbf{f}}_d, \tilde{\mathbf{e}}_d) \gg = \langle \mathbf{f}_d \mid \tilde{\mathbf{e}}_d \rangle + \langle \tilde{\mathbf{f}}_d \mid \mathbf{e}_d \rangle, \quad (\mathbf{f}_d, \mathbf{e}_d), (\tilde{\mathbf{f}}_d, \tilde{\mathbf{e}}_d) \in \mathcal{F}_d \times \mathcal{E}_d. \quad (52)$$

The action of the operator $D(A_g\cdot)\&D(A_\ell\cdot)$ is given by

$$D(A_g\cdot)\&D(A_\ell\cdot) \begin{pmatrix} e_{m_g,d} \\ e_{m_\ell,d} \end{pmatrix} = \partial_x(A_g e_{m_g,d} + A_\ell e_{m_\ell,d}) - e_{m_g,d}\partial_x A_g - e_{m_\ell,d}\partial_x A_\ell. \quad (53)$$

Remark 4.7. *This can be considered as a special case of the extended structure shown in [26] in the context of spatially-varying cross-section. We skip the proof of Theorem 4.6 and instead refer to [26] and use similar lines of reasoning.*

We have shown Stokes-Dirac structure representations for both dissipative Hamiltonian formulations of the mathematical models under consideration.

5. Special case considerations for the DFM

In this section, we disqualify the DFM with the Zuber-Findlay slip conditions as an energy consistent model for two-phase flow, and, thus, motivate the reasons behind considering the DFM without slip.

We recall the dissipation inequality obeyed by the TFM (see Theorem 3.3). Under the imposition of periodic boundary conditions, the time derivative of the Hamiltonian (18) can be expressed using (3d) as follows:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= -\int_\Omega (\delta_q \mathcal{H}(q))^T (\mathcal{R}_T) \delta_q \mathcal{H}(q)\, \mathrm{d}x, \\ &= -\int_\Omega b^M_g \left(v_g - v_\ell\right)^2 \, \mathrm{d}x, \\ &= -\int_\Omega M_{ig}(v_\ell - v_g)\, \mathrm{d}x \le 0. \end{aligned} \quad (54)$$

The equivalence between the TFM and DFM, discussed in Section 2, gives a better understanding of the DFM, especially when comparing the energy considerations between these two models since the only difference is how the term M_{ig} is chosen. In the TFM, it is chosen to be proportional to the slip velocity $v_\ell - v_g$ with a non-negative coefficient of proportionality b^M_g. This linear relationship has been chosen to enforce an entropy inequality [27] and it is the basic ingredient to show that the Hamiltonian is non-increasing along solutions, see Theorem 3.3. However, to imitate the behavior of the DFM from the TFM, the expression for M_{ig} in (11) is much more complex and it is challenging to analytically investigate the sign of the term $\int_\Omega M_{ig}(v_\ell - v_g)\, \mathrm{d}x$ that appears in (54).

If the term $\int_\Omega M_{ig}(v_\ell - v_g)\, \mathrm{d}x$ is always positive, it can be claimed that the dissipation inequality $\mathrm{d}\mathcal{H}/\mathrm{d}t \le 0$ also holds for the (general) DFM (using (54)). It is worth recalling that the dissipation inequality $\mathrm{d}\mathcal{H}_D/\mathrm{d}t \le 0$ holds for the DFM under zero slip considerations (see Theorem 3.4).

As the theoretical assessment of the term $\int_\Omega M_{ig}(v_\ell - v_g)\, \mathrm{d}x$ for the model with non-zero slip is highly involved, we investigate its behavior numerically. In order to calculate μ_g, μ_ℓ and ζ as in Theorem 2.4, the same expressions as computed in [15] are used. The Rusanov scheme [28] together with Zuber-Findlay slip (with $K = 1.07$ and $S = 0.216$ m/s cf. (8)) is used to solve the DFM numerically in a horizontal 1000 m-long spatial domain with the spatial and temporal step size of 0.5 m and 0.0005 s, $p_{\ell 0} = 1$ bar, $\rho_{\ell 0} = 1000$ kg/m^3, $c_\ell = 1000$ m/s, and $c_g = 316$ m/s. We consider periodic boundary conditions with the initial condition as shown in Figure 1. We use this test case to draw a concrete conclusion on the sign of $\int_\Omega M_{ig}\left(v_\ell - v_g\right) \mathrm{d}x$. As obvious from Figure 1, we have found a counter example for which this integral is negative for all time instants.

The numerical results indicate that the proposed Hamiltonian $\mathcal{H}_D$ with periodic boundary considerations does not guarantee the non-increasing behavior of the Hamiltonian functional along solutions of the DFM. A possible underlying reason for this effect could be that the Hamiltonian (22) (under zero gravitational contribution) is not suitable for the DFM with the Zuber-Findlay slip. However, the Hamiltonian $\mathcal{H}_D$ has the interpretation of the energy. The increment in this energy along the solutions in principle disqualifies the DFM for such slip conditions as an energy-consistent model for two-phase flow. Hence, we do not consider the

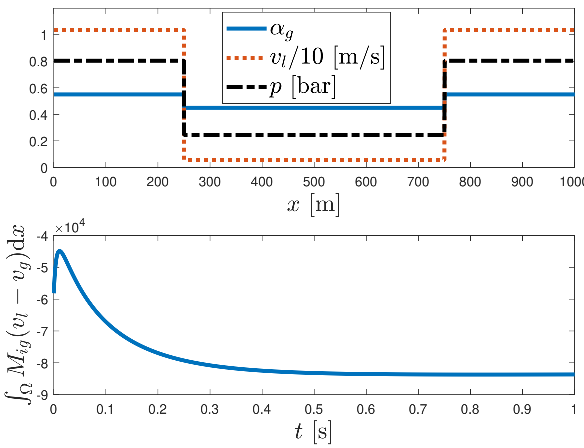

Figure 1: (top) Initial condition and (bottom) the temporal evolution of $\int_\Omega M_{ig}(v_\ell - v_g)\mathrm{d}x$ for the DFM with periodic boundary conditions.

general case of the DFM and only focus on a special case of the model, i.e., the model without slip.

6. Conclusions

We introduced a dissipative Hamiltonian formulation for two variants of multi-phase flow models, i.e., the Two-Fluid Model (TFM) and the no-slip Drift Flux Model (DFM) across a constant cross-section. Moreover, we presented Stokes-Dirac structure representations corresponding to the skew-adjoint operators obtained both for the TFM and for the DFM without slip (under certain choice of state-variables) along with the proof of corresponding representation for the TFM. Port-Hamiltonian representations for the multi-phase models are implicitly represented in terms of the Stokes-Dirac structures. Additionally, we numerically reasoned, by exploiting a connection to the TFM, to support the consideration of the DFM without slip.

Elegantly parametrizing the boundary port-variables for a class of state-dependent Stokes-Dirac structures is one important research direction for the future. The construction of structure-preserving surrogate models will be another focus of future work. This will open up possibilities for the analysis and control of complex physical systems.

References

[1] U. Aarsnes, Modeling of two-phase flow for estimation and control of drilling operations, PhD thesis, NTNU, 2016.

[2] U. J. F. Aarsnes, F. Di Meglio, S. Evje, O. M. Aamo, Control-Oriented Drift-Flux Modeling of Single and Two-Phase Flow for Drilling, American Society of Mechanical Engineers, San Antonio, Texas, USA, 2014, p. V003T37A003.

[3] A. Poullikkas, Effects of two-phase liquid-gas flow on the performance of the nuclear reactor cooling pumps, Progress in Nuclear Energy 42 (1) (2003) 3–10.

[4] B. Maschke, A. van der Schaft, Port-Controlled Hamiltonian Systems: modelling origins and system-theoretic properties, IFAC Proceedings Volumes 25 (1992) 359 – 365.

[5] A. van der Schaft, B. Maschke, Hamiltonian formulation of distributed-parameter systems with boundary energy flow, J. Geom. Phys 42 (2002) 166–194.

[6] S. Chaturantabut, C. Beattie, S. Gugercin, Structure-Preserving Model Reduction for Nonlinear Port-Hamiltonian Systems, SIAM J. Sci. Comput 38 (5) (2016) B837–B865.

[7] R. Pasumarthy, A. van der Schaft, A Finite Dimensional Approximation of the shallow water Equations: The port-Hamiltonian Approach, Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, CA, USA, 2006, pp. 3984–3989.

[8] A. Macchelli, C. Melchiorri, Modeling and control of the timoshenko beam. the distributed port hamiltonian approach, SIAM J. Control. Optim 43 (2) (2004) 743–767.

[9] R. Altmann, P. Schulze, A port-Hamiltonian formulation of the Navier-Stokes equations for reactive flows, Syst. Control. Lett 100 (2017) 51–55.

[10] W. Zhou, B. Hamroun, Y. Le Gorrec, F. Couenne, Infinite Dimensional Port Hamiltonian Representation of reaction diffusion processes, IFAC-PapersOnLine 48 (2015) 476–481.

[11] F. Cardoso-Ribiero, D. Matignon, V. Pommier-Buddinger, Modeling of a fluid-structure coupled system using port-Hamiltonian formulation, IFAC-PapersOnLine 48 (13) (2015) 217–222.

[12] V. Martins, B. Maschke, Y. Le Gorrec, Hamiltonian approach to the stabilization of systems of two conservation laws, IFAC Proceedings Volumes 43 (14) (2010) 581–586.

[13] N. M. Trang VU, L. Lefevre, B. Maschke, Port-Hamiltonian formulation for systems of conservation laws: application to plasma dynamics in Tokamak reactors, IFAC Proceedings 45 (2012) 108–113.

[14] D. D. Holm, B. A. Kupershmidt, Hydrodynamics and electrohydrodynamics of adiabatic multiphase fluids and plasmas, Int. J. Multiphas. Flow 12 (4) (1986) 667–680.

[15] S. Evje, T. Flatten, On the wave structure of two-phase flow models, SIAM J. Appl. Math 67(2) (2007) 487–511.

[16] H. de Wilde, Port-Hamiltonian discretization of gas pipeline networks, Masters thesis, University of Groningen, 2015.

[17] Y. Le Gorrec, H. Zwart, B. Maschke, Dirac structures and boundary control systems associated with skew-symmetric differential operators, SIAM J. Control. Optim 44 (5) (2005) 1864–1892.

[18] B. Jacob, H. Zwart, Linear port-Hamiltonian systems on infinite-dimensional spaces, Springer, New York, 2012.

[19] S. Evje, K. K. Fjelde, Hybrid Flux-Splitting Schemes for a Two-Phase Flow Model, J. Comput. Phys 175 (2002) 674–701.

[20] V. Duindam, A. Macchelli, S. Stramigioli, H. Bruyninckx (Eds.), Modeling and control of complex physical systems: the port-Hamiltonian approach, Springer, Berlin, 2009.

[21] Y. Cengel, M. A. Boles, Thermodynamics: An Engineering Approach, New York City, NY, USA, 2006.

[22] E. Tassi, Hamiltonian closures in fluid models for plasmas, Eur. Phys. J. D 71 (269).

[23] M. Abbasi, H. Bansal, H. Zwart, L. Iapichino, W. Schilders, N. van de Wouw, Power-preserving interconnection of single- and two-phase flow models for managed pressure drilling, Accepted for American Control Conference (ACC 2020).

[24] M. Kurula, A. van der Schaft, H. Zwart, Composition of infinite-dimensional dirac structures, 17th International Symposium on Mathematical Theory of Networks and Systems, MTNS (2006) 27–32.

[25] J. Villegas, A Port-Hamiltonian approach to distributed parameter systems, PhD thesis, 2007.

[26] H. Bansal, H. Zwart, L. Iapichino, W. Schilders, N. van de Wouw, Port-Hamiltonian modelling of fluid dynamics models with variable cross-section, Submitted to 24th International Symposium on Mathematical Theory of Networks and Systems, MTNS (2020).

[27] M. Ishii, T. Hibiki, Thermo-Fluid Dynamics of Two-Phase Flow, Springer, New York, NY, USA, 2006.

[28] V. Rusanov, Calculation of Interaction of Non-steady Shock Waves with Obstacles, 1961.