Applied Logic, Practice Set 3 2021–2022 Q2

The exercises in this set allow you to practice with the course material. The first three are intended to do yourself using Z3, the remaining ones are particularly suitable as a preparation for the tutorial meetings on December 17 and January 12, and for the third interim test on January 14.

Exercise 1.

Start with 100 marbles.
At every step either 3 marbles are removed (if possible) or the number of marbles is doubled.
Use Z3 to establish what is the least number of steps to reach exactly 1000 marbles.

Exercise 2.

Consider the following program:
for $i := 1$ to 10 do
if $a > b$ then $\langle b := 2b;\ a := a - 3\rangle$
else $\langle a := 2a;\ b := b - 5\rangle$
Use Z3 to find initial values for a, b such that after running this program one has $a = 1000$ and $b = 999$.

Exercise 3.

One dimensional solitaire. On a sequence of n positions in a row, pegs may be placed or not. If from three consecutive positions A, B and C, A and B are occupied by a peg and C is not, then the peg on A may jump over B to C, by which the peg on B is removed. So after the move only C is occupied by a peg and A and B are not. Such moves may be done both from left to right and from right to left. The goal is to repeat such moves until a small number of pegs remains.

(a) Take the case $n = 15$, while in the initial configuration all positions are occupied except for the middle one. Use Z3 to establish the least number of pegs that may remain in this game.

(b) Take the case $n = 12$, while in the initial configuration all positions are occupied except for one. Use Z3 to establish that it is possible to choose this initial free position in such a way that it is possible that only one peg may remain in this game.

Exercise 4.

Formulate by a Hoare triple that the program S (that does not change the values of the integer array A) computes the maximum of all values in $A[1..10]$. In the postcondition you may use quantifications $\forall$, $\exists$, but not the maximum operator.

Exercise 5.

Compute $wp(a := a + 1; b := a + b, a + b = 100)$ and $wp(b := a + b; a := a + 1, a + b = 100)$.

Exercise 6.

Compute $wp(\langle$ if $a < 10$ then $a := a + 1\rangle; b := b + 1, a = b)$ and
$wp($ if $a < 10$ then $\langle a := a + 1; b := b + 1\rangle, a = b)$.

Exercise 7.

Prove that

$$\{a > 0\} \text{ if } a \leq 10 \text{ then } a := a + 10 \; \{a > 10\}.$$

Exercise 8.

Prove that $wp($ if $a < b$ then $a := b - a$ else $a := a - b, a = 10) \equiv (a = b - 10 \vee a = b + 10)$.
Here you may use basic properties like $p \to q \equiv \neg p \vee q$, $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.

Exercise 9.

We start with 100 black marbles and 100 white marbles. We repeat the following step 199 times.

Take two arbitrary marbles.

- If they have different colors, then the white one will be put back, and the black one will be removed.
- If both are black, then one will be put back, and the other will be removed.
- If both are white, then both will be removed, while one black marble from some external repository will be put back.

So at every step the number of marbles decreases by one, and after 199 steps onely one marble remains. What is the color of this remaining marble?

Exercise 10.

European peg solitaire (see wikipedia) is played on the 37 positions marked by a letter in the following 7×7 square:

```
- - A B C - -
- A B C A B -
A B C A B C A
B C A B C A B
C A B C A B C
- B C A B C -
- - A B C - -
```

Initially all positions are occupied by a peg, except for the middle one (a `B`). Moves are done just as in Exercise 3, but now both horizontally and vertically. Prove by an invariant that it is not possible to end in only one peg.

Hint: look at the marking of the positions by `A, B, C`.

Exercise 11.

Prove by an invariant that one-dimensional solitaire as presented in Exercise 3cannot end in a single peg for $n = 85$ and the initial configuration in which all positions are occupied except for the middle one.

Exercise 12.

Prove partial correctness of

$\{a \geq 0 \wedge a = A \wedge b = 1\}$
while $a \neq 0$ do $\langle a := a - 1;\ b := b + b\rangle$
$\{b = 2^A\}$

by means of an invariant.

Exercise 13.

Prove partial correctness of

$\{a = 0 \wedge b = 1 \wedge n \geq 0\}$
while $a \neq n$ do $\langle a := a + 1;\ b := 3b\rangle$
$\{b = 3^n\}$

by means of an invariant.

Exercise 14.

Prove partial correctness of

$\{a = A \wedge b = B \wedge a \geq 0\}$
while $a \neq 0$ do
if $a > 15$ then $\langle a := a - 10;\ b := b + 20\rangle$
else $\langle a := a - 1;\ b := b + 2\rangle$
$\{b = 2A + B\}$

by means of an invariant.

Exercise 15.

Prove partial correctness of

$\{n = N \wedge a = 0 \wedge N > 0\}$
while $n \neq 1$ do $\langle n := n - 1;\ a := a + n\rangle$
$\{a = \frac{N(N-1)}{2}\}$

by means of the invariant $a = \frac{N(N-1)-n(n-1)}{2}$.