Discrete Structures 2IT50

Interim test October 8, 2014

Including solutions

This interim test is the second of three, of which the best two count for 30% of the final grade.

In giving proofs you may use theorems and lemmas from the lecture notes (not exercises), as long as you indicate that you use them.

The problems may be solved either in English or in Dutch.

This test will be graded according the given percentages.

Please indicate in which of the following instruction groups you are:

- Bas Joosten, AUD 9,
- Wieger Wesselink, LaPlace 1.105,
- Jaap van der Woude, Potentiaal 1.05.

Problem 1.

a. (20 %) Give an example of a set A and an injective function $f : A \to A$ for which $a \notin f(A)$ for some $a \in A$.

Solution:

Take A to be the set of natural numbers $\mathbf{N}$ and define $f : A \to A$ by $f(x) = x + 1$ for all $x \in A$. Then f is injective since from $f(x) = f(x')$ we can conclude $x = f(x) - 1 = f(x') - 1 = x'$.

Since there is no $x \in A$ such that $f(x) = 0$, we conclude $0 \notin f(A)$.

Other solutions include f defined by $f(x) = 100x$ for all $x \in A$; now every number not divisible by 100 is not in $f(A)$.

b. (20 %) Prove that this is not possible if A is finite.

Solution:

Recall the theorem:

Theorem For A finite, a function $f : A \to A$ is injective if and only if it is surjective.

If A is finite then by this theorem from $f : A \to A$ and f is injective, we conclude that f is surjective. That means that $a \in f(A)$ for all $a \in A$.

Problem 2.

a. (20 %) Consider the poset of natural numbers with the divisibility relation $|$. Determine all elements that are both minimal and maximal in $\{x \mid 4 \leq x \leq 15\}$.

Solution:

From $4 \mid 8$, $5 \mid 10$, $6 \mid 12$, $7 \mid 14$, $5 \mid 15$, we conclude that $4, 5, 6, 7$ are not maximal and $8, 10, 12, 14, 15$ are not minimal. For the remaining elements $9, 11, 13$ no proper divisor and no proper multiple is contained in the given set, so these are the elements that are both minimal and maximal.

b. (40 %) Let $(U, \sqsubseteq)$ be a lattice and $a, b, c, d \in U$.

Prove that $(a \sqcap c) \sqcup (b \sqcap d) \sqsubseteq (b \sqcup c) \sqcap (a \sqcup d)$.

Solution:

Recall the defining properties of $\sqcup$ and $\sqcap$:

(1): $x \sqsubseteq x \sqcup y$ and $y \sqsubseteq x \sqcup y$ for all x, y,

(2): If $x \sqsubseteq z$ and $y \sqsubseteq z$ then $x \sqcup y \sqsubseteq z$, for all x, y, z,

(1′): $x \sqcap y \sqsubseteq x$ and $x \sqcap y \sqsubseteq y$ for all x, y,

(2′): If $z \sqsubseteq x$ and $z \sqsubseteq y$ then $z \sqsubseteq x \sqcap y$, for all x, y, z.

According to (2) we have to prove

A: $a \sqcap c \sqsubseteq (b \sqcup c) \sqcap (a \sqcup d)$, and

B: $b \sqcap d \sqsubseteq (b \sqcup c) \sqcap (a \sqcup d)$.

For proving A, according to (2′) we have to prove

A1: $a \sqcap c \sqsubseteq b \sqcup c$ and

A2: $a \sqcap c \sqsubseteq a \sqcup d$.

A1 follows from (1) and (1′) and transitivity: $a \sqcap c \sqsubseteq c \sqsubseteq b \sqcup c$.

A2 follows from (1) and (1′) and transitivity: $a \sqcap c \sqsubseteq a \sqsubseteq a \sqcup d$.

For proving B, according to (2′) we have to prove

B1: $b \sqcap d \sqsubseteq b \sqcup c$ and

B2: $b \sqcap d \sqsubseteq a \sqcup d$.

B1 follows from (1) and (1′) and transitivity: $b \sqcap d \sqsubseteq b \sqsubseteq b \sqcup c$.

B2 follows from (1) and (1′) and transitivity: $b \sqcap d \sqsubseteq d \sqsubseteq a \sqcup d$.