Applied Logic, Practice Set 1 — 2021–2022 Q2

The exercises in this set allow you to practice with the course material. They are particularly suitable as a preparation for the tutorial meetings on November 24 and December 1 (where they will be discussed), and for Assignment 1 and the Interim Test on December 3.

Exercise 1.

Use Z3 to find integers a, b, c, d such that $3a > b + 2c$, $2b > c + d$, $2c > 3d$ and $3d > a + c$. Are there integers a, b, c, d such that moreover $a = 42$ holds? Are there integers a, b, c, d such that moreover $a < 42$ holds?

Exercise 2.

Use Z3 to find the highest possible value of $f(5) - f(2)$ for integers $f(1), f(2), f(3), f(4), f(5)$ for which

- $f(i) < f(i + 1)$ for $i = 1, 2, 3, 4$, and
- $2f(i) > f(i + 1)$ for $i = 1, 2, 3, 4$, and
- $f(1) + f(2) + f(3) + f(4) + f(5) = 100$.

Exercise 3.

Use Z3 to find integers $f(1), f(2), f(3), f(4), f(5)$ for which

- $2f(i) > f(i + 1)$ for $i = 1, 2, 3, 4$, and
- $f(1) + f(2) + f(3) + f(4) + f(5) = 110$, and
- $f(1) - f(3) = 9$, and
- $f(2) + f(4) = 59$, and
- exactly three of $f(1), f(2), f(3), f(4), f(5)$ have value 20.

Exercise 4.

(a) Give a propositional formula in the boolean variables $a_1, a_2, \ldots, a_n$ describing that at least one of the variables $a_1, a_2, \ldots, a_n$ is true.

(b) Give a propositional formula in the boolean variables $a_1, a_2, \ldots, a_n$ describing that at most one of the variables $a_1, a_2, \ldots, a_n$ is true.

(c) Give a propositional formula in the boolean variables $a_1, a_2, \ldots, a_n$ describing that exactly one of the variables $a_1, a_2, \ldots, a_n$ is true.

(d) Give a formula in SMT syntax describing that exactly one of the variables a, b, c is true.

Exercise 5.

For a propositional formulas Φ we have to prove that it is a tautology, that is, for every assignment of the variables the formula yields true. Explain how this problem can be solved by SAT solving.

Exercise 6.

For two propositional formulas Φ and Ψ we have to prove that they are equivalent, that is, for every assignment of the variables they give the same value. Explain how this problem can be solved by SAT solving.

Exercise 7.

Explain how Sudoku puzzles can be expressed in SMT format without using the `distinct` operator.

Exercise 8.

In *diagonal Sudoku*, apart from the usual Sudoku rules it is also required that for both diagonals all values are distinct. Explain how this requirement can be expressed in SMT format, extending the encoding as presented on the slides.

Exercise 9.

A *magic square* is an $n \times n$ square filled by the numbers 1 to n^2, all occurring exactly once, such that all rows and all columns sum up to the same value.

(a) Describe how finding a 3×3 magic square can be expressed in SMT syntax.

(b) Use Z3 to find an $n \times n$ magic square for n as big as possible.

Exercise 10.

A sequence $a_1 a_2 \ldots a_n$ of boolean variables $a_1, a_2, \ldots, a_n$ represents a number a in n-bit binary notation, that is,

$$a = \sum_{i=1}^{n} a_i * 2^{n-i}$$

in which false is identified with the number 0 and true is identified with the number 1. Similarly, boolean variables $b_1, b_2, \ldots, b_n$ represent a number b. Give propositional formulas in the variables $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ describing that

(a) a is even;

(b) $a > 0$;

(c) $a > 1$;

(d) $a > 2$;

(e) $a < 2^{n-1}$;

(f) $a * b = 0$;

(g) $a = b$;

(h) $a > b$.

Remark: the MOOC contains two lectures (the last two of the first week) that explain how addition and multiplication can be expressed in this binary encoding.

Exercise 11.

Give a CNF in two boolean variables p, q that is equivalent to $p \leftrightarrow q$.

Exercise 12.

Consider a CNF on n variables with k distinct clauses all consisting of n distinct literals, and no clause contains both a variable and its negation. Argue that the truth table of the CNF contains exactly k zeros and $2^n - k$ ones.

Exercise 13.

Consider the CNF consisting of the following eight clauses

$$\begin{array}{llll} (1) & p \vee w & (5) & u \vee \neg w \\ (2) & \neg r \vee v & (6) & \neg p \vee \neg v \\ (3) & \neg p \vee r & (7) & p \vee \neg s \\ (4) & \neg t \vee \neg u & (8) & s \vee t \end{array}$$

For both a and b: indicate for every step the number of the clause that is used.

(a) Prove that this CNF is unsatisfiable by resolution.

(b) Prove that this CNF is unsatisfiable by DPLL.

Exercise 14.

Consider the CNF consisting of the following eight clauses

$$\begin{array}{llll} (1) & \neg p \vee t & (5) & p \vee \neg r \\ (2) & \neg q \vee r & (6) & \neg q \vee s \vee \neg t \\ (3) & p \vee q & (7) & r \vee \neg s \vee \neg t \\ (4) & \neg r \vee \neg s & (8) & \neg p \vee q \vee s \end{array}$$

For both a and b: indicate for every step the number of the clause that is used.

(a) Prove that this CNF is unsatisfiable by resolution.

(b) Prove that this CNF is unsatisfiable by DPLL.

Exercise 15.

Find a satisfying assignment of the CNF consisting of the following eight clauses

$$\begin{array}{llll} (1) & \neg p \vee t & (5) & p \vee \neg r \\ (2) & \neg q \vee r & (6) & \neg q \vee s \vee \neg t \\ (3) & p \vee q \vee s & (7) & r \vee \neg s \vee \neg t \\ (4) & \neg r \vee \neg s & (8) & \neg p \vee q \vee s \end{array}$$

Exercise 16.

Compute the Tseitin transformation of $(p \wedge q) \leftrightarrow (r \rightarrow s)$.

Exercise 17.

Compute how many clauses are in the Tseitin transformation of

$$((r \wedge q) \vee (r \rightarrow s)) \leftrightarrow ((r \vee p) \rightarrow (p \wedge s)).$$