Stream representations of real numbers

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Overview

- Real numbers
- Representaions of real numbers via streams
- Stream computability: TTE
- Computability of $f : \mathbb{R} \to \mathbb{R}$
- Concrete representations and algorithms

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Based on (a.o.)
Weihrauch - Computable Analysis (An Introduction), Springer EATCS series 2000.
Niqui - Formalising Exact Real Arithmetic (Representations, Algorithms, Proofs), PhD thesis, Radboud University Nijmegen, 2004.

Related work / what I will not talk about

Detailed algorithms for exact real arithmetic

- Edalat, Potts et al.

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Other approaches

- O'Connor: use $f : \mathbb{Q}^+ \to \mathbb{Q}$ in stead of $f : \mathbb{N} \to \mathbb{Q}$
- Pasca: formalize interval arithmetic
- Work with reals axiomatically, e.g. Mayero (Coq standard lib, classical); Cruz-Filipe (CoRN, constructive analysis)

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- only countably many are representable via syntax
- $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.
- So: use a countable set of approximations to elements in $\mathbb{R}$.

Defining real numbers via rational approximations

For example:

$$
\begin{aligned}
e &:= \sum_{i=0}^{\infty} \frac{1}{i!} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} \\
e &:= \lim_{n\to\infty} (1 + \frac{1}{n})^n \\
\frac{\pi^2}{6} &:= \sum_{i=0}^{\infty} \frac{1}{i^2} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i^2}
\end{aligned}
$$

Defining $\mathbb{R}$ out of $\mathbb{Q}$ (I): Cauchy sequences

Sequence of rational numbers $q_0, q_1, q_2, \ldots$ that converges.

$$\forall k \in \mathbb{N}^+ \exists N \in \mathbb{N} \forall m \geq N (|q_m - q_N| < \tfrac{1}{k})$$
$$\forall \epsilon \in \mathbb{Q}^+ \exists N \in \mathbb{N} \forall m \geq N (|q_m - q_N| < \epsilon)$$
$$\forall k \in \mathbb{N}^+ \exists N \in \mathbb{N} \forall m, p (|q_{N+m} - q_{N+p}| < \tfrac{1}{k})$$

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Option: Express N in terms of k via a function $\beta(k)$

$$\forall k \in \mathbb{N}^+ \forall m, p (|q_{\beta(k)+m} - q_{\beta(k)+p}| < \frac{1}{k})$$

Fundamental sequence: pair of $(q_i)_{i \in \mathbb{N}}$ and $\beta : \mathbb{N} \to \mathbb{N}$.

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Decimal notation

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ \ldots$$

Defining $\mathbb{R}$ out of $\mathbb{Q}$ (II): Decreasing intervals

Sequence of rational intervals $[p_0, q_0] \supseteq [p_1, q_1] \supseteq [p_2, q_2], \ldots$ that converges.

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Decimal notation can be seen as an interval representation

3	3.1	3.14	3.141	3.1415
$[3, 4]$	$[3.1, 3.2]$	$[3.14, 3.15]$	$[3.141, 3.142]$	$[3.1415, 3.1416]$

Defining $\mathbb{R}$ out of $\mathbb{Q}$ (III): Dedekind Cuts

A partition (A, B) of $\mathbb{Q}$ such that

- $\forall x \in A\, \forall y \in B (x < y)$,
- $\forall x \in A\, \exists z \in A (x < z)$,
- $A \neq \emptyset$, $B \neq \emptyset$

(Or variations)

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In concrete situations, a real number is defined via a property.
We need logic to prove equality of real numbers.

Equality on real numbers

Real number equality $\neq$ Stream equality

- $([-\frac{1}{2^k}, \frac{1}{2^k}])_{k \in \mathbb{N}} = ([-\frac{1}{3^k}, \frac{1}{3^k}])_{k \in \mathbb{N}}$
- $0.999\ldots = 1.000\ldots$

There is no hope in deciding that two real number representations are equal.

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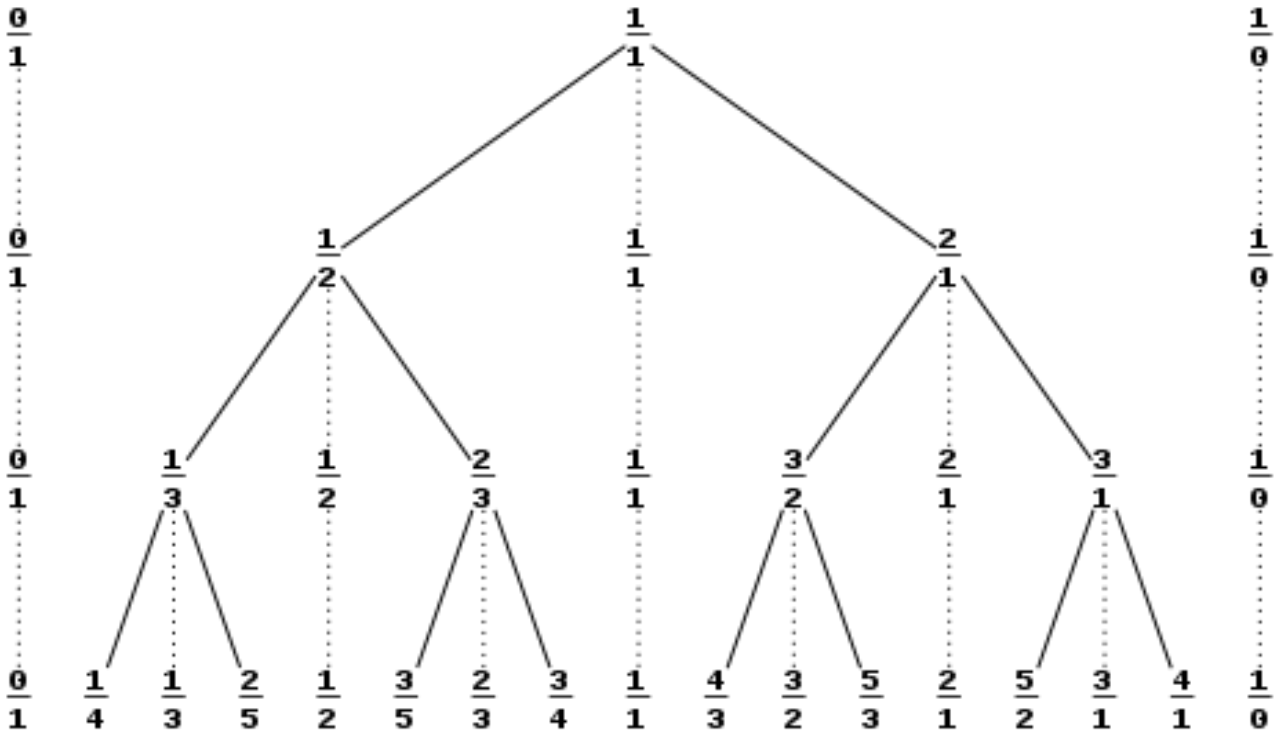
There is no hope in deciding that two real number representations are equal.
Would it help to remove redundancy in representations?

Stern-Brocot tree: unique representation of rational numbers



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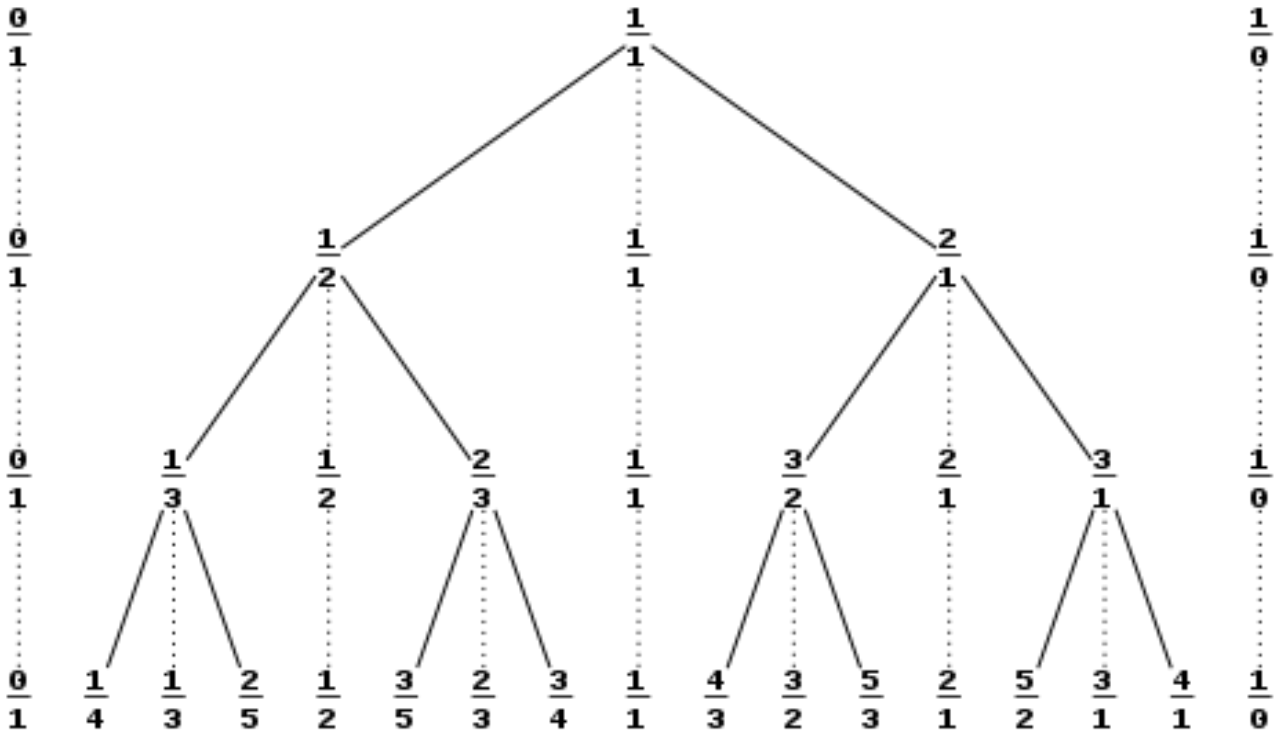

- Compute new element from "ancestors" $\frac{p}{q}$ and $\frac{n}{m}$ by taking the mediant: $\frac{p+n}{q+m}$

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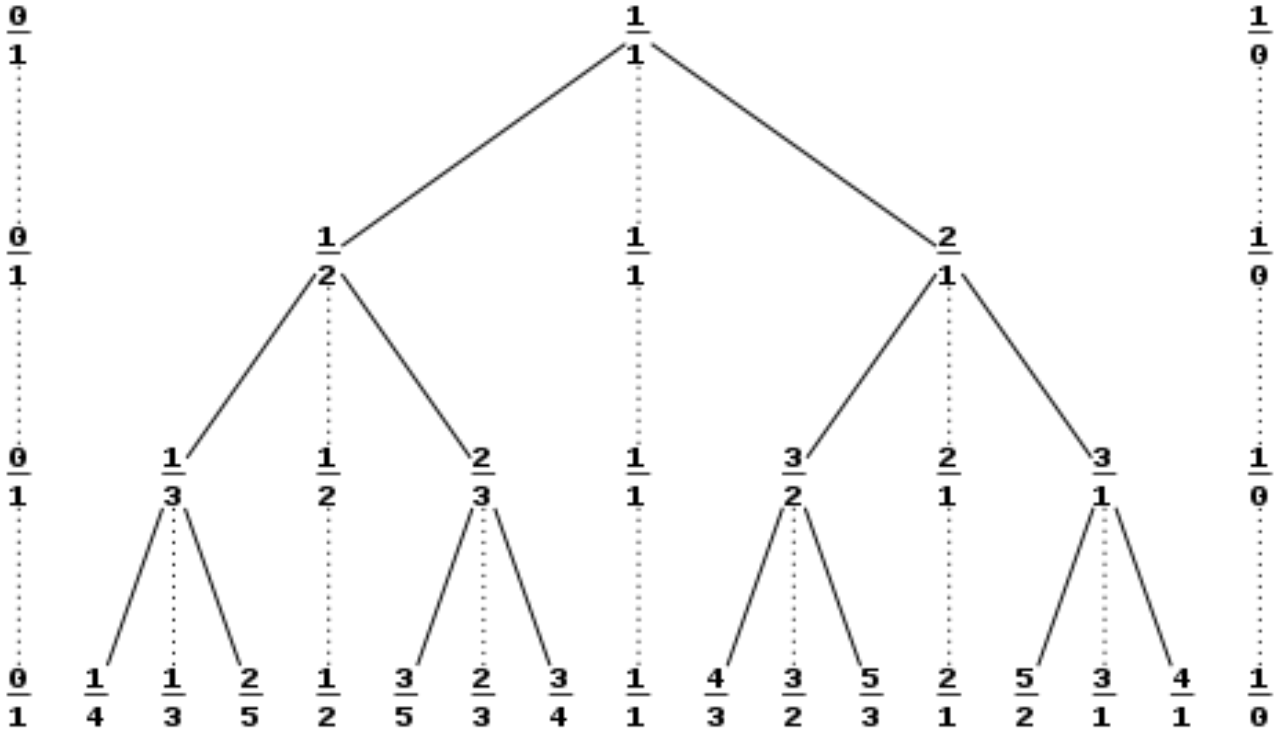
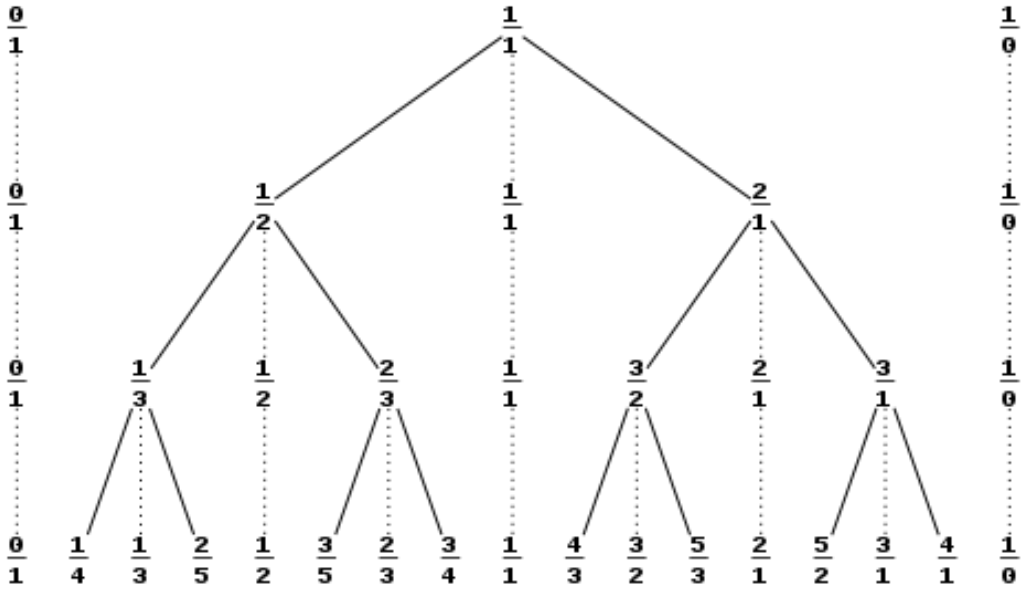


The Stern-Brocot tree gives a unique representation of $\mathbb{Q}^+$:

- For every $\frac{p}{q}$ in the tree, p and q are coprime.
- Every $x \in \mathbb{Q}^+$ occurs exactly once in the tree.

Stern-Brocot tree for real numbers



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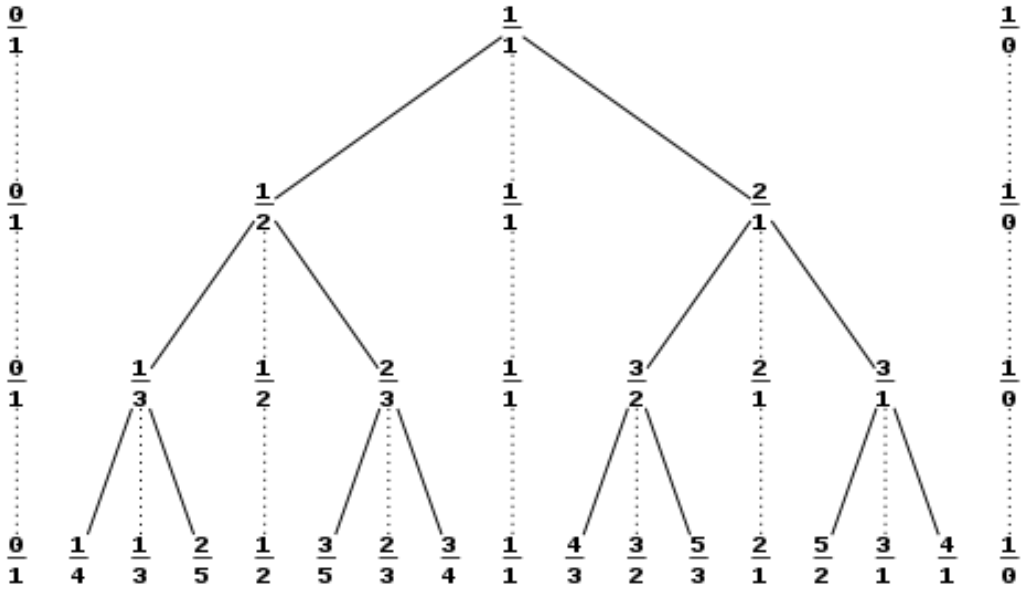

- A path in the tree gives a rational number via $[\![\langle - \rangle]\!] = 1$, $[\![L\sigma]\!] = \frac{[\![\sigma]\!]}{[\![\sigma]\!]+1}$, $[\![R\sigma]\!] = [\![\sigma]\!] + 1$.

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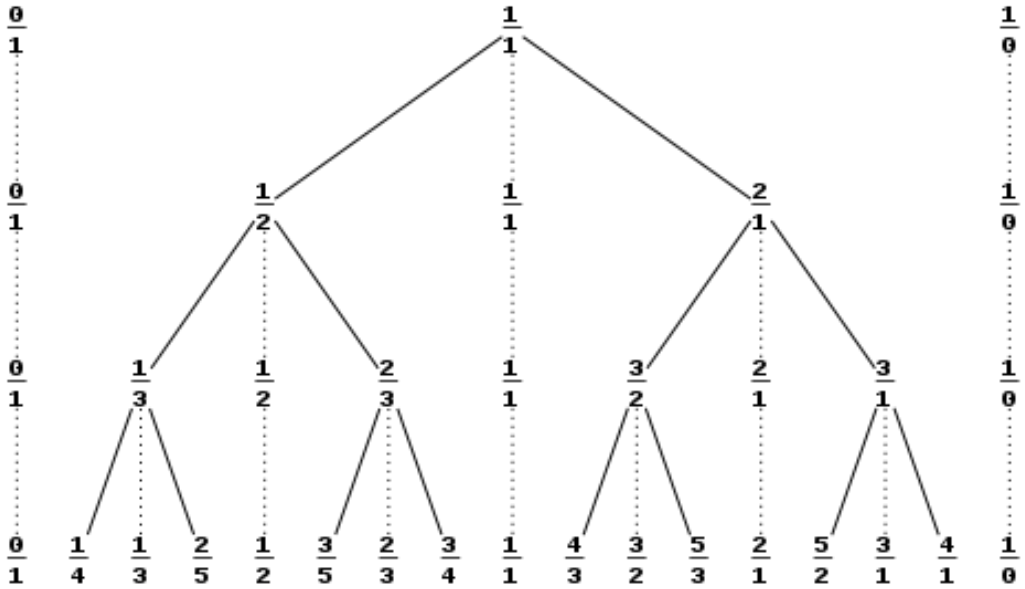

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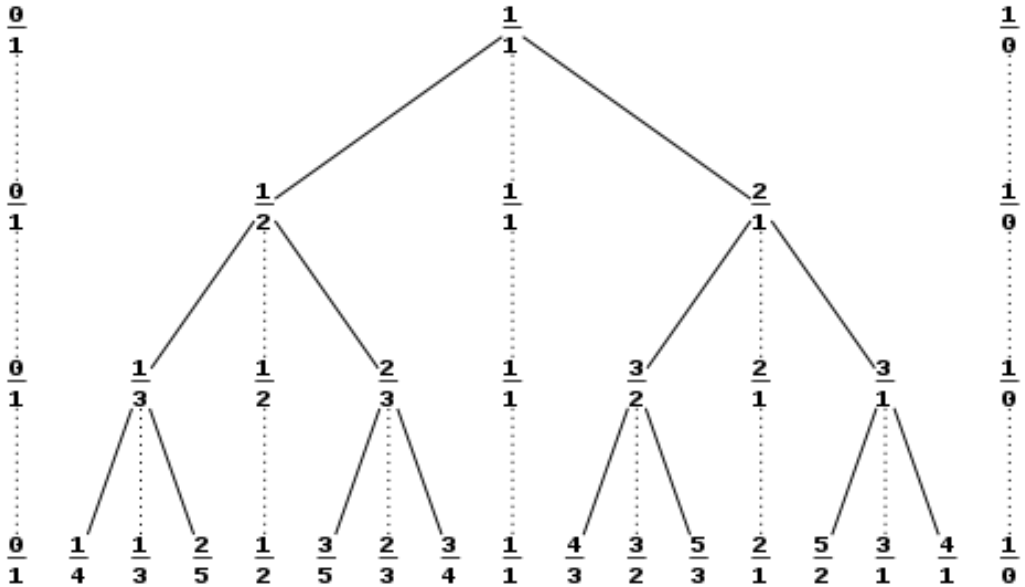

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- Two "base functions" $L(x) = \frac{x}{x+1}$ and $R(x) = x + 1$,
- An infinite path in the tree $d_0, d_1, d_2, \ldots$ represents $\bigcap_{i \in \mathbb{N}} d_0(d_1 \ldots d_i([0, \infty]) \ldots)$

Redundancy in representation

Are representations with little redundancy better?
E.g. Stern Brocot tree, continued fractions (with certain side conditions) have little redundancy.

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No!

- Too little redundancy makes certain functions non-computable.
- We want every $x \in \mathbb{R}$ to have countably many representations.

What is stream computability?

Type-2 Turing Machines and TTE, Type-2 Theory of Effectivity [Weihrauch, Grzegorczyk]

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- Normal multi-tape Turing machine over finite input alphabet Σ
- Special input tape with infinite stream of symbols as input,
- Special output tape where symbols from Σ can only be written. (Never erase.)
  Here the output $\in \Sigma^\omega$ is generated.

TTE computability

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- **Def** $f : \Sigma^\omega \to \Sigma^\omega$ is computable if there is a Turing machine M that computes it, etc.

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$$\begin{aligned} d(s,t) &:= 2^{-n} \text{ if } n \text{ is the smallest for which } s(n) \neq t(n) \\ d(s,s) &:= 0 \text{ otherwise} \end{aligned}$$

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- In general, $\mathrm{dom}(f)$ is the infinite intersection of open sets.
  E.g. $f(0:s) = f(s)$, $f(1:s) = 1:f(s)$,
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- NB. X is a Π^0_2-set iff $X = \mathrm{dom}(f)$ for some $f : \Sigma^* \to \Sigma^\omega$.

Every computable function is continuous

Theorem Every $f : \Sigma^\omega \to \Sigma$ that is computable, is also continuous.

Proof Let $s \in \Sigma^\omega, \epsilon > 0$. To prove:
$\exists \delta \forall t(|s - t| < \delta \to |f(s) - f(t)| < \epsilon)$.
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Take p with $2^{-p} < \epsilon$ and compute $f(s)$ until p outputs have been written on the output tape.
We have now read a finite number of inputs, say the first q.
Take $\delta := 2^{-q}$
Now for all t, if $|s - t| < \delta$, then t begins with the same q inputs as s, so $|f(s) - f(t)| < 2^{-p} < \epsilon$.

Domain theory: a slightly more general view

Domain D of finite and infinite sequences over Σ.
($D := \Sigma^* \cup \Sigma^\omega$.)

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(Proof of $f(\sqcup_i t_i) \sqsubseteq \sqcup_i f(t_i)$ is similar to proof of continuity via metric space.)

Computing with infinite streams that represent reals

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We take the decimal representation and compute the "double" function:

$$0,1234567\ldots \xrightarrow{\times 2} ??$$

$$\begin{aligned}
0.1 &\xrightarrow{\times 2} 0. \\
0.12 &\xrightarrow{\times 2} 0.2 \\
0.123 &\xrightarrow{\times 2} 0.24 \\
0.1234 &\xrightarrow{\times 2} 0.246 \\
0.12345 &\xrightarrow{\times 2} 0.2469 \\
0.123456 &\xrightarrow{\times 2} 0.24691 \\
&\ldots
\end{aligned}$$

"Look ahead" $L_f(y)(k) = k + 1$

Problem with computing with decimal representation

We compute the "triple function:

$$0,3333\ldots \xrightarrow{\times 3} ??$$

$$\begin{aligned} 0.3 &\xrightarrow{\times 3} ? \\ 0.33 &\xrightarrow{\times 3} ? \\ 0.333 &\xrightarrow{\times 3} ? \\ &\ldots \end{aligned}$$

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$$0.333 \xrightarrow{\times 3} \text{0.9 wrong in case 0.3334}$$

$$0.333 \xrightarrow{\times 3} \text{1. wrong in case 0.3332}$$

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So, representing reals via their infinite decimal representation is not good.
Brouwer knew this already:
"Besitzt jede reelle Zahl eine Dezimalbruchentwikklung?"
(Mathematische Annalen, 1921)

A good representation has redundancy

Decimal representation:

$$\ldots \mid \quad 0.0 \quad \mid \quad 0.1 \quad \mid \quad 0.2 \quad \mid \ldots$$

0.0 denotes [0.0, 0.1] and 0.1 denotes [0.1, 0.2].
What if the number we are trying to approximate turns out to be 0.1?

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Solution:: Add one extra digit: -1

$$\begin{array}{ccccccccc} \ldots & & [& 0.0 &] & & [& 0.2 &] & & \ldots \\ \hline \ldots & & & [& 0.1 &] & & [& 0.3 &] & \ldots \end{array}$$

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$$\begin{array}{ccccccc} \ldots & & [\quad 0.0 \quad] & & [\quad 0.2 \quad] & & \ldots \\ \hline \ldots & & [\quad 0.1 \quad] & & [\quad 0.3 \quad] & & \ldots \end{array}$$

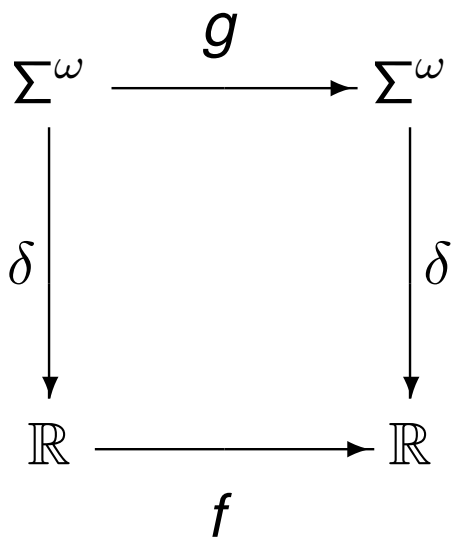
$$[\![d_0 d_1 d_2 d_3 \ldots]\!] := \sum_{i=0}^{\infty} d_i \times 10^{-i}$$

Representation systems

Definition A representation is a surjective $\delta : \Sigma^\omega \to \mathbb{R}$ (possibly partial).
Definition $x \in \mathbb{R}$ is δ-computable if $x = \delta(s)$ for some $s \in \Sigma^\omega$.

Representation systems

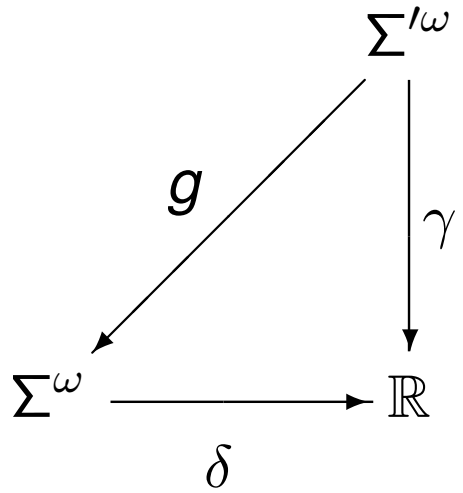
Definition A representation is a surjective $\delta : \Sigma^\omega \to \mathbb{R}$ (possibly partial).
Definition $x \in \mathbb{R}$ is δ-computable if $x = \delta(s)$ for some $s \in \Sigma^\omega$.
Definition $f : \mathbb{R} \to \mathbb{R}$ is δ-computable if there is a computable $g : \Sigma^\omega \to \Sigma^\omega$ such that $f(\delta(s)) = \delta(g(s))$ for all $s \in \Sigma^\omega$.



Good representation systems

Definition A representation system $\delta : \Sigma^\omega \to \mathbb{R}$ is admissible if

- δ is continuous
- δ is maximal: for every continous representation $\gamma : \Sigma'^\omega \to \mathbb{R}$, there is a continuous $g : \Sigma'^\omega \to \Sigma^\omega$ such that $\gamma = \delta \circ g$.



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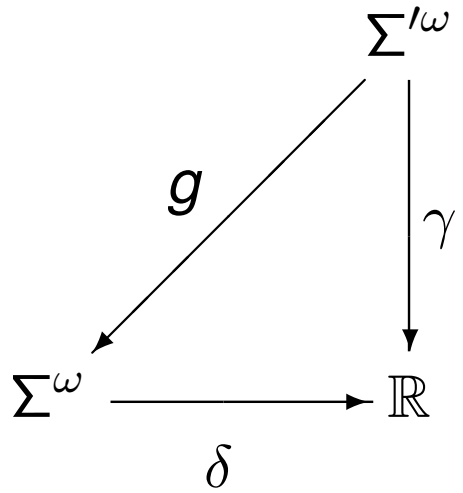

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Everything can be restricted to a closed subinterval of $\mathbb{R} \cup \{-\infty, +\infty\}$.

Computable function are continuous

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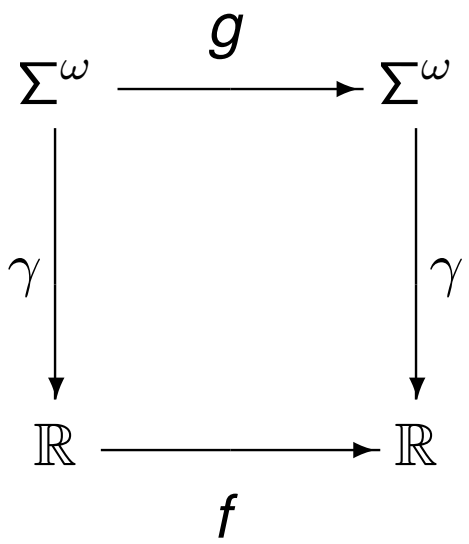

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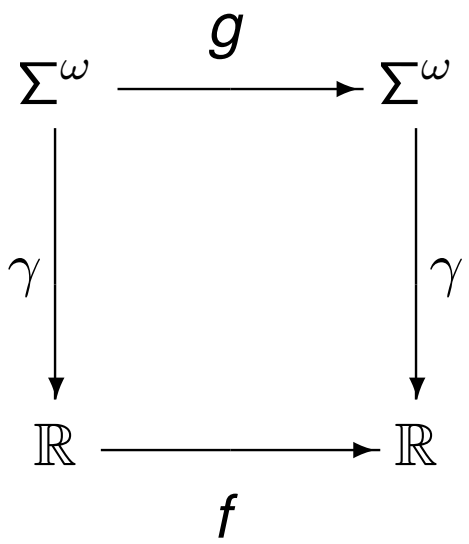

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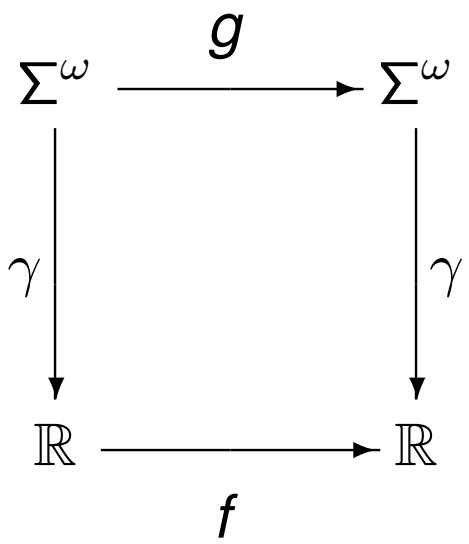

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So, inputs that are “close” produce outputs that are “close”.

Comparisons and step functions are not computable

- $c(x, y) := 0$ if $x \leq y$, $c(x, y) := 1$ if $x > y$ is not computable,
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NB. In case (1), different representations of the same x may produce different outputs.

An admissible representation system for $[0, 1]$

For $[0, 1]$, we let $\delta : \{0, 1\}^\omega \longrightarrow [0, 1]$ be:

$$[\![d_0 d_1 d_2 \ldots]\!] = \sum_{i=0}^{\infty} d_i \times 2^{-i}$$

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Admissible digit sets

We observe a general procedure for defining a representation for an interval I.

- Define functions $\varphi_1, \ldots, \varphi_k : I \to I$,
- Use them as "digits" for representing the elements of I by

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Under which conditions does this method work?

Definition [Niqui] A digit set $\varphi_1, \ldots, \varphi_k : I \to I$ is admissible if

- $\bigcap_{i=0}^{\infty} \varphi_{d_0}(\ldots \varphi_{d_i}(I) \ldots)$ is always a singleton
- $\bigcup_{j=0}^{k} \varphi_j(I^o) = I^o$.

(I^o is the interior of I)

Admissible digit sets and Möbius maps

Theorem [Niqui] An admissible digit set yields an admissible representation.

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There is still a choice which functions $\varphi_1, \ldots, \varphi_k : I \to I$ we take.
A popular way to choose them is via Möbius maps, maps of the shape

$$\frac{ax + b}{cx + d}$$

All the maps we have seen are Möbius maps.
They are usually viewed as matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Möbius maps

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Möbius maps $\varphi_0, \ldots, \varphi_k$ give a very concrete way of computing with infinite sequences. Requirements:

- non-singular: $ad - bc \neq 0$
- increasing: $x < y \to \varphi(x) < \varphi(y)$
- refining: $\varphi(I) \subseteq I$
- shrinking: $\bigcap_{i=0}^{\infty} \varphi_{d_0}(\ldots \varphi_{d_i}(I) \ldots)$ is a point,
- maximal $\bigcup_{j=0}^{k} \varphi_j(I^o) = I^o$.

Stern Brocot Möbius maps

$$L(x) = \frac{x}{x+1}, R(x) = x+1, M(x) = \frac{2x+1}{x+2}$$

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

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Now we can define a function as another Möbius map, for example $D(x) = 2x$:

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An actual algorithm has to compute output (digits) from Ds, with $s \in \{L, R, M\}^\omega$

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- In case s starts with R, we can output R and continue. (Note that the function D has changed.)
- In the other cases we (may) have to consume more input first