

# A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control<sup>☆</sup>

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## Abstract

In this paper, popular model reduction techniques from the fields of structural dynamics, numerical mathematics and systems and control are reviewed and compared. The motivation for such a comparison stems from the fact the model reduction techniques in these fields have been developed fairly independently. In addition, the insight obtained by the comparison allows for making a motivated choice for a particular model reduction technique, on the basis of the desired objectives and properties of the model reduction problem. In particular, a detailed review is given on mode displacement techniques, moment matching methods and balanced truncation, whereas important extensions are outlined briefly. In addition, a qualitative comparison of these methods is presented, hereby focussing both on theoretical and computational aspects. Finally, the differences are illustrated on a quantitative level by means of application of the model reduction techniques to a common example.

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## 1. Introduction

An important tool in the design of complex high-tech systems is the numerical simulation of predictive models. However, these dynamical models are typically of high order, i.e. they are described by a large number of ordinary differential equations. This results from either the inherent complexity of the system or the discretization of partial differential equations. Model reduction can be used to find a low-order model that approximates the behavior of the original high-order model, where this low-order approximation facilitates both the computationally efficient analysis and controller design for the system to induce desired behavior.

The earliest methods for model reduction belong to the field of structural dynamics, where the dynamic analysis of structures is of interest. Typical objectives are the identification of eigenfrequencies or the computation of frequency response functions. Besides the mode displacement reduction method and extensions thereof (see e.g. [1, 2]), important techniques are given by component mode synthesis techniques [3, 4], which started to emerge in the 1960s.

The model reduction problem has also been studied in the systems and control community, where the analysis of dynamic systems and the design of feedback controllers are of interest. Some of the most important contributions were made in the 1980s by the development of balanced truncation [5, 6] and optimal Hankel norm approximation [7].

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<sup>☆</sup>This research is financially supported by the Dutch Technology Foundation STW.

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Finally, numerically efficient methods for model reduction have been developed in the field of numerical mathematics in the 1990s. Important techniques are asymptotic waveform evaluation [8], Padé-via-Lanczos [9] and rational interpolation [10]. These methods are often applied in the design and analysis of large electronic circuits.

Despite the fact that the above techniques essentially deal with the same problem of model reduction, the results in the fields of structural dynamics, systems and control and numerical mathematics have largely been developed independently. This paper aims at providing a thorough comparison between the model reduction techniques from these three fields, facilitating the choice of a suitable reduction procedure for a given reduction problem. To this end, the most popular methods from the fields of structural dynamics, systems and control and mathematics will be reviewed. Then, the properties of these techniques will be compared, where both theoretical and numerical aspects will be discussed. In addition, these differences and commonalities will be illustrated by means of application of the model reduction techniques to a common example.

Reviews of model reduction techniques exist in literature. However, these reviews mainly focus on methods from the individual fields, i.e. they focus on methods from structural dynamics [11, 12], systems and control [13] or numerical mathematics [14, 15] only. Nonetheless, methods from systems and control and numerical mathematics are reviewed and compared in [16, 17, 18], where the comparison is mainly performed by the application of the methods to examples. In the current paper, popular model reduction techniques from all the three fields mentioned above will be reviewed. Additionally, both a qualitative and quantitative comparison will be provided. The focus of this paper is on this comparison; it does not aim at presenting a full comprehensive historical review of all method in these three domains.

In this paper, the scope will be limited to *model-based* reduction techniques for *linear time-invariant systems*. Consequently, the data-based model reduction technique of proper orthogonal decomposition [19, 20] will not be discussed. For an overview of proper orthogonal decomposition, see [21, 22]. Also, reduction methods for nonlinear systems (see e.g. [23, 24]) fall outside the scope of this paper.

The outline of this paper is as follows. First, the most important model reduction techniques from the fields of structural dynamics, numerical mathematics and systems and control will be reviewed in Section 2. In Section 3, a qualitative comparison between these methods will be provided, focussing on both theoretical and numerical aspects. This comparison will be illustrated by means of examples in Section 4, which further clarifies differences and commonalities between methods. Finally, conclusions will be stated in Section 5.

*Notation.* The field of real numbers is denoted by  $\mathbb{R}$ , whereas  $\mathbb{C}$  represents the field of complex numbers. Boldface letters are used to represent vectors and matrices, where the latter are printed in upper case. For a vector  $\mathbf{x}$ , the Euclidian norm is denoted by  $\|\mathbf{x}\|$ , i.e.  $\|\mathbf{x}\|^2 = \mathbf{x}^T \mathbf{x}$ . The  $\mathcal{H}_\infty$  norm of a system is denoted by  $\|\cdot\|_\infty$ .

## 2. Review of model reduction techniques

In this section, popular model reduction techniques from different fields are discussed. In Section 2.1, methods from structural dynamics are discussed, whereas model reduction techniques from the fields of numerical mathematics and systems and control are discussed in Sections 2.2 and 2.3, respectively.

### 2.1. Mode displacement methods

In the field of structural dynamics, the design and performance evaluation of mechanical systems is of interest. Herein, the computation of deformations, internal stresses or dynamic properties are subject of analysis. Even though the goal of analysis might differ from one specific application to another, important objectives are the prediction of regions with high stress, prediction of the eigenvalues (related to resonance frequencies) and eigenvectors (related to structural eigenmodes) and the computation of the system's response to a certain excitation.

All of the above mentioned goals share a common property. Namely, the models used in the design must, generally, contain detailed information for the precise description of the response properties of the structure. The mathematical models are basically constructed in terms of partial differential equations. These equations might be solved exactly only for simple problems and one has to resort to discretization-based approaches, such as the finite element method (FEM) or the boundary element method (BEM). In the context of this section, the finite element related concepts are of interest.

Model reduction methodologies are efficiently used in the structural field since the 19th century. The most common methods are mode superposition methods [1], in which a limited number of free vibration modes of the structure is used to represent the displacement pattern [25]. There are also improvements of the original mode superposition method by the addition of different vectors to the expansion procedure, such as the mode acceleration or modal truncation augmentation [1, 26]. Mode superposition methods are generally considered for the complete structure. However, it is common to partition the structure in some components, on which model reduction is performed individually. Then, these reduced-order component models are coupled to represent the global behavior. These methods are all together named component mode synthesis techniques. These methods are extensions of common mode superposition methods to the partition level where the forces on the partition boundaries replace the general forces on the whole structure. In [27] and [3], Hurty provided a general method for component mode synthesis techniques. Craig and Bampton, in [4], used the static deformation shapes of the structure with respect to its boundary loads and enriched this space with the internal dynamic mode shape vectors to increase the accuracy. This method is known as the *fixed-interface reduction method* because the modes of the system are found while all the boundaries are fixed. Later on, the works of MacNeal [28] and Rubin [29] extended these methods to a class of methods known as *free-interface methods*. In these methods, the dynamic mode shape vectors used in the basis are computed without the application of any restraints on the component boundaries, where in fixed-interface methods the boundary degrees of freedom are all fixed. A recent general overview on dynamic substructuring methods can be found in [12]. Another overview that summarizes the component mode synthesis approaches can be found in [11].

Discretization-based methods, such as FEM, analyze complex engineering problems by constructing piece-wise approximation polynomials over the spatial domain and solve for the unknown variables at specific locations of the discretization, known as node points [30]. This representation might already be considered to be a model order reduction process in itself. Namely, the displacement  $u(z, t)$ , which is dependent on the spatial variable  $z$  and time  $t$ , is represented by the finite expansion

$$u(z, t) = \sum_{j=1}^N \Psi_j(z) q_j(t). \quad (1)$$

Herein,  $\Psi_j(z)$  are linearly independent functions representing the displacement shape of the structure, where it is noted that they satisfy the essential boundary conditions of the problem. Next,  $q_j(t)$  are the unknown functions of time, whereas  $N$  represents the number of functions exploited in the representation. Since the representation of a body consists of infinitely many points (and therefore infinitely many degrees of freedom), the finite expansion (1) has already accomplished the task of reducing the system to a finite number of degrees of freedom.

The discretization of the differential equations of the problem results in the equations of motion of the system, which are typically of the following form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}, \quad (2)$$

where  $\mathbf{M} \in \mathbb{R}^{N \times N}$  and  $\mathbf{K} \in \mathbb{R}^{N \times N}$  represent the mass and stiffness matrices, respectively. Furthermore,  $\mathbf{q} \in \mathbb{R}^N$  represents the unknown displacements of the structure and  $\mathbf{f} \in \mathbb{R}^N$  is the externally applied generalized force vector. Structural systems possess, most of the time, light damping and the reduction typically is based on the undamped system. Therefore, undamped

systems of the form (2) are considered in this section. However, it is stressed that this is only suitable when the system is lightly damped and the eigenfrequencies are well separated [2].

In general, a detailed problem representation and the use of a high number of elements in the discretization result in large matrices and, hence, in long computation times. Model order reduction methods are used to efficiently reduce the system size and, as a consequence, achieve acceptable computation times. Reduction methods in structural dynamics may be classified into two classes, namely, methods related to mode superposition and methods related to component mode synthesis techniques. In this section, the context is limited to mode superposition methods, since they apply to the full system. This enables a comparison with methods from the fields of numerical mathematics and systems and control. More information on component mode synthesis can be found in [4].

Mode superposition methods share the common property that they use a small number of free vibration modes to represent the dynamics of the structure with some reduced number of generalized degrees of freedom. With this selection, one represents the solution vectors as a summation of free vibration modes that form a linearly independent set. This operation therefore reduces the system size to be solved and could result in important computational gains. However, there are some important points to note on the expansion procedures used in practice [31], namely:

1. the used mode shape vectors do not span the complete space;
2. the computation of eigenvectors for large systems is very expensive and time consuming;
3. the number of eigenmodes required for satisfactory accuracy is difficult to estimate a priori, which limits the automatic selection of eigenmodes;
4. the eigenbasis ignores important information related to the specific loading characteristics such that the computed eigenvectors can be nearly orthogonal to the applied loading and therefore do not participate significantly in the solution.

Three different main variants can be considered which are often used in structural dynamics community. These are the mode displacement method, mode acceleration method and modal truncation augmentation method. The latter two methods are enhancements of the mode displacement method with the addition of the contribution of the omitted parts in an expansion process.

Generally, these methods do not propose the computation of an error bound for the response studies. Consequently, the success of the methods is established on the basis of a posteriori error comparisons. Typically, either the errors on the eigenfrequencies or the errors on the input-output representation are used to show the success of the applied method. In the following sections, the mode displacement method, the mode acceleration methods and the modal truncation augmentation method will be treated in more detail.

### 2.1.1. Mode displacement method

The equation of motion of the structure (2) is recalled:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}.$$

Then, the mode displacement method is based on the free vibration modes of the structure, which can be found by using a time-harmonic representation for the displacement of the unforced system (i.e.  $\mathbf{f} = \mathbf{0}$ ). This leads to the generalized eigenvalue problem

$$(\mathbf{K} - \omega_j^2 \mathbf{M}) \phi_j = \mathbf{0}, \quad (3)$$

where  $\phi_j$  is the mode shape vector corresponding to the eigenfrequency  $\omega_j$ , with  $j \in \{1, \dots, N\}$ . Using the expansion concept along with the mode shape vectors  $\phi_j$ , the displacement can be represented as follows:

$$\mathbf{q} = \sum_{j=1}^N \phi_j \eta_j, \quad (4)$$

where it is recalled that  $N$  is the size of the system. Here,  $\eta_j$  is typically referred to as a set of modal coordinates. It is a common practice to mass-orthogonalize the mode shape vectors, resulting in

$$\begin{aligned}\phi_i^T \mathbf{M} \phi_j &= \delta_{ij}, \\ \phi_i^T \mathbf{K} \phi_j &= \delta_{ij} \omega_j^2,\end{aligned}\tag{5}$$

where  $\delta_{ij}$  denotes the Kronecker delta. These orthogonality relations are used to decouple the coupled equations of motion (2). Using (5), the decoupled equations are represented in modal coordinates as

$$\ddot{\eta}_j + \omega_j^2 \eta_j = \phi_j^T \mathbf{f}, \quad j \in \{1, \dots, N\}.\tag{6}$$

An important practical point on the expansion method is related to the computation of the expansion vectors. The computation of the mode shape vectors that are used in the mode superposition methods can be an expensive task and, in practice, all the computational methods extract a limited number  $K$  of vectors of the eigenvalue problem. The general idea of the expansion procedure is to keep the first  $K$  vectors in the representation, that correspond to the lowest eigenfrequencies. This results in a truncation, namely,

$$\mathbf{q} = \sum_{j=1}^K \phi_j \eta_j + \underbrace{\sum_{j_t=K+1}^N \phi_{j_t} \eta_{j_t}}_{\text{truncated}},\tag{7}$$

where the indices  $j$  and  $j_t$  represent the kept mode and the truncated mode indices, respectively.

Since the displacement is represented as a linear combination of  $K$  linearly independent vectors, it can also be given in matrix notation, leading to the approximation

$$\mathbf{q} = \Phi \boldsymbol{\eta}, \quad \Phi = [\phi_1 \ \phi_2 \ \dots \ \phi_K].\tag{8}$$

Using (2) and (8) and projecting the resulting equations of motion on the expansion basis  $\Phi$  results in the following reduced-order dynamics

$$\mathbf{M}_r \ddot{\boldsymbol{\eta}} + \mathbf{K}_r \boldsymbol{\eta} = \mathbf{f}_r,\tag{9}$$

where

$$\mathbf{M}_r = \Phi^T \mathbf{M} \Phi = \mathbf{I},\tag{10}$$

$$\mathbf{K}_r = \Phi^T \mathbf{K} \Phi = \text{diag}\{\omega_1^2, \dots, \omega_K^2\},\tag{11}$$

$$\mathbf{f}_r = \Phi^T \mathbf{f}.\tag{12}$$

In general, the analysts are interested in the response properties of the system for the lower frequency range and therefore, the lowest modes are typically chosen. The reason behind this selection is the fact that most structures are operated at low frequencies.

The importance of a mode is mostly related to two concepts. First, the orthogonality of the mode with respect to the excitation, as given by  $\phi_j^T \mathbf{f}$ , is of importance. Secondly, the closeness of the eigenfrequency of the mode with respect to the excitation spectrum is of interest.

### 2.1.2. Mode acceleration method

The mode acceleration method is a computational variant of the static correction method. The static correction method aims at taking into account the contribution of the omitted modes. The driving idea of the static correction concept is to be able to include the effects of the truncated modes statically into the summation procedure. Namely, truncated modes have a static contribution on the response for low frequencies. This results in an improvement for the response studies

in the lower frequency range. The response might be represented as before but with a correction term  $\mathbf{q}_{cor}$ , such that

$$\mathbf{q} = \Phi \boldsymbol{\eta} + \mathbf{q}_{cor}. \quad (13)$$

To obtain the static correction term  $\mathbf{q}_{cor}$  (with  $\dot{\mathbf{q}}_{cor} = \ddot{\mathbf{q}}_{cor} = \mathbf{0}$ ), the truncated representation for the acceleration is substituted in the equation of motion (2), leading to

$$\mathbf{M} \sum_{j=1}^K \phi_j \ddot{\eta}_j + \mathbf{K} \mathbf{q} = \mathbf{f}. \quad (14)$$

Then, the use of the (reduced-order) dynamics in modal coordinates (6) leads to

$$\begin{aligned} \mathbf{q} &= \mathbf{K}^{-1} \left( \mathbf{f} - \mathbf{M} \sum_{j=1}^K \phi_j (\phi_j^T \mathbf{f} - \omega_j^2 \eta_j) \right), \\ &= \sum_{j=1}^K \phi_j \eta_j + \left( \mathbf{K}^{-1} - \sum_{j=1}^K \frac{\phi_j \phi_j^T}{\omega_j^2} \right) \mathbf{f}, \end{aligned} \quad (15)$$

where the relation imposed by the eigenvalue problem (3) is used in the latter step. When comparing (15) to the truncation (7), it is observed that the correction term is given as

$$\mathbf{q}_{cor} = \left( \mathbf{K}^{-1} - \sum_{j=1}^K \frac{\phi_j \phi_j^T}{\omega_j^2} \right) \mathbf{f}. \quad (16)$$

It is noted that, by using all eigenmodes, the inverse of the stiffness matrix can be represented as [2]

$$\mathbf{K}^{-1} = \sum_{j=1}^N \frac{\phi_j \phi_j^T}{\omega_j^2}, \quad (17)$$

such that the use of (17) in (15) results in

$$\mathbf{q} = \sum_{j=1}^K \phi_j \eta_j + \sum_{j=K+1}^N \frac{\phi_j \phi_j^T}{\omega_j^2} \mathbf{f}. \quad (18)$$

Even though this last form is not applicable in practice, since it requires the computation of all model vectors, it clearly shows that only the static contribution of the omitted modes  $\phi_j$ ,  $j \in \{K+1, \dots, N\}$ , is taken into account in the correction term  $\mathbf{q}_{cor}$ .

### 2.1.3. Modal truncation augmentation method

The modal truncation augmentation method is an extension of the mode acceleration method. Its main principle depends on the use of the static correction as an additional direction for the truncation expansion [32, 33, 34]. Inclusion of the correction in a modal expansion results in the modal truncation augmentation method, such that  $\mathbf{q}$  is approximated as

$$\mathbf{q} = \sum_{j=1}^K \phi_j \eta_j + \mathbf{q}_{cor} \xi, \quad (19)$$

where  $\mathbf{q}_{cor}$  is given by the mode acceleration method in (18) and  $\xi$  is an additional coordinate in the reduced-order system. This correction vector is included in the reduction basis, such that the new reduction basis reads

$$\Psi = [\Phi \ \mathbf{q}_{cor}]. \quad (20)$$

Here, it is noted that  $\Psi$  is generally orthogonalized.

Modal truncation augmentation methods are mostly used when there are multiple forcing vectors acting on the system. Therefore, these correction vectors are not used a posteriori as in the mode acceleration method but they really become a part of the reduction space.

There exist also further extensions of the common mode superposition methods which include higher-order correction vectors. These methods are outlined in [33] and references therein. See Section 3.3 for further details.

## 2.2. Krylov subspace based model order reduction

Krylov subspace based model order reduction (MOR) methods are methods which reduce a system with many degrees of freedom (i.e. states) to a system with few(er) degrees of freedom but with similar *input-output behavior*. Typical applications are large electronic circuits with large linear subnetworks of components (see e.g. [35, 36]) and micro-electro-mechanical systems (MEMS). The main purpose of Krylov methods is the construction of an approximation of the system's transfer function which (accurately enough) describes the dependence between the input and the output of the original system, e.g. in some range of the (input) frequency domain. Methods of this type are based on projections onto a Krylov subspace and are (relatively) computationally cheap compared to other reduction techniques, for instance because they can effectively exploit parallel computing. The objective is the derivation of a smaller system with similar input-output behavior and with similar properties such as stability, passivity or a special structure of the matrices in the model description. The quality of the reduced-order approximation can be assessed by studying norms of the difference between the outputs of the unreduced and reduced models applied for the same inputs. Preservation of additional properties is of importance if the reduced system has to exhibit some physical properties of the model; for instance, when the reduced system has to be a (realizable) circuit consisting out of resistors, inductors and capacitors (a RLC network), just as the original system. So far, there have been no proven a priori error-bounds for the Krylov based reduction techniques, see [37] for more details and for alternative approaches, to ensure a good (application domain dependent) approximation.

The first reduction method involving the usage of the Krylov subspace, called asymptotic waveform evaluation, was described in 1990, see [8]. However, the main focus of this paper was on finding a Padé approximation of the transfer function rather than on the construction of a Krylov subspace. Later, in 1995, in [9] a method called Padé via Lanczos (PVL) was proposed and the relation between the Padé approximation and Krylov subspace was shown. In 1998, a new reduction technique, PRIMA, was introduced in [35], that uses the Arnoldi algorithm instead of Lanczos to build the reduction bases. These and later developments of Krylov based reduction techniques focus not only on the improvement of the accuracy of the approximation, but also on the preservation of the properties of the system to be reduced.

In this section, the basic ideas of model reduction by projection onto the Krylov subspace are explained and the application of some common reduction techniques based on Arnoldi and Lanczos algorithms (see e.g. [38] for more details), is briefly discussed.

Linear time-invariant state-space systems of the form

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y = \mathbf{c}^T\mathbf{x}, \end{cases} \quad (21)$$

are considered, with  $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ , the input variable  $u \in \mathbb{R}$ , the output variable  $y \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$  being a vector of the state variables. For the sake of simplicity, SISO systems (with scalar input and scalar output) are considered. However, the methods discussed in this section have been extended to multi-input-multi-output (MIMO) cases (see e.g. [39]).

If the system (21) is transformed to the Laplace domain, then, for an arbitrary  $s \in \mathbb{C}$ , the dependence between its input and its output is given by a transfer function  $H(s)$  defined as follows

$$H(s) = \mathbf{c}^T (s\mathbf{E} - \mathbf{A})^{-1} \mathbf{b}. \quad (22)$$

In this section, it is assumed that the pencil  $(s\mathbf{E} - \mathbf{A})$  is regular, i.e. it is singular only for a finite number of  $s \in \mathbb{C}$ . For an arbitrary  $s_0 \in \mathbb{C}$ , the transfer function (22) may be rewritten in a polynomial form, using a so-called moment expansion:

$$H(s) = \sum_{n=0}^{\infty} (-1)^n M_n(s_0) (s - s_0)^n. \quad (23)$$

Here, the coefficients  $M_n(s_0)$ , called moments of the transfer function, are calculated using the Taylor expansion formula and given by

$$M_n(s_0) = \mathbf{c}^T [(s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}]^n (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}. \quad (24)$$

Expansion around  $s_0 = \infty$  is evaluated based on Laurent series, and the moments then are called Markov parameters (see [16] for more details). The accuracy of the moment expansion depends on the choice of the expansion point  $s_0$ . It is also possible to use a multipoint expansion choosing multiple expansion points.

The goal of the Krylov subspace model order reduction is to find a projection-based approximation of the original transfer function, that matches the first  $k$  moments of the original transfer function. In other words, the objective is to calculate the reduced-order transfer function  $\hat{H}(s)$ , whose moment expansion is given by

$$\hat{H}(s) = \sum_{n=0}^{\infty} (-1)^n \hat{M}_n(s_0) (s - s_0)^n, \quad (25)$$

with

$$\hat{M}_n(s_0) = M_n(s_0), \quad \text{for } n = 1, \dots, k, \quad (26)$$

and  $M_n(s_0)$  being the moments of the original transfer function defined in (24). This is called the moment matching property of the reduction method.

In case of the reduction methods studied in this section, the reduced-order model is calculated using a projection  $\mathbf{\Pi} = \mathbf{V}\mathbf{W}^T \in \mathbb{R}^{n \times n}$ , with  $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times k}$  being biorthogonal matrices, i.e.  $\mathbf{W}^T\mathbf{V} = \mathbf{I}$ . Application of the projection  $\mathbf{\Pi}$  to the original system (21) gives

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\hat{\mathbf{x}}} = \mathbf{W}^T\mathbf{A}\mathbf{V}\hat{\mathbf{x}} + \mathbf{W}^T\mathbf{b}u, \\ \hat{\mathbf{y}} = \mathbf{c}^T\mathbf{V}\hat{\mathbf{x}}, \end{cases} \quad (27)$$

where the reduced-order state vector  $\hat{\mathbf{x}} \in \mathbb{R}^k$  results from the state transformation

$$\mathbf{x} \approx \mathbf{V}\hat{\mathbf{x}}. \quad (28)$$

The choice of the spaces  $\mathbf{V}$  and  $\mathbf{W}$  depend on the goal of the reduction procedure. In case of the Krylov subspace based methods, the aim is to approximate the input-output behavior of the system. This is done by matching the moments of the original transfer function. This means that the reduced-order transfer function corresponding to system (27), which results from applying matrices  $\mathbf{V}$  and  $\mathbf{W}$  to the original system matrices, has the property (26). To ensure the satisfaction of the moment matching property (26), one can choose  $\mathbf{V}$  and  $\mathbf{W}$  such that the columns of these matrices span so-called Krylov subspaces. The  $k$ -th Krylov subspace induced by a matrix  $\mathbf{P}$  and a vector  $\mathbf{r}$  is defined as

$$\mathcal{K}_k(\mathbf{P}, \mathbf{r}) = \text{span}\{\mathbf{r}, \mathbf{P}\mathbf{r}, \dots, \mathbf{P}^{k-1}\mathbf{r}\}. \quad (29)$$

The choice of the starting matrix  $\mathbf{P}$  and the starting vector  $\mathbf{r}$  depends on the value  $s_0$  around which the transfer function should be approximated. If the approximation of the transfer function (22) around  $s_0 = 0$  is to be found, the matrices  $\mathbf{V}$  and  $\mathbf{W}$  are chosen as follows:

$$\mathbf{V} \text{ is a basis of } \mathcal{K}_{k_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b}), \quad (30)$$

$$\mathbf{W} \text{ is a basis of } \mathcal{K}_{k_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c}). \quad (31)$$

The sizes of the subspaces,  $k_1$  and  $k_2$ , should assure that  $\mathbf{V}$  and  $\mathbf{W}$  are both of rank  $k$ . If  $\mathbf{V}$  and  $\mathbf{W}$  are built in the way defined in (30-31), the model reduction method is called a two-sided method. If only one of the projection matrices ( $\mathbf{V}$  or  $\mathbf{W}$ ) is built in that way, the method is called one-sided. Application of the two-sided method results in a reduced model that matches the first  $2k$  moments of the original transfer function. In case of one-sided methods,  $k$  moments are matched.

The general proof of the moment matching property can be found in [10]. To illustrate the idea behind this proof, the matching of the zeroth moment of the system (21) for  $s_0 = 0$  is shown following [40].

According to the formula (24), the zeroth moment for  $s_0 = 0$  is equal to

$$M_0(0) = -\mathbf{c}^T \mathbf{A}^{-1} \mathbf{b}. \quad (32)$$

With  $\mathbf{V}$  chosen as in (30) and the fact that  $\mathbf{A}^{-1} \mathbf{b}$  belongs to the Krylov subspace  $\mathcal{K}_{k_1}(\mathbf{A}^{-1} \mathbf{E}, \mathbf{A}^{-1} \mathbf{b})$ , one can find a vector  $\mathbf{r}_0$  such that  $\mathbf{V} \mathbf{r}_0 = \mathbf{A}^{-1} \mathbf{b}$ . Then, using the reduction procedure defined in (27), it can be shown that

$$\begin{aligned} \hat{M}_0(0) &= -\mathbf{c}^T \mathbf{V} (\mathbf{W}^T \mathbf{A} \mathbf{V})^{-1} \mathbf{W}^T \mathbf{b} = -\mathbf{c}^T \mathbf{V} (\mathbf{W}^T \mathbf{A} \mathbf{V})^{-1} \mathbf{W}^T \mathbf{A} \mathbf{V} \mathbf{r}_0 \\ &= -\mathbf{c}^T \mathbf{V} \mathbf{r}_0 = -\mathbf{c}^T \mathbf{A}^{-1} \mathbf{b} = M_0(0). \end{aligned} \quad (33)$$

In case the approximation around  $s_0 \neq 0$  or for  $s_0 = \infty$  is needed, the starting matrix and vector for building the Krylov subspace have to be modified. One can also build a subspace using different values of  $s_0$  at the same time. More details on how to do this and suggestions for starting values for different  $s_0$  can be found in [10].

Besides the difference in the number of moments matched, the choice to use either one- or two-sided methods influences also some other properties of the reduced system. Two-sided methods may lead to better approximations of the output  $y$  and deliver a reduced-order model, whose input-output behavior does not depend on the state space realization of the original model. In case of the one-sided techniques with  $\mathbf{W} = \mathbf{V}$  and  $\mathbf{V}$  defined as in (30), for certain original models, one can also prove the preservation of the passivity property.

The process of constructing the reduction matrices,  $\mathbf{V}$  and  $\mathbf{W}$ , is not straightforward and requires the use of special techniques. Because of round-off errors, the vectors building a Krylov subspace may quickly become linearly dependent. To avoid this problem, one usually constructs an orthogonal basis of the appropriate Krylov subspace. This can be achieved using e.g. Arnoldi or Lanczos algorithms (explanation of these algorithms and implementation details are given in [38]). The classical Arnoldi algorithm generates a set  $\mathbf{V}$  of orthonormal vectors, i.e.

$$\mathbf{V}^T \mathbf{V} = \mathbf{I}, \quad (34)$$

that form a basis for a given Krylov subspace. The Lanczos algorithm finds two sets of basis vectors,  $\mathbf{V}$  and  $\mathbf{W}$ , that span an appropriate Krylov subspace and have property

$$\mathbf{W}^T \mathbf{V} = \mathbf{I}. \quad (35)$$

Two sets of basis vectors  $\mathbf{V}$  and  $\mathbf{W}$  for Krylov subspaces may also be computed using a two-sided Arnoldi algorithm (see [40]). In this case, both  $\mathbf{V}$  and  $\mathbf{W}$  are orthonormal,

$$\mathbf{V}^T \mathbf{V} = \mathbf{I}, \quad \mathbf{W}^T \mathbf{W} = \mathbf{I}. \quad (36)$$

As a result, each of the above mentioned techniques generates a Krylov subspace. The choice of the subspace depends on the type of algorithm and the expansion point  $s_0$  around which the approximation is of interest. A more detailed explanation on how to choose the proper subspaces can be found in [10].

The ideas of the Krylov subspace based reduction presented in this section can be further modified, depending on e.g. the application or the specific criteria that the reduced-order model should fulfill. In electronic circuit design, there exist methods especially suited for reducing specific

types of systems that exploit the characteristic structure of the underlying matrices, see e.g. [41]. In case of coupled or interconnected systems, the goal may be to preserve the interpretation of the different physical domains. More details on this topic can be found in [36, 42]. There exist also modifications that aim at preserving other properties of the original system, such as stability or passivity. In case of symmetric matrices, the algorithm SyPVL was proposed in [43] that guarantees stability. A stability and passivity preserving technique, PRIMA, is presented in [35].

### 2.3. Balanced truncation

The field of systems and control focusses on the analysis of dynamical systems and design of feedback controllers for these systems. Herein, the objective of controller design is to change the dynamics of the system to induce desired behavior. Typical examples are the stabilization of unstable systems, tracking of a reference trajectory or the rejection of external disturbances on a system.

These control strategies are applied in a broad range of practical engineering problems, such as control of mechanical or electrical systems. These applications have in common that they deal with systems with inputs and outputs. On the one hand, a dynamical system can often only be influenced by a limited number of actuators, which are represented as inputs. Additionally, external disturbances, such as e.g. measurement noise, often act only locally as well. On the other hand, only a limited number of sensors (i.e. outputs) is available in practical engineering systems. For these systems, it is thus particularly relevant to have an accurate model for their input-output behavior. Even though this model does not need to describe the global behavior of a system, complex dynamics can still yield large models of orders up to  $\mathcal{O}(10^3)$ . To facilitate controller design and/or analysis for these systems, model reduction is needed. Here, it is noted that a controller needs to be implemented in real-time, which also requires a controller realization of relatively low-order.

Model reduction procedures in the field of systems and control therefore aim at approximating the *input-output behavior* of a high-order model. The quality of the reduced-order model can thus be assessed by comparing the outputs of the high-order and reduced-order models for given inputs, where the magnitude of the output error is measured using some signal norm.

Balanced truncation is the most popular method in systems and control addressing this model reduction problem. It mainly owes its popularity due to the fact that it preserves stability of the high-order model and provides an error bound, which gives a direct measure of the quality of the reduced-order model.

The balanced truncation method was first presented by Moore [5], where results of Mullis and Robberts [44] were exploited. Later, the stability preservation property was found by Pernebo and Silverman [45], whereas the error bound was derived by Enns [6] and Glover [7].

. Linear dynamical models with inputs and outputs in state-space form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \quad (37)$$

are considered. Here,  $\mathbf{u} \in \mathbb{R}^m$  denotes the input whereas  $\mathbf{y} \in \mathbb{R}^p$  represents the output. The internal state is given by  $\mathbf{x} \in \mathbb{R}^n$  and the system matrices are of corresponding dimensions. Throughout this section, it is assumed that the model (37) is asymptotically stable (i.e. all eigenvalues of  $\mathbf{A}$  have negative real part) and is a minimal realization, where the latter guarantees that all state components contribute to the input-output behavior. The transfer function of (37) is given as

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}, \quad s \in \mathbb{C}. \quad (38)$$

In balanced truncation, a reduced-order model is obtained in two steps. First, a so-called balanced realization is found, in which the states are ordered according to their contribution to the input-output behavior. Second, a reduced-order model is obtained on the basis of this balanced realization by discarding the states with the smallest influence.

In order to find the balanced realization, the input-output behavior of the system (37) has to be quantified. To this end, the so-called controllability and observability functions are defined. First, the controllability function  $\mathcal{E}_c(\mathbf{x}_0)$  gives the smallest input energy required to reach the state  $\mathbf{x}_0$  from the zero state in infinite time, given as

$$\mathcal{E}_c(\mathbf{x}_0) = \inf_{\substack{\mathbf{u} \in \mathcal{L}_2(-\infty, 0) \\ \mathbf{x}(-\infty) = \mathbf{0}, \mathbf{x}(0) = \mathbf{x}_0}} \int_{-\infty}^0 \|\mathbf{u}(t)\|^2 dt, \quad (39)$$

where  $\mathcal{L}_2(-\infty, 0)$  denotes the space of square integrable functions, defined on the domain  $(-\infty, 0)$ . Second, the observability function  $\mathcal{E}_o(\mathbf{x}_0)$  is defined by

$$\mathcal{E}_o(\mathbf{x}_0) = \int_0^{\infty} \|\mathbf{y}(t)\|^2 dt, \quad \mathbf{x}(0) = \mathbf{x}_0, \mathbf{u}(t) = \mathbf{0} \quad \forall t \in [0, \infty), \quad (40)$$

and gives the future output energy of the system when released from an initial condition  $\mathbf{x}_0$  for zero input. It is well-known (see e.g. [5, 46]) that for linear systems as in (37) the controllability and observability functions in (39) and (43) can be written as the quadratic forms

$$\mathcal{E}_c(\mathbf{x}_0) = \mathbf{x}_0^T \mathbf{P}^{-1} \mathbf{x}_0, \quad \mathcal{E}_o(\mathbf{x}_0) = \mathbf{x}_0^T \mathbf{Q} \mathbf{x}_0, \quad (41)$$

where  $\mathbf{P}$  and  $\mathbf{Q}$  are the controllability and observability gramian, given by

$$\mathbf{P} = \int_0^{\infty} e^{\mathbf{A}t} \mathbf{B} \mathbf{B}^T e^{\mathbf{A}^T t} dt \quad (42)$$

and

$$\mathbf{Q} = \int_0^{\infty} e^{\mathbf{A}^T t} \mathbf{C}^T \mathbf{C} e^{\mathbf{A}t} dt, \quad (43)$$

respectively. From (42) and (43), it is easily observed that the controllability and observability gramians are only finite when the system is asymptotically stable, which explains the assumption stated before. In addition,  $\mathbf{P}$  and  $\mathbf{Q}$  are symmetric and positive definite, where the latter is guaranteed by the assumption that the system (37) is minimal, i.e. controllable and observable. The controllability and observability gramian can be obtained as the unique positive definite solutions of the respective Lyapunov equations (see e.g. [46])

$$\mathbf{A} \mathbf{P} + \mathbf{P} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T = \mathbf{0} \quad (44)$$

and

$$\mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A} + \mathbf{C}^T \mathbf{C} = \mathbf{0}, \quad (45)$$

which makes balanced truncation computationally feasible. Nonetheless, solving the Lyapunov equations is computationally costly, such that balanced truncation is limited to systems of orders up to  $\mathcal{O}(10^3)$ .

Since both the controllability and observability gramian characterize the in- or output energy associated to a state  $\mathbf{x}_0$ , they are dependent on the realization of the system (37). Stated differently, a change of coordinates  $\bar{\mathbf{x}} = \mathbf{T} \mathbf{x}$ , with  $\mathbf{T} \in \mathbb{R}^{n \times n}$  a nonsingular matrix, results in a realization with system matrices

$$\bar{\mathbf{A}} = \mathbf{T} \mathbf{A} \mathbf{T}^{-1}, \quad \bar{\mathbf{B}} = \mathbf{T} \mathbf{B}, \quad \bar{\mathbf{C}} = \mathbf{C} \mathbf{T}^{-1}, \quad \bar{\mathbf{D}} = \mathbf{D}. \quad (46)$$

Then, the new controllability and observability gramians are given as

$$\bar{\mathbf{P}} = \mathbf{T} \mathbf{P} \mathbf{T}^T, \quad \bar{\mathbf{Q}} = \mathbf{T}^{-T} \mathbf{Q} \mathbf{T}^{-1}. \quad (47)$$

Nonetheless, the product of  $\bar{\mathbf{P}}$  and  $\bar{\mathbf{Q}}$  yields

$$\bar{\mathbf{P}} \bar{\mathbf{Q}} = \mathbf{T} \mathbf{P} \mathbf{Q} \mathbf{T}^{-1}, \quad (48)$$

indicating that the eigenvalues of the product of the controllability and observability gramian are independent of the set of coordinates and thus system invariants. These eigenvalues equal the (squared) Hankel singular values  $\sigma_i$  [7], such that

$$\sigma_i = \sqrt{\lambda_i(\mathbf{P}\mathbf{Q})}, \quad i = 1, \dots, n, \quad (49)$$

where  $\lambda_i(\mathbf{X})$  denotes the  $i$ -th eigenvalue of the matrix  $\mathbf{X}$ , ordered as  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$ .

At this point, it is recalled that the observability gramian  $\mathbf{Q}$  characterizes the output energy associated to a given initial state  $\mathbf{x}_0$  and thus provides a measure of the importance of state components with respect to the output  $\mathbf{y}$ . Hence, states generating high output energy can be considered more important than states generating little output energy, since the former are easy to observe. On the other hand, the controllability gramian  $\mathbf{P}$  gives a measure of the importance of state components  $\mathbf{x}_0$  with respect to the input  $\mathbf{u}$ , in the sense that states that require little input energy to reach are more relevant than states that require high input energy. States that require little energy to reach are thus easy to control. Clearly, the combination of the controllability and observability gramians gives a characterization of the importance of state components from an input-output perspective. However, in an arbitrary coordinate system, a state  $\bar{\mathbf{x}}_0^1$  that requires little energy to reach might also generate little output energy. On the other hand, a different state  $\bar{\mathbf{x}}_0^2$  might exist that requires a lot of energy to reach, but generates high output energy. In this case, it is not easy to decide which of  $\bar{\mathbf{x}}_0^1$  and  $\bar{\mathbf{x}}_0^2$  is the most important component from an input-output perspective. To facilitate this analysis, the balanced realization is introduced. Namely, there exists a state-space realization such that the corresponding controllability and observability gramians are equal and diagonal, where the entries on the diagonal are given by the Hankel singular values [5]:

$$\bar{\mathbf{P}} = \bar{\mathbf{Q}} = \bar{\Sigma} := \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}. \quad (50)$$

In addition, the Hankel singular values are ordered as  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$ . In this balanced realization, the controllability and observability function are given as

$$\mathcal{E}_c(\bar{\mathbf{x}}_0) = \bar{\mathbf{x}}_0^T \bar{\Sigma}^{-1} \bar{\mathbf{x}}_0, \quad \mathcal{E}_o(\bar{\mathbf{x}}_0) = \bar{\mathbf{x}}_0^T \bar{\Sigma} \bar{\mathbf{x}}_0. \quad (51)$$

Now, the form (51) allows for a clear interpretation. Namely, the realization is balanced in the sense that states that are easy to control are also easily observed. In fact, due to the ordering of the Hankel singular values, the state  $\bar{\mathbf{x}}_0 = \mathbf{e}_1 := [1, 0, \dots, 0]^T$  requires the least energy to reach ( $\mathcal{E}_c(\mathbf{e}_1) = \sigma_1^{-1}$  is small) and gives the highest output energy ( $\mathcal{E}_o(\mathbf{e}_1) = \sigma_1$  is large). Stated differently, this state is easy to control and easy to observe. Hence,  $\bar{\mathbf{x}}_0 = \mathbf{e}_1$  has the largest contribution to the input-output behavior of the system. On the other hand, the state  $\bar{\mathbf{x}}_0 = \mathbf{e}_n := [0, \dots, 0, 1]^T$  is both difficult to control and difficult to observe, such that it has the smallest contribution to the input-output behavior.

The coordinate transformation  $\mathbf{T}$  to obtain the balanced realization can be obtained on the basis of the controllability and observability gramians (42-43). Thereto, the Cholesky factor  $\mathbf{U}$  of  $\mathbf{P}$  is used, as well as the eigenvalue decomposition of  $\mathbf{U}^T \mathbf{Q} \mathbf{U}$ :

$$\mathbf{P} = \mathbf{U} \mathbf{U}^T, \quad \mathbf{U}^T \mathbf{Q} \mathbf{U} = \mathbf{K} \mathbf{S} \mathbf{K}^T. \quad (52)$$

In the latter, it is noted that  $\mathbf{U}^T \mathbf{Q} \mathbf{U}$  is a positive definite symmetric matrix, such that the matrix of eigenvectors  $\mathbf{K}$  is orthonormal. Additionally, the eigenvalues are real and, when ordered, are equal to the squared Hankel singular values such that  $\mathbf{S} = \bar{\Sigma}^2$  with  $\bar{\Sigma}$  as in (50). Then, the balancing transformation and its inverse are given as

$$\mathbf{T} = \bar{\Sigma}^{\frac{1}{2}} \mathbf{K}^T \mathbf{U}^{-1}, \quad \mathbf{T}^{-1} = \mathbf{U} \mathbf{K} \bar{\Sigma}^{-\frac{1}{2}} \quad (53)$$

as can be checked by substitution of (53) in (47), while using the relations (52). An overview of alternative algorithms to obtain the balanced realization can be found in [16].

. So far, a balanced realization is found, but no model reduction has been performed yet. However, the balanced realization gives a representation in which the states are ordered according to their contribution to the input-output behavior. Hence, a reduced-order model of order  $k$  can be obtained by partitioning the state  $\bar{\mathbf{x}}$  of the balanced realization as  $\bar{\mathbf{x}}^1 = [\bar{x}_1, \dots, \bar{x}_k]^T \in \mathbb{R}^k$  and  $\bar{\mathbf{x}}^2 = [\bar{x}_{k+1}, \dots, \bar{x}_n]^T \in \mathbb{R}^{n-k}$ , such that  $\bar{\mathbf{x}}^1$  contains the state components with the largest influence on the input-output behavior. When the system matrices are partitioned accordingly,

$$\Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{bmatrix}, \quad \bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix}, \quad \bar{\mathbf{C}} = [\bar{\mathbf{C}}_1 \quad \bar{\mathbf{C}}_2], \quad \bar{\mathbf{D}} = \mathbf{D}, \quad (54)$$

a reduced-order system can be obtained by truncation, i.e. by setting  $\bar{\mathbf{x}}_2 = \mathbf{0}$ . The resulting reduced-order model (with  $\hat{\mathbf{x}} \in \mathbb{R}^k$  an approximation of  $\bar{\mathbf{x}}^1 \in \mathbb{R}^k$ ) is given by the state-space realization

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \bar{\mathbf{A}}_{11}\hat{\mathbf{x}} + \bar{\mathbf{B}}_1\mathbf{u}, \\ \hat{\mathbf{y}} = \bar{\mathbf{C}}_1\hat{\mathbf{x}} + \bar{\mathbf{D}}\mathbf{u}. \end{cases} \quad (55)$$

Here, it can be observed that the reduced-order state-space system (55) is itself a balanced realization, with the controllability and observability gramians given by  $\Sigma_1$  (see [45]). In addition, when  $\Sigma_1$  and  $\Sigma_2$  have no diagonal entries in common (i.e. when  $\sigma_k > \sigma_{k+1}$ ), the reduced-order system is asymptotically stable [45].

The reduced-order system thus preserves stability of the original model, and its output  $\hat{\mathbf{y}}$  serves as an approximation for the output of the high-order system  $\mathbf{y}$ . The quality of this approximation can be assessed by means of a bound on the error. Namely, an error bound can be expressed in terms of the discarded Hankel singular values [6, 7] as

$$\|\mathbf{H}(s) - \hat{\mathbf{H}}(s)\|_\infty \leq 2 \sum_{i=k+1}^n \sigma_i, \quad (56)$$

where  $\mathbf{H}(s)$  and  $\hat{\mathbf{H}}(s)$  are the transfer functions of the full-order system (37) and the reduced-order system (55), respectively. Furthermore,  $\|\cdot\|_\infty$  denotes the  $\mathcal{H}_\infty$  norm defined as

$$\|\mathbf{H}(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\mathbf{H}(j\omega)), \quad (57)$$

with  $\bar{\sigma}(\cdot)$  the largest singular value. The error bound (56) confirms the intuitive idea that the states corresponding to the largest Hankel singular values are the most important from the perspective of input-output behavior. Namely, a good approximation (i.e. a low error bound) will be obtained when the Hankel singular values in  $\Sigma_2$  are small. Since these Hankel singular values are only dependent on the high-order model (37), they can be computed a priori and allow for control over the reduction error by selection of the order  $k$ . Finally, it is noted that in (56) it is assumed that all Hankel singular values are distinct. When Hankel singular values with multiplicity larger than one occur, they only need to be counted once, leading to a tighter bound (see e.g. [7]).

In the model reduction procedure presented here, a reduced-order system is obtained by truncation (i.e. setting  $\bar{\mathbf{x}}^2 = 0$ ) of a balanced realization. An alternative approach is given by singular perturbation [47] of this realization. Herein, it is assumed that the dynamics describing the evolution of  $\bar{\mathbf{x}}^2$  is very fast (and asymptotically stable). Then, this dynamics can be assumed to be in its equilibrium position at all time, which is obtained by setting  $\dot{\bar{\mathbf{x}}^2} = 0$  and solving for  $\bar{\mathbf{x}}^2$  as a function of  $\bar{\mathbf{x}}^1$  and  $\mathbf{u}$ . Contrary to balanced truncation, the singular perturbation approach guarantees that the steady-state gains of the high-order system are matched in the reduced-order system. The reduced-order model is controllable, observable, asymptotically stable and the error bound (56) also holds [48].

. Balanced truncation aims at approximating a high-order system by selecting the state components that have the largest contribution in the input-output behavior, according to the energy in the input and output signals. The entire frequency range is considered in this approach. However, in many practical applications, a good approximation is only required in a specific frequency

range. To this end, frequency-weighted balanced truncation can be used [6], which is an extension of the method discussed in the previous paragraphs. In frequency-weighted balanced truncation, the objective is to find a reduced-order system such that the error

$$\|\mathbf{H}_o(s)(\mathbf{H}(s) - \hat{\mathbf{H}}(s))\mathbf{H}_i(s)\|_\infty \quad (58)$$

is small, where  $\mathbf{H}_i(s)$  and  $\mathbf{H}_o(s)$  denote the transfer functions of an input and output frequency weight, respectively. These weights can be designed by the user to emphasize specific regions in the frequency domain. To obtain the frequency-weighted reduced-order model, controllability and observability gramians are computed on the basis of the frequency weighted high-order system, which are simultaneously diagonalized. Details can be found in [6].

When the original system is asymptotically stable, observable and controllable, and only one-sided weighting is applied (i.e. either  $\mathbf{H}_i(s) = \mathbf{I}$  or  $\mathbf{H}_o(s) = \mathbf{I}$ ), asymptotic stability of the reduced-order system is guaranteed. However, in the case of general two-sided weighting, stability of the reduced-order approximant can not be guaranteed. Nonetheless, when the reduced-order model is stable, an error bound is given in [49].

. In the preceding paragraphs, the standard balanced truncation technique for asymptotically stable systems as well as an extension to frequency-weighted balanced truncation is presented. Several extensions of balanced truncation exists. For example, balanced truncation of the coprime factorization applies to unstable systems [50, 51], whereas a method preserving passivity is given in [52, 53].

Besides these methods based on balanced truncation, a popular alternative is optimal Hankel norm approximation [7], which is also based on the balanced realization. For an overview of model reduction in systems and control, see e.g. [16, 13].

### 3. Qualitative comparison on model reduction methods

In this section, the methods as discussed in Section 2 will be compared. First, the common feature of projection is presented in Section 3.1. Then, a general comparison will be given in Section 3.2. A close connection between moment matching and modal truncation augmentation is discussed in Section 3.3. Computational aspects and the preservation of properties will be discussed in Sections 3.4 and 3.5, respectively.

#### 3.1. Projection

Before discussing differences between the methods from the fields of structural dynamics, numerical mathematics and systems and control, an important similarity is discussed. Namely, the methods discussed in Section 2 have in common that the reduced-order models are obtained by projection. Here, the reduced-order model is obtained by application of the projection  $\mathbf{\Pi} = \mathbf{V}\mathbf{W}^T$  to the original model. In numerical mathematics, the projection matrices might be chosen as  $\mathbf{V} = \mathbf{V}$  and  $\mathbf{W} = \mathbf{W}$  with  $\mathbf{V}$  and  $\mathbf{W}$  as in (30) and (31), respectively. This specific choice ensures moment matching around  $s_0 = 0$ . For balanced truncation, as used in systems and control, the matrices  $\mathbf{V}$  and  $\mathbf{W}$  are given as the first  $k$  columns of the transformation matrices  $\mathbf{T}^{-1}$  and  $\mathbf{T}^T$  as in (53), respectively. Hence, they project on the subspace of  $\mathbf{PQ}$  corresponding to the largest Hankel singular values (see (48)), which yields the subspace with the largest contribution in the input-output behavior. Similarly, in the mode superposition techniques in structural dynamics, the projection is given as  $\mathbf{\Pi} = \mathbf{\Phi}\mathbf{\Phi}^T$ . Here, the projection basis  $\mathbf{\Phi}$  forms a basis for the space spanned by the  $k$  most relevant eigenvectors (see (3)), which are typically chosen as the eigenvectors corresponding to the lowest eigenfrequencies. Here, it is noted that the state-space form is used in the fields of numerical mathematics and systems and control, whereas a second-order form is exploited in structural dynamics when no damping is present or when the damping can be considered as small.

### 3.2. General comparison

Besides the common feature of projection, the reduction techniques in Section 2 have important differences, as listed below.

- **First-order form versus second-order form**

The most apparent distinction is the type of model under consideration. In the field of structural dynamics, models of second-order form are usually studied, whereas first-order models are examined in the fields of numerical mathematics and systems and control. Even though the use of this symmetric second-order form seems limiting, it is noted that the mechanical structures studied in this field can indeed be modeled as second-order systems. In addition, these mechanical structures typically have little damping, which motivates the exploitation of undamped vibration modes for model reduction. Nonetheless, due to the specific structure of these models, the model reduction techniques from structural dynamics can in general not be applied to other application domains. On the other hand, any model that can be written in the first-order form can be handled by the reduction techniques from numerical mathematics and systems and control, although asymptotic stability is assumed in the latter.

- **Input-output behavior versus global behavior**

A second difference is given by the objective of the approximation. In numerical mathematics and systems and control, a reduced-order model is sought which approximates the input-output behavior of the original system. On the other hand, this input-output behavior is of less relevance in the field of structural dynamics, where the approximation of the global dynamics is of interest. Again, this results from the specific objectives in structural dynamics. Namely, typical interest is in the identification of the regions where the highest stresses or maximum displacements occur, whose locations are not known beforehand. Hence, the modeling of the global dynamical behavior is the main goal. However, extensions to mode superposition methods (see e.g. Section 2.1.2) provide techniques of incorporating the (static) influence of input forces in the reduction basis, partially taking input-output behavior into account.

In numerical mathematics and systems and control, the internal behavior of the model is of little interest. In control design, the system behavior from the control input to the measured outputs is of relevance and this directly forms the basis for the model reduction procedure. In the analysis of large-scale electrical circuits, where moment matching methods from the field of numerical mathematics are typically applied, interest is in the reduction of linear subcircuits. Here, its influence on the total circuit is described by the inputs and outputs, such that the approximation of input-output behavior is of interest. Nonetheless, input-output behavior is only truly taken into account in the latter when two-sided projection techniques are used. Namely, when using one-sided projection techniques, either the input matrix  $\mathbf{B}$  or output matrix  $\mathbf{C}$  is discarded, such that the focus of the reduction is limited to the state-to-output or input-to-state behavior, respectively. In this case, the number of moments matched is independent of the choice of input or output matrix. Nonetheless, the number of moments matched for a given reduction order  $k$  is doubled in the two-sided case, when input-output behavior is fully taken into account.

- **Interpretation of reduction space**

Model reduction techniques from structural dynamics are largely based on physical properties of mechanical systems. Therefore, the reduction space resulting from modal approximation has a useful engineering interpretation. Namely, it consists of the modes of the system, which represent the typical vibration pattern of a structure at a given frequency. The most important modes and the corresponding eigenfrequencies are preserved in the reduced-order system. Since these modes are obtained via an eigenvalue decomposition, the system in modal coordinates is in diagonal form, as discussed in Section 2.1. Here, it is recalled that this only holds when the system is undamped or has proportional (Rayleigh) damping or

modal damping. In this diagonal form, the equations describing the dynamics of the modes are uncoupled, which means that no error is introduced in the dynamics of the modes that are kept in the reduced-order model. In fact, the reduction error is due to the deletion of modes, rather than errors in the dynamics of the modes themselves.

In the reduction techniques from numerical mathematics and systems and control, the reduction space does not have a clear physical interpretation. Of course, this is largely due to the fact that these procedures are not limited to mechanical systems and are based on system-theoretic properties instead, as discussed in Section 2.3.

- **Global versus local approximation in frequency domain**

The modal truncation and moment matching model reduction techniques from structural dynamics and numerical mathematics have in common that they can be considered as frequency-domain-based (or Laplace-domain-based) techniques. Therefore, they give a good approximation in some part of the frequency-domain only. This is directly apparent in the modal reduction techniques from structural dynamics, where the modes as used in the reduced-order model are selected by their corresponding eigenfrequency. Here, the modes are typically selected from the lower end of the frequency spectrum. On the other hand, moment matching in numerical mathematics is based on the Taylor series expansion of a transfer function at a specific point (or multiple points) in Laplace domain. Since the moments around this expansion point form the basis for the reduced-order model, this approximation can only be expected to be accurate around the expansion point, leading to a local approximation in frequency domain.

In balanced truncation, as used in systems and control, the behavior in frequency domain does not form the basis of the model reduction procedure. Instead, the transfer of energy from the input to the output is used as a tool for model reduction, which can be considered as a time-domain approach. Nonetheless, specific regions in frequency domain can be emphasized by the extension to frequency-weighted balanced truncation, as noted in Section 2.3.

- **Automatic versus user-dependent model reduction**

A final general difference can be found in the level of automation of the model reduction techniques from the different fields. Here, only the balanced truncation method in systems and control is fully automatic when a requirement on the quality of the reduced-order model is given. Namely, the existence of an a priori error bound (56) allows for the automatic choice of the reduction order. The methods from structural dynamics and numerical mathematics lack such an error bound.

Even when the reduction order is chosen beforehand, the methods from structural dynamics and numerical mathematics are heuristic. That is, the mode superposition techniques from structural dynamics are dependent on the frequency range of interest, which needs to be specified a priori. Herein, typically the modes corresponding to the lowest frequencies are chosen. Similarly, the reduction procedure in the moment matching techniques from numerical mathematics is dependent on the choice of expansion points. However, the influence of this choice on the properties of the reduced-order system is largely an open problem and very few guidelines for this selection exists. Therefore, these expansion points are typically chosen as  $s_0 = 0$  or  $s_0 = \infty$ . Of course, the computational procedure in mode superposition and moment matching is fully automatic as soon as a choice is specified for the frequency range of interest and the expansion point, respectively.

### 3.3. *Moment matching and model truncation augmentation*

A close link exists between modal truncation augmentation techniques used in structural dynamics (Section 2.1.3) and the moment matching methods (Section 2.2). This can be understood by considering the series expansion of the (non-damped) structural equations (2) in the Laplace

domain for  $s^2$  as

$$\mathbf{q} = (\mathbf{K} + s^2\mathbf{M})^{-1}\mathbf{f} = \sum_{i=0}^{\infty} ((\mathbf{K} + s_0^2\mathbf{M})^{-1}\mathbf{M})^i (\mathbf{K} + s_0^2\mathbf{M})^{-1}\mathbf{f}(s^2 - s_0^2)^i, \quad (59)$$

where  $s_0$  is a chosen expansion point. Clearly this expansion is similar to the moment matching expansion (23) except that here it is written for the second-order form. The reduction basis suggested by this expansion is the Krylov series

$$\mathcal{K}_k((\mathbf{K} + s_0^2\mathbf{M})^{-1}\mathbf{M}, (\mathbf{K} + s_0^2\mathbf{M})^{-1}\mathbf{f}) \quad (60)$$

In the modal truncation augmentation approach the reduction basis consists of some eigenmodes of the system and modal truncation corrections as described in (20). Recalling the definition (16) of the correction vectors, it can be seen that the reduction basis (20) for the modal truncation augmentation is

$$\text{span} \{ [\Phi \mathbf{q}_{cor}] \} = \text{span} \{ [\Phi \mathbf{K}^{-1}\mathbf{f}] \}, \quad (61)$$

indicating that it includes the zero-order expansion term around  $s_0 = 0$ . Thus, it conserves the zero-order moment of the second-order problem around  $s_0 = 0$ , which is a direct consequence of the fact that the reduction basis includes the exact static solution. Through a similar reasoning one could say that substructuring methods that include the interface static modes (like the Craig-Bampton, the Rubin/MacNeal and the Dual Craig-Bampton methods) are matching the zero-order moments for the interface forces.

The modal truncation augmentation form presented in Section 2.1.3 includes only the zeroth-order correction as indicated by the basis (61). Higher-order corrections as suggested in the Krylov sequence (60) can also be included in the reduction space as proposed in [54, 26, 55, 56, 57, 33], which guarantees matching higher-orders moments and thus leads to an approach similar to the moment matching technique. Higher-order correction modes have also been used in the context of substructuring and mode component synthesis [32, 58, 59]. Note that the high-order corrections for structural problems can be obtained as a by-product of the Lanczos algorithm used to compute the eigenmodes [60] and that one can also consider quasi-static corrections (i.e. for  $s_0 \neq 0$ ) in case one is interested in a specific frequency range [61].

It is important to observe that the modal truncation augmentation uses a reduction basis that, in addition to the moments, also includes true eigenmodes of the system. In that sense this method differs from the usual moment matching techniques and it accounts both for the global behavior of the system (through its eigenmodes) and for input-specific components (through the moments).

### 3.4. Computational aspects

A general comparison of properties of model reduction techniques from the fields of structural dynamics, numerical mathematics and systems and control was given in Section 3.2. Computational aspects are addressed in the current section.

From a computational point of view, the methods from systems and control have the highest cost. In these methods, the computational complexity is mainly due to the solution of two Lyapunov equations (see (44) and (45)), which are of the size of the original high-order model. This seriously hinders the applicability of balanced truncation to systems of very high order. In addition, a full eigenvalue decomposition (see (52)) is required, such that the total computational cost associated to balanced truncation is high. Finally, a full coordinate transformation has to be computed, before reduction can be performed by means of truncation.

The computational cost for reduction techniques from the fields of structural dynamics and numerical mathematics is significantly lower. First, these methods do not require the computation of a full coordinate transformation. Instead, only the reduction space is computed, which is given by only  $k$  basis vectors. Furthermore, the computations are less costly since the matrix operations that are required are relatively cheap when compared to those needed for the solution of Lyapunov equations. In the mode displacement techniques from structural dynamics, only the most important eigenvalues and eigenvectors need to be computed. Since the frequency domain of interest is

typically known beforehand, efficient iterative methods can be used to find the eigenfrequencies in this range. For the Krylov-subspace based moment matching techniques from numerical mathematics, the numerical cost is even less. Namely, the application of the Arnoldi or Lanczos methods only requires the solutions of linear sets of equations or matrix-vector multiplications. Therefore, moment matching methods by Krylov subspaces can be applied to systems of very high order.

Even though the application of balanced truncation seems limited from a computational point of view, it is remarked that the perception of "high-order" differs in the three different fields. Especially, models of very low order (i.e.  $\mathcal{O}(10^0 - 10^1)$ ) are of interest in the field of systems and control. This is mainly due to the fact that controllers have to be implemented in real-time, which provides a limit on the order of the controller. Furthermore, low-order controllers are preferred because of their limited complexity. Hence, even though the computation of Lyapunov equations limits the applicability of balanced truncation to systems of order  $\mathcal{O}(10^3)$ , it still provides a solution to relevant model reduction problems in practice. On the other hand, the models describing mechanical structures in the field of structural dynamics typically result from finite element procedures, leading to models of orders up to  $\mathcal{O}(10^6)$ . Similarly, the moment matching techniques from numerical mathematics typically find application in the analysis of large-scale electrical circuits, leading to models of order up to  $\mathcal{O}(10^6)$ . From these applications, the need for numerically efficient model reduction procedures is clear.

### 3.5. Preservation of properties

In model reduction, the objective is the construction of a reduced-order model that gives a good approximation of the original high-order model. Herein, it is of crucial importance that the reduced-order model preserves some properties of the original system, among which stability is the most important. If the high-order system is asymptotically stable, balanced truncation (see Section 2.3) indeed preserves this property, which is due to the fact that the (diagonal) gramians act as Lyapunov equations. The moment matching techniques from Section 2.2 do not satisfy such a property, such that stability of the reduced-order model can not be guaranteed in general. Nonetheless, methods exist that preserve stability for classes of linear systems (see e.g. [35]).

In the mode superposition techniques outlined in Section 2.1, stability of the reduced-order model can not be guaranteed when the original high-order system exhibits general damping. However, in the important cases of undamped systems or systems with positive definite symmetric damping matrix (which includes the cases of proportional (Rayleigh) and modal damping), the stability properties are indeed preserved. In fact, since the reduced-order model is based on the computation of the undamped vibration modes, reduction of an undamped system leads to an undamped reduced-order system, where the most important eigenfrequencies are preserved. Stated differently, the pole locations of the most important poles remain unchanged. This property does in general not hold for balanced truncation and moment matching techniques.

Furthermore, it is obvious that modal superposition techniques preserve the second-order form in the reduced-order model. Nonetheless, this is an important feature in the field of structural dynamics since it implies that the kinematic relation between displacement and velocity is preserved. This does not hold for balanced truncation and moment matching, even if the models stem from a second-order form.

Next, it is remarked that the existence of an error bound, as discussed in Section 3.2, is closely related to stability preservation. Namely, a bound on the difference of solutions from the high-order and reduced-order systems can only be expected to exist when both systems are stable, making stability a prerequisite for the existence of an error bound.

Finally, it is often important to preserve other system properties besides stability. Herein, passivity and bounded realness are the most notable. Even though the methods as discussed in Section 2 do not generally preserve these properties, it is noted that extensions exist that do. For the different fields, some references to the literature are given in the corresponding parts in Section 2.

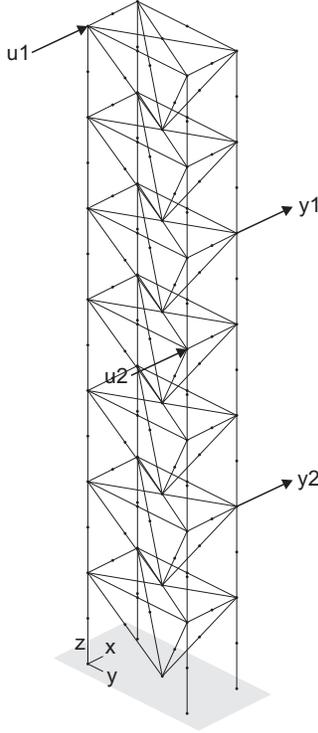


Figure 1: Truss frame system.

#### 4. Illustrative example

To illustrate the differences between methods as discussed in Section 3, the model reduction procedures of Section 2 are applied to a common benchmark example.

The benchmark example is chosen from the domain of structural dynamics, to allow for application of all model reduction techniques discussed in Section 2. More specifically, the truss frame system as depicted in Figure 1 is considered. Here, the nodes, as indicated by points, are connected by beam elements, leading to a model of the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \tilde{\mathbf{b}}_1 u_1 + \tilde{\mathbf{b}}_2 u_2, \quad (62)$$

$$y_1 = \tilde{\mathbf{c}}_1^T \mathbf{q}, \quad (63)$$

$$y_2 = \tilde{\mathbf{c}}_2^T \mathbf{q}, \quad (64)$$

where  $\mathbf{q} \in \mathbb{R}^N$  is the vector of the displacements and rotations of the nodes, with  $N = 714$ . The truss frame is subject to forces acting (in  $x$ -direction) on the positions as shown in Figure 1, which are modeled as the inputs  $u_1$  and  $u_2$  in (62). In addition, the displacement (in  $x$ -direction) of the truss frame is measured at two locations, leading to the outputs  $y_1$  and  $y_2$ . Finally, the truss frame model is lightly damped, which is modeled using modal damping.

In order to apply model reduction techniques from the fields of numerical mathematics and systems and control, the truss frame model (62) has to be written in state-space form. By choosing the state vector as  $\mathbf{x}^T = [\mathbf{q}^T \ \dot{\mathbf{q}}^T]$ , the dynamics is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2, \quad (65)$$

$$y_i = \mathbf{c}_i^T \mathbf{x}, \quad i \in \{1, 2\}, \quad (66)$$

where it is noted that  $\mathbf{x} \in \mathbb{R}^n$  with  $n = 2N$ . The system matrices read

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \quad \mathbf{b}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\tilde{\mathbf{b}}_i \end{bmatrix}, \quad \mathbf{c}_i = \begin{bmatrix} \tilde{\mathbf{c}}_i \\ \mathbf{0} \end{bmatrix}, \quad i \in \{1, 2\}. \quad (67)$$

Alternatively, the moment matching methods discussed in Section 2.2 can also be applied using system descriptions of the form

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1u_1 + \mathbf{b}_2u_2, \quad (68)$$

$$y_i = \mathbf{c}_i^T \mathbf{x}, \quad i \in \{1, 2\}, \quad (69)$$

with matrices

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}, \quad \mathbf{b}_i = \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_i \end{bmatrix}, \quad i \in \{1, 2\}, \quad (70)$$

which avoids the need for inversion of the matrix  $\mathbf{M}$ . In this form, the output vectors  $\mathbf{c}_i$  remain unchanged.

The model reduction techniques from Section 2 are applied to this example. From the field of structural dynamics, the mode displacement method is used. Here, it is recalled that this method is based on the undamped system (i.e.  $\mathbf{D} = \mathbf{0}$ ) and the location of the inputs and outputs (i.e. knowledge on  $\tilde{\mathbf{b}}_i$  and  $\tilde{\mathbf{c}}_i$ ) is not taken into account. Nonetheless, the (static) influence of the locations of the inputs can be taken into account by the extensions given by mode acceleration and modal truncation augmentation (see Section 2.1). On the other hand, model reduction with respect to input  $u_1$  and output  $y_1$  is performed using moment matching and balanced truncation. Since these methods are based on the state-space form (66) (or (69)), damping can be included. In moment matching, the expansion point is chosen as  $s_0 = 0$  and a one-sided projection is used, based on the input only (see (30)). In reduction, the reduced-order size of the first-order models  $k$  is chosen as  $2K$ , with  $K$  the number of modes taken into account in the mode displacement methods. This choice is motivated by the fact that the representation of a second-order model in first-order form doubles the number of equations. Hereby, the reduced-order models in first-order and second-order form have a similar approximation accuracy. Choosing  $K = 2$  ( $k = 4$ ) leads to the frequency response functions, with input  $u_1$  and output  $y_1$ , as depicted in Figures 2 and 3.

In the mode displacement method (MD), the  $K$  lowest eigenvectors are chosen in the reduction basis. Therefore, the first two resonance peaks are captured by this reduced-order model. On the other hand, the method of balanced truncation (BT) does not capture the second resonance peak. Instead, the third resonance is approximated. This is caused by the fact that balanced truncation takes the location of the input and output into account. Specifically, the second and third resonance peak correspond to a bending mode around the  $y$ -axis and a torsional mode, respectively, where the latter has a larger influence on the input-output behavior from input  $u_1$  to output  $y_1$ . Finally, the moment matching technique (MM) gives a good approximation at low frequencies, which originates from the choice of the expansion point as  $s_0 = 0$ . Therefore, for  $k = 4$ , only the first resonance peak is captured. This is most clearly shown in the error magnitude in Figure 3, where the moment matching techniques gives the best approximation for low frequencies. However, moment matching gives the largest  $\mathcal{H}_\infty$  norm of the error system. The lowest norm is obtained for balanced truncation, which outperforms the mode displacement method by the selection of the third rather than the second resonance peak.

To illustrate the influence of the locations of the inputs and outputs on the reduced-order model, the frequency response functions for input  $u_2$  and output  $y_2$  are depicted in Figure 4, whereas the corresponding error is given in Figure 5. Here, the same reduction bases were used as in Figure 2. Hence, the input  $u_2$  and output  $y_2$  are not taken into account in the model reduction procedure. Since the mode displacement method is based on the global dynamics rather than specific inputs and outputs, it also gives a good approximation for these new inputs. On the contrary, balanced truncation and moment matching are dependent on the input-output behavior, where it is recalled that reduction was based on input  $u_1$  and output  $y_1$ . Therefore, they do not

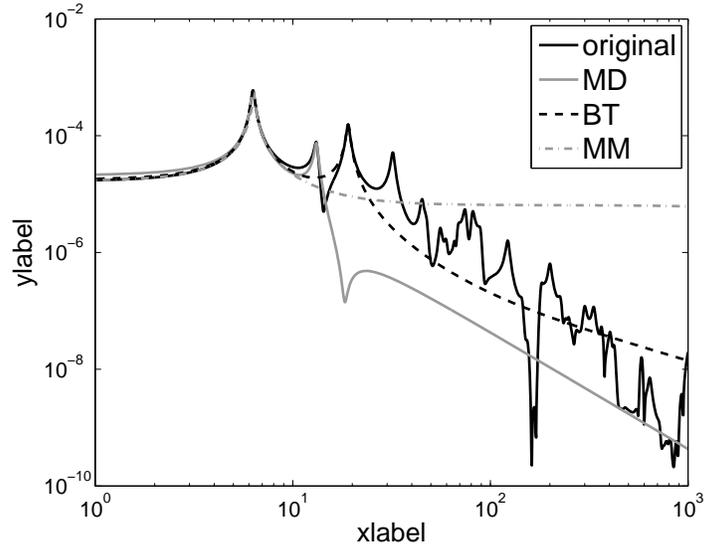


Figure 2: Comparison of the modal displacement method (MD), balanced truncation (BT) and moment matching (MM) for reduction to  $K = 2$  ( $k = 4$ ): magnitude of the frequency response function for input  $u_1$  and output  $y_1$ .

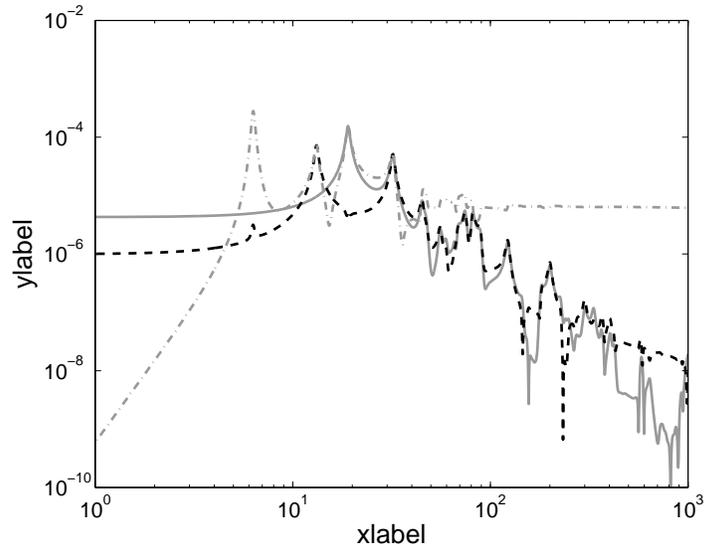


Figure 3: Magnitude of the error for reduction to  $K = 2$  ( $k = 4$ ) for input  $u_1$  and output  $y_1$ . Line styles as in Figure 2.

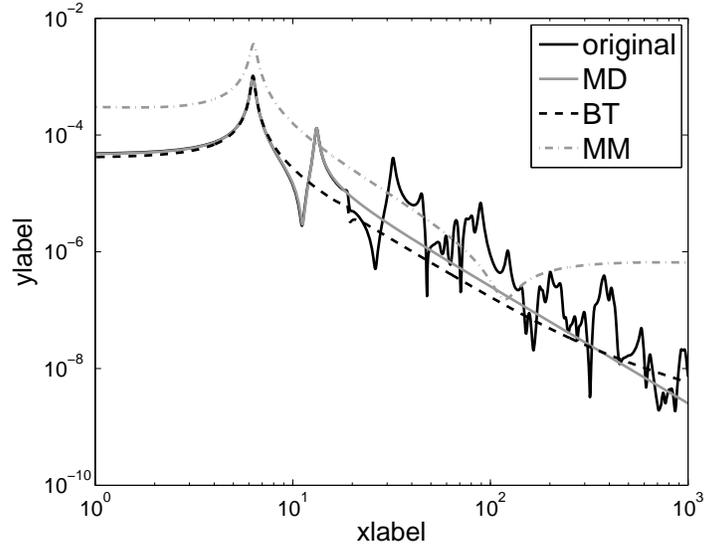


Figure 4: Comparison of the modal displacement method (MD), balanced truncation (BT) and moment matching (MM) for reduction to  $K = 2$  ( $k = 4$ ): magnitude of the frequency response function for input  $u_2$  and output  $y_2$ . For balanced truncation and moment matching, the reduced-order model is based on input  $u_1$  and output  $y_1$ .

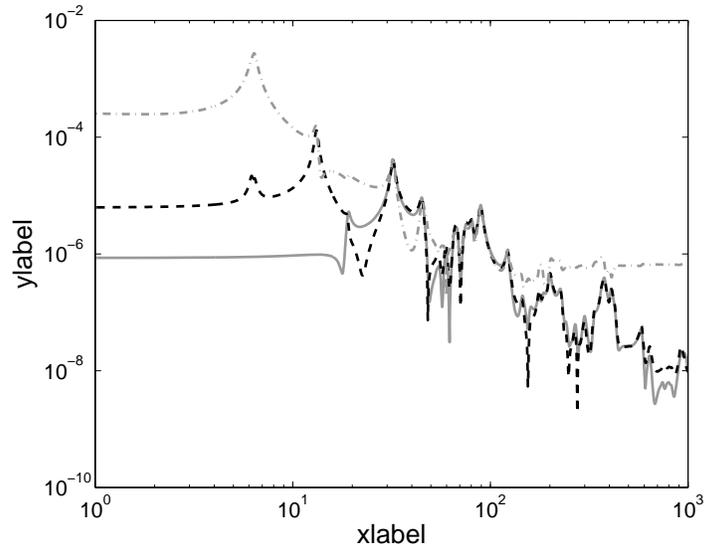


Figure 5: Magnitude of the error for reduction to  $K = 2$  ( $k = 4$ ) for input  $u_2$  and output  $y_2$ . Line styles as in Figure 4. For balanced truncation and moment matching, the reduced-order model is based on input  $u_1$  and output  $y_1$ .

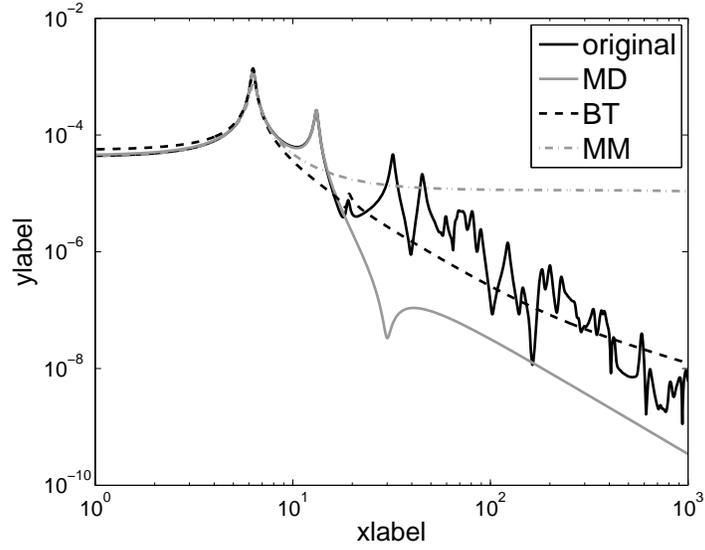


Figure 6: Comparison of the modal displacement method (MD), balanced truncation (BT) and moment matching (MM) for reduction to  $K = 2$  ( $k = 4$ ): magnitude of the frequency response function for input  $u_1$  and output  $y_2$ . For balanced truncation and moment matching, the reduced-order model is based on input  $u_1$  and output  $y_1$ .

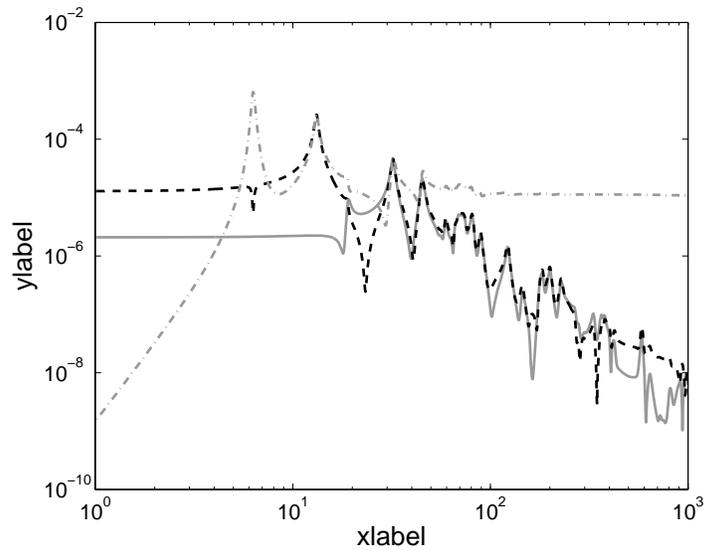


Figure 7: Magnitude of the error for reduction to  $K = 2$  ( $k = 4$ ) for input  $u_1$  and output  $y_2$ . Line styles as in Figure 6. For balanced truncation and moment matching, the reduced-order model is based on input  $u_1$  and output  $y_1$ .

give a good approximation for the input-output behavior from input  $u_2$  to output  $y_2$ , as is clear from the large errors in Figure 5.

However, the moment matching technique used here is one-sided. Hence, only the input matrix is taken into account and, in this case, moment matching (of  $k$  moments) can be proven for any output. To illustrate this, the input-output behavior from input  $u_1$  to output  $y_2$  is shown in Figures 6 and 7, again using the reduction basis generated for input  $u_1$  and output  $y_1$ . As expected, the performance obtained by moment matching is similar to that of Figure 2 (and Figure 3), where the same input  $u_1$  was used. For the mode displacement method and balanced truncation, the conclusions as stated before hold.

Finally, stability of the reduced-order models is checked. Since the truss frame system exhibits modal damping, the reduced-order model obtained by the mode displacement technique is guaranteed to be stable. Stability is also guaranteed in the case of balanced truncation. For moment matching, stability can not be guaranteed a priori. However, for  $k = 4$ , the reduced-order model obtained by moment matching is stable, as follows from an a posteriori check. On the other hand, for  $k = 6$ , stability is not preserved using moment matching.

## 5. Conclusions

In this paper, an overview and comparison of popular model reduction methods from the fields of structural dynamics, numerical mathematics and systems and control are provided. A detailed review is given on mode displacement techniques, moment matching methods and balanced truncation, whereas important extensions are outlined briefly.

The differences and similarities between presented methods are discussed, both qualitatively and quantitatively. Here, an important difference is the fact that the global dynamics is taken into account in the mode displacement methods, whereas moment matching and balanced truncation aim at the approximation of input-output behavior. Moreover, the computational cost of the methods differs, which limits the application of balanced truncation to systems of moderate size. On the other hand, balanced truncation has an a priori error bound, which is not the case for the mode displacement and moment matching techniques. Also, balanced truncation and the mode displacement method preserve stability of the high-order model, whereas stability is not guaranteed when applying moment matching.

The overview of the differences and commonalities between the different reduction method facilitates the choice of the reduction technique with the desirable properties for a given reduction problem.

Finally, these differences are illustrated by means of application of the different methods to a common benchmark example.

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