Discrete Structures 2IT50

Interim test September 24, 2014, 13:45 - 14:45
including solutions

This interim test is the first of three, of which the best two count for 30% of the final grade.

In giving proofs you may use theorems and lemmas from the lecture notes (not exercises), as long as you indicate that you use them.

The test consists of four problems all having the same weight.

Please indicate in which of the following instruction groups you are:

- Bas Joosten, AUD 9,
- Wieger Wesselink, LaPlace 1.105,
- Jaap van der Woude, Potentiaal 1.05.

Problem 1.

Give an example of an irreflexive relation on $\{1, 2, 3\}$ that is not transitive.

Solution:

For R being not transitive there should be x, y, z such that xRy and yRz but not xRz. Choosing $x = 1, y = 2, z = 3$ the smallest such relation is

$$R = \{(1, 2), (2, 3)\}.$$

It is irreflexive, so it satisfies both requirements.

Another, even smaller, solution is

$$R = \{(1, 2), (2, 1)\}.$$

Problem 2.

Let R, S be relations on a set U. Prove that $(R; S)^n; R \subseteq R; (S; R)^n$ for all $n \geq 0$.

Solution:

We apply induction on n. For $n = 0$ we obtain

$$(R; S)^0; R \;=\; I; R \;=\; R \;=\; R; I \;=\; R; (S; R)^0.$$

It remains to prove $(R; S)^{n+1}; R \subseteq R; (S; R)^{n+1}$, assuming the induction hypothesis $(R; S)^n; R \subseteq R; (S; R)^n$.

We give two separate proofs for this: one referring to separate elements, and a shorter proof exploiting monotonicity. In all cases we leave applications of associative implicit, that is, leave away parentheses if possible.

Proof 1:

Let $(x, y) \in (R; S)^{n+1}; R$.
By definition $(R; S)^{n+1} = R; S; (R; S)^n$, so $(x, y) \in R; S; (R; S)^n; R$.
Then there exists z such that $(x, z) \in R; S$ and $(z, y) \in (R; S)^n; R$.
From the induction hypothesis we conclude $(z, y) \in R; (S; R)^n$.
Combining this with $(x, z) \in R; S$ yields $x, y) \in R; S; R; (S; R)^n = R; (S; R)^{n+1}$, proving

$$(R; S)^{n+1}; R \subseteq R; (S; R)^{n+1}.$$

A lemma states that $R_1^{m+k} = R_1^k + R_1^m$ for all $m, k \geq 0$, so

$$(x, y) \in (R; S)^{n+1}; R = (R; S)^n; R; S; R.$$

Proof 2:

Now we will use monotonicity, that is, for all R_1, R_2, R_3 satisfying $R_2 \subseteq R_3$ we have $R_1; R_2 \subseteq R_1; R_3$.

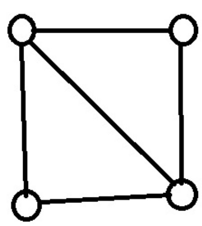
$$\begin{array}{rcll} (R; S)^{n+1}; R & = & R; S; (R; S)^n; R & \text{(Definition)} \\ & \subseteq & R; S; R; (S; R)^n & \text{(Induction hypothesis, monotonicity)} \\ & = & R; (S; R)^{n+1} & \text{(Definition).} \end{array}$$

Problem 3.

Give an example of an undirected graph with four nodes that has a Hamilton cycle but does not have an Euler cycle.

Solution:

Start by a cycle of length four on the four nodes, then by definition it is a Hamilton cycle. It is also an Euler cycle, but by adding one more edge there are two nodes of degree 3, so odd, by which according to Euler's Theorem no Euler cycle exists any more. The resulting graph is shown below.



Problem 4.

Let (V, E) and (V', E') be undirected trees for which $V \cap V' = \emptyset$, let $v_1, v_2 \in V$ and $v'_1, v'_2 \in V'$, $v_1 \neq v_2$, $v'_1 \neq v'_2$.

Prove that $(V \cup V', E \cup E' \cup \{(v_1, v'_1), (v_2, v'_2)\})$ is not a tree.

Solution: We give two separate proofs.

Proof 1:

Being a tree means that the graph is connected and does not contain cycles. So for proving not being a tree it suffices to show that the graph contains a cycle.

Let $u_0, \ldots, u_n$ be a shortest path from $v_1 = u_0$ to $v_2 = u_n$ in (V, E), this exists since (V, E) is a tree, so connected.

Let $u'_0, \ldots, u'_m$ be a shortest path from $v'_2 = u'_0$ to $v'_1 = u'_m$ in (V', E'), this exists since (V', E') is a tree, so connected.

Next consider the path

$$v_1 = u_0, \ldots, u_m = v_2, v'_2 = u'_0, \ldots, u'_m = v'_1, v_1.$$

It is a cycle in $(V \cup V', E \cup E' \cup \{(v_1, v'_1), (v_2, v'_2)\})$ since it is a path from the node v_1 to itself, and since we took the shortest paths and $V \cap V' = \emptyset$, for no node in the path its successor is equal to its predecessor.

Proof 2:

We use the lemma stating that every tree (V, E) satisfies $\#E = \#V - 1$.

Using this lemma for the two given trees (V, E) and (V', E') yields $\#E = \#V - 1$ and $\#E' = \#V' - 1$.

Next we count the numbers of nodes and edges in the new graph $(V \cup V', E \cup E' \cup \{(v_1, v'_1), (v_2, v'_2)\})$.

Since $V \cap V' = \emptyset$ we have $\#(V \cup V') = \#V + \#V'$.

Since $E \cap E' = \emptyset$ we have

$$\#(E \cup E' \cup \{(v_1, v'_1), (v_2, v'_2)\}) = \#E + \#E' + 2 = \#V - 1 + \#V' - 1 + 2 = \#V + \#V'.$$

So

$$\#(E \cup E' \cup \{(v_1, v'_1), (v_2, v'_2)\}) = \#V + \#V' \neq \#(V \cup V') - 1,$$

so according the lemma $(V \cup V', E \cup E' \cup \{(v_1, v'_1), (v_2, v'_2)\})$ is not a tree.